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Many Hypotheses LAO Testing With Rejection of Decision for Arbitrarily Varying Object

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Abstract

The model of multiple statistical hypotheses testing with possibility rejecting to make choices between hypotheses concerning discrete arbitrarily varying object is investigated. The optimal procedure of decisions is shown. The matrix of optimal asymptotical interdependencies (reliability-reliability functions) of all possible pairs of the error probability exponents (reliabilities) is studied for arbitrarily varying object with the current states sequence known to the statistician.

1. Introduction

The classical problem of hypothesis testing refers to the case of two hypotheses. Based on data sample a statistician makes decision on which of the two proposed hypotheses H_1 and H_2 is correct. The problem was studied for the case of an asymptotic test, where the probabilities of error decrease exponentially when the number of observations tends to the infinity. The goal of research was to express the functional relation between the error probabilities exponents of the first and the second types of the optimal test sequence. That test was considered first by Hoffding [1], examined later by Csiszár and Longo [2], Tusnady [3], (he called such test series exponentially rate optimal (ERO)), Longo and Sgarro [4]. Following Birgé [5] we called the sequence of such tests logarithmically asymptotically optimal (LAO).

In the information theory literature we can also note works of Natarajan [6], Gutman [7], Anantharam [8], Han [9] and of many others.

Haroutunian [10, 11] investigated the problem of many hypotheses LAO testing. The multiple hypotheses testing problem for arbitrarily varying object with side information was examined by Haroutunian and Hakobyan [12]. The case of two hypotheses when state sequences are not known to the decision maker was studied by Fu and Shen [13] and when states sequence is known to statistician was examined by Ahlswede, Aloyan and Haroutunian [14].

In this paper the problem of a many hypotheses asymptotically optimal testing has been explored for a model with possibility to withdraw a judgement. The problem solved here is induced by the paper of Nikulin [15] concerning two hypothesis testing with refusal to take decision. Asymptotically optimal classification, in particular hypotheses testing problems with rejection were considered by Gutman [7].

Information theoretic methods are used. in particular. the method of types of Csiszár and Körner [16]. Applications of methods of information theory in mathematical statistics. specifically in hypotheses testing. are reflected also in the monographs by Blahut [17]. Cover and Thomas [18], Csiszár and Shields [19].

2. Many Hypothesis Testing With Rejection of Decision by Informed Statistician for Arbitrarily Varying Object

An arbitrarily varying object is a generalized model of the discrete memoryless one. Let $\mathcal{X}$ be a finite set of values of random variable X characterizing the object in consideration and $\mathcal{S}$ be an alphabet of states of the object. which changing each moment of time n. M possible probability distributions (PD) X are known:

$$G_m = \{G_m(x|s),\ x \in \mathcal{X},\ s \in \mathcal{S}\}.\ m = \overline{1, M},$$

but it is not known which of these alternative hypotheses $H_m : G = G_m$, $m = \overline{1, M}$, is in reality. The source of states produces sequence $\mathbf{s} = (s_1, s_2, ... s_N)$, $s_n \in \mathcal{S}$. The statistician must select one among M hypotheses or he can withdraw any judgement. An answer must be defined using vector of results of N independent experiments $\mathbf{x} \triangleq (x_1, x_2, ... x_N)$ and state sequence $\mathbf{s} = (s_1, s_2, ... s_N)$. The procedure of decision making is a non-randomized test $\varphi_N(\mathbf{x}, \mathbf{s})$, it can be defined by division of the sample space $\mathcal{X}^N$ on $M + 1$ disjoint subsets $\mathcal{A}_m^N(\mathbf{s}) = \{\mathbf{x} : \varphi_N(\mathbf{x}, \mathbf{s}) = m\}$, $m = \overline{1, M+1}$. The set $\mathcal{A}_m^N(\mathbf{s})$, $m = \overline{1, M}$, consists of vectors $\mathbf{x}$ for which the hypothesis H_m is adopted, and $\mathcal{A}_{M+1}^N(\mathbf{s})$ includes all vectors for which we refuse to take certain answer.

