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On the Powers of Dead-end Recognizing Systems in the Class of Two-Element Sets Concerning Operations of Intersection and Complement

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Abstract

The notion of n-recognizing system in the class of two-element sets concerning the operations of intersection and complement is defined. The set of powers of dead-end n-recognizing systems of such kind is estimated.

A similar task was considered by the author for the class of two-element sets concerning the operations of intersection [1]. In this paper an analogous problem concerning the operations of intersection and complement is considered. Let us give some definitions by analogy with those given in [1-4].]. Let us consider a finite set $[n] = \{1, 2, ..., n\}$, when $n \geq 3$. By $R[n]$ we denote the set of all subsets of $[n]$. Let $n^* = \{A_1, A_2, ..., A_k\}$ be a subset of the set $R[n]$.

Definition 1: *We say that an element $i \in [n]$ is recognizable by n^* if the set $\{i\}$ can be obtained by the operations of intersection and complement (with respect to $[n]$) from the sets $A_1, A_2, ..., A_k$.*

Definition 2: *We say that $n*$ is n- recognizable if any $i \in [n]$ is recognizable by $n*$.*

By $|A|$ we denote the power of the set A. Obviously the class of n-recognizing systems is not empty. Let us define the notions of dead-end system and minimal system in the class of n-recognizing systems.

Definition 3: *An n-recognizing system S is said to be dead-end if any proper subset of S is not an n-recognizing system.*

Definition 4: *An n-recognizing system S is said to be minimal if there is no n-recognizing system having the power less than $|S|$.*

If maxima and minima are considered not in the class of all recognizing systems but in one of its subclasses then they are called local maxima and minima. In this paper we consider n-recognizing systems containing only two-element subsets of the set $[n]$. All considered definitions refer to this subclass of the class of n-recognizing systems.

For every n-recognizing system S belonging to the mentioned subclass we consider some graph associated with it. This graph is defined as follows. To any number $i \in [n]$ we associate some vertex of the graph; we associate different vertices to different numbers. Two different vertices we connect by an edge if and only if the two-element set of numbers associated to these vertices belongs to the considered n-recognizing system. The graph constructed in such a way is said to be the representing graph of the considered n-recognizing system S. Sometimes we shall identify an n-recognizing system with its representing graph and describe the properties of n-recognizing systems in terms of their representing graphs.

The definitions which are not given here can be found in [5].

It is proved in [4] that the power of any minimal n-recognizing system in the class of two-element subsets is as follows.

$$|T_2| = \begin{cases} \frac{2}{3}n & \text{if } n \equiv 0(mod3); \\ \frac{2}{3}(n-1) & \text{if } n \equiv 1(mod3); \\ \frac{2}{3}(n-2)+1 & \text{if } n \equiv 2(mod3). \end{cases}$$

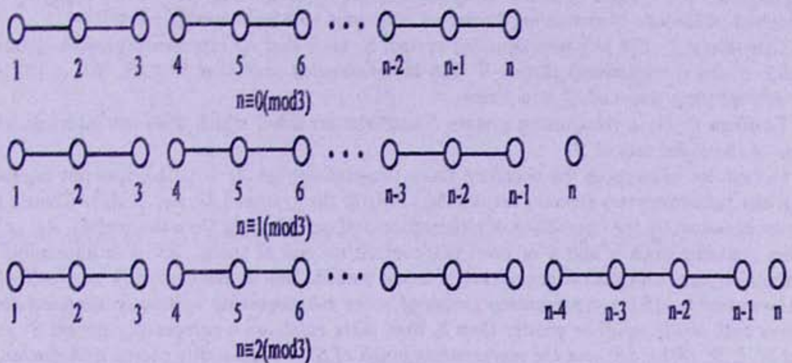

Figure 1:

The examples of minimal n-recognizing systems in the cases $n \equiv 0(mod3)$, $n \equiv 1(mod3)$, $n \equiv 2(mod3)$ are given on the Fig. 1.

Definition 5 (cfr.[5]): *A tree S having m vertices is said to be a star (or m-star) if there exists a vertex in S adjacent with all other $(m-1)$ vertices.*

For any $n \geq 3$ let us consider a forest containing two components of connectivity; one of them is an $(n-1)$-star and the other one is an isolated vertice. We denote such a forest by L_n^2. It is easy to see that the n-recognizing system for which L_n^2 is the representing graph is an dead-end system having the power $(n-2)$. Indeed, if we remove some edge from L_n^2 then the remaining graph will contain two isolated vertices and such a graph cannot be representing graph for an n-recognizing system (see below, lemma1).

Theorem 1: *If $n \geq 4$ then the power of any dead-end n-recognizing system in the subclass of two-element subsets is less or equal to $n-2$.*

Theorem 2: *If $n \geq 7$ then for any j such that $|T_2| \leq j \leq (n-2)$ there exists a dead-end n-recognizing system having the power j.*

Lemma 1: *If $n^* = \{A_1, A_2, \dots, A_k\}$ is n-recognizing system then the union of all sets A_i can exclude maximum one element of the set $[n]$.*

Indeed, if we suppose the contrary, i.e. $\bigcup_{i=1}^{k} A_i$ does not contain some different x and y belonging to $[n]$, then x and y do not belong to any A_i, $i = \overline{1,k}$ but belong to all $B_i = \overline{A_i}$. It is evident that applying any operation of intersection, and complement over the system $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k$ we will obtain sets which either contain both x and y or don't contain any of them. So it is impossible to distinguish one from the other and hence x and y could not be recognized. Lemma 1 is proved.

Lemma 2: *If the representing graph of some n-recognizing system S contains cycles, then there exists an n-recognizing system S^* such that $S^* \subset S$, $|S^*| < |S|$ and the representing graph of S^* does not contain cycles.*

Indeed, assume that the representing graph of S contains a cycle $\{v_1, v_2, ..., v_{m-1}, v_m, v_1\}$. Let us remove the edge $\{v_1, v_2\}$ from it. Since the removing of $\{v_1, v_2\}$ provides impact only on vertices v_1 and v_2 then the recognition of the remaining vertices will not change. But $\{v_2\} = \{v_2, v_3\} \bigcap \{v_3, v_4\}$ (it is worth to mention that v_4 can coincide with v_1 if the cycle is of length 3). By analogy $\{v_1\} = \{v_1, v_m\} \bigcap \{v_m v_{m-1}\}$. So, the resulting system is recognizing and contains a set and a cycle less than the original n-recognizing system. By applying the described procedure to remaining cycles we will come to acyclic graph.

Corollary 1: *For any n-recognizing system S, such that its representing graph contains cycles, a new n-recognizing system S^* can be constructed such that $S^* \subset S$, $|S^*| < |S|$ and the representing graph of S^* is a forest.*

Lemma 3: No n-recognizing system S can contain a set, which does not intersect with none of the other sets of S.

Indeed, let us suppose the contrary, i.e. a two-element set $A_1 = \{x, y\}$ does not intersect with any remaining two-element set $A_2, A_3, ..., A_k$ of the system $\{A_1, A_2, ..., A_k\}$. Clearly every set obtained by the operations of intersection and complement from the sets $A_1, A_2, ..., A_k$ either contains both x and y or does not contain no one of them. So, it is impossible to distinguish one from the other and hence x and y could not be recognized.

Lemma 4: *If the representing graph of some n-recognizing system S contains some chains with length equal or greater than 5, then there exists an n-recognizing system S^* such that $S^* \subset S$, $|S^*| < |S|$ and the representing graph of S^* contains only chains with the length less or equal to 4.*

Indeed, let us consider a chain of length ≥ 5 in the n-recognizing system S. Let's remove the edge which is on the distance ≥ 3 from both its ends(it is always possible since the length of chain is ≥ 5). Denote by u and v the ends of the removed edge, then there exists a chain of length ≥ 2 starting from v (correspondingly the chain of length ≥ 2 starting from u exists). Let the chain be $\{v_1, v_2, ...\}$ (or $\{u_1, u_2, ...\}$).Then $\{v\} = \{vv_1\} \bigcap \{v_1 v_2\}$, $\{u\} = \{uu_1\} \bigcap \overline{\{u_1 u_2\}}$ and consequently the obtained system is recognizing.(The state of remaining vertices will not change and hence their recognition will not be violated). Applying the described procedure to all chains of length ≥ 5, we obtain the final system S^*. Proceeding from lemmas proved above and the corollary1 we obtain the following consequence.

Corollary 2: *Any dead-end n-recognizing system is a forest satisfying the following conditions:*

- *It contains no more than one isolated vertices.*
- *It does not contain isolated chains containing only one edge.*
- *It does not contain chains of length ≥ 5.*

Let us denote the number of connectivity components of the forest S by p. Let x_i be the number of vertices in the i-th component of connectivity of S. Then the numbers x_i, $i = \overline{1, p}$ satisfy the equality $x_1 + x_2 + \cdots + x_p = n$. Since any component of connectivity is a tree then the number of edges in the i-th component is $x_i - 1$ and the number of edges in the total forest is $(x_1 - 1) + (x_2 - 1) + \cdots + (x_p - 1) = x_1 + x_2 +) + \cdots + x_p - p = n - p$.

Thus, the number of edges depends only on p and n. It is clear than $p \geq 1$. If $p = 1$ then we have a tree which is a representing graph of an n-recognizing but not dead-end system (because any edge leading to an hanging vertice of this tree may be removed. and the property of being representing graph of an n-recognizing system will not be preserved). Hence the greatest value for $(n-p)$ is obtained when $p=2$ and in this case we can take L_n^2 (Fig. 2a) as the representing graph of an n-recognizing system having the power $n-2$. So. Theorem 1 is proved.