We study for all pairs the probabilities of the erroneous acceptance of hypothesis H_l provided that H_m is true

$$\alpha_{l|m}(\varphi_N) \triangleq \max_{\mathbf{s} \in \mathcal{S}^N} G_m^N\left(\mathcal{A}_l^N\right).\ m, l = \overline{1, M},\ m \neq l. \tag{2}$$

When we decline decision, but hypothesis H_m is true. we consider the following probability of error:

$$\alpha_{M+1|m}(\varphi_N) \triangleq \max_{\mathbf{s} \in \mathcal{S}^N} G_m^N\left(\mathcal{A}_{M+1}^N\right).$$

If the hypothesis H_m is true, but it is not accepted, or equvalently while the statistician accepted one of hypotheses H_l, $l = \overline{1, M}$, $l \neq m$. or refused to make decision, then probability of error is the following:

$$\alpha_{m|m}(\varphi_N) \triangleq \sum_{l \neq m} \alpha_{l|m}(\varphi_N) = \max_{\mathbf{s} \in \mathcal{S}^N} G_m^N\left(\overline{\mathcal{A}_m^N}\right),\ m = \overline{1, M}. \tag{3}$$

Corresponding error probability exponents. called "reliabilities". are defined as follows:

$$E_{l|m}(\varphi) \triangleq \overline{\lim_{N \to \infty}} \left\{-\frac{1}{N} \log \alpha_{l|m}(\varphi_N)\right\}.\ m = \overline{1, M}.\quad l = \overline{1, M+1}. \tag{4}$$

In this paper functions exp and log are considered at the base 2.

It follows from (3) that for every test φ

$$E_{m|m}(\varphi) = \min_{l = \overline{1, M+1},\ l \neq m} E_{m|l}(\varphi).\ m = \overline{1, M}. \tag{5}$$

The matrix

$$\mathbf{E}(\varphi) = \begin{pmatrix} E_{1|1} \dots E_{l|1} \dots E_{M|1} \; E_{M+1|1} \\ \dots\dots\dots\dots\dots\dots \\ E_{1|m} \dots E_{l|m} \dots E_{M|m} \; E_{M+1|m} \\ \dots\dots\dots\dots\dots\dots \\ E_{1|M} \dots E_{l|M} \dots E_{M|M} \; E_{M+1|M} \end{pmatrix}$$

is called the reliability matrix of the tests sequence φ.

We call tests LAO for this model if for given positive values of M elements of the matrix $\mathbf{E}$ the procedure provides maximal values for other elements of it.

We exploit the method of types [16]. For $\mathbf{s} = (s_1, \dots, s_N)$, $\mathbf{s} \in \mathcal{S}^N$, let $N(s \mid \mathbf{s})$ be the number of occurrences of $s \in \mathcal{S}$ in the vector $\mathbf{s}$. The type (or empirical distribution) of $\mathbf{s}$ is the distribution $P_{\mathbf{s}} = \{P_{\mathbf{s}}(s), s \in \mathcal{S}\}$ defined by

$$P_{\mathbf{s}}(s) = \frac{1}{N} N(s \mid \mathbf{s}), \qquad s \in \mathcal{S}.$$

For a pair of sequences $\mathbf{x} \in \mathcal{X}^N$ and $\mathbf{s} \in \mathcal{S}^N$, let $N(x, s|\mathbf{x}, \mathbf{s})$ be the number of occurences of $(x, s) \in \mathcal{X} \times \mathcal{S}$ in the pair of vectors $(\mathbf{x}, \mathbf{s})$. The joint type of the pair $(\mathbf{x}, \mathbf{s})$ is the distribution $Q_{\mathbf{x},\mathbf{s}} = \{Q_{\mathbf{x},