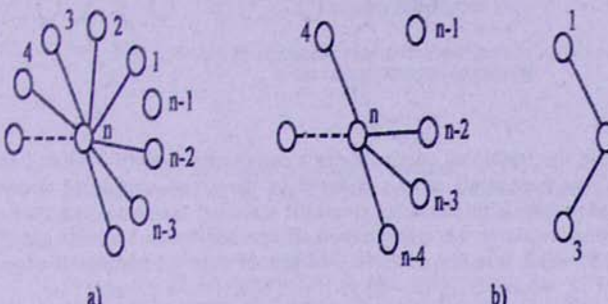

Figure 2:

Let us define the operation of partition for the graph L_m^2 in the case $m \geq 7$.

Definition 6: *The replacement of the graph L_m^2 with graphs L_{m-3}^2 and z_3 is called a partition of the graph L_m^2. The partition of the graph L_n^2 into $L_{n-3}^2 + z_3$ is shown on the Fig. 2b.*

Lemma 5: *The result of the partition of the representing graph L_m^2 of m-recognizing system with $m \geq 7$ is the representing graph of an m-recognizing dead-end system and the number of edges in it is less than the number of edges in L_m^2 by one.*

It has been already mentioned above that L_m^2 is m-recognizing and dead-end. If $m \geq 7$ then $m-3 \geq 4$ and hence the number of edges in L_{m-3}^2 will be greater than 2. which ensures recognition of all elements from L_{m-3}^2. Considering the elements of z_3 it is easy to notice that they are also m-recognizable. The graph is dead-end since the removing of any edge leads to the second isolated vertex. which is not allowed. The change of the number of edges is defined by the fact that we removed 3 from L_m^2 and added 2 edges.

Corollary 3: *The result of the partition of the graph L_n^2 is an n-recognizing dead-end system with $n-3$ edges. (sets).*

Applying the operation of partition on the component L_i^2 unless $i \geq 7$ we get n-recognizing dead-end systems containing one edge less than L_i^2

The components of z_3 do not change during partition, meanwhile a new component z_3 is obtained at every step of partition. The process of partition is continued until the result is an n-recognizing system and the obtained components of z_3 type are the following subgraphs (Fig. 3).

In the first situation the remaining part contains only one component z_3 and an isolated vertex. i.e. the final result coincides with the graph considered above in the case $n \equiv 1$ $(mod3)$ (Fig..1) which is minimal.

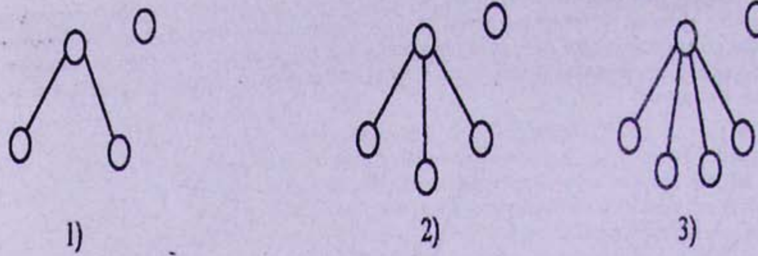

Figure 3:

In the second situation the remaining part contains a component z_4 with 3 edges (as in chains of length 3), i.e. the final result can be replaced by the graph considered above in case $n \equiv 0\ (mod3)$ (Fig.1) which is minimal. In the third situation the remaining part can be replaced with two components z_3, i.e. the final result can be replaced by the graph in case $n \equiv 0\ (mod3)$ (Fig.1) which is minimal. In fact we proved that the number of edges in the result of partition of L_n^2 reduces from $(n-2)$ to $|T_2|$. Theorem 2 is proved.

Remark: *When n takes values 3,4,5 and 6 then the greatest possible length of dead-end systems coincides with the length of minimal n-recognizing system and is equal to 2,2,3 and 4 correspondingly.*

References

[1] С. М. Варданян, "О длинах тупиковых распознающих систем в классе двухэлементных подмножеств", *ДАН НАН РА*, том 107, N 1, с с. 37 - 43, 2007.

[2] С. М. Варданян, "Об одной задаче распознавания множеств", *ДАН Арм. ССР*, том 72, с с.141- 143, 1981.

[3] S. M. Vardanyan, "Recognizig sets (systems) Computer Science and Information Technologies", *Procedings of the conference CSIT05*, pp.161 - 162, Yerevan, Armenia 2005.

[4] С. М. Варданян, "О минимальности некоторых распознающих систем в классе двухэлементных подмножеств относительно операций пересечения и дополнения", (to appear in the Reports of NAS of RA).

[5] F. Harary, Graph Theory, Addison - Wesley. Reading MA 1969.

Հատման և լրացման գործողությունների նկատմամբ երկտարրանի բազմությունների դասում ճանաչող փակուղային համակարգերի հզորությունների մասին

Ս. Վարդանյան

Ամփոփում

Աշխատանքում սահմանվում է n-տարրանի բազմության n-ճանաչող համակարգ հասկացությունը ըստ բազմությունների հատման ու լրացման գործողությունների և երկտարրանի ենթաբազմությունների դասում գնահատվում է փակուղային n-ճանաչող համակարգերի հզորությունների բոլոր հնարավոր արժեքները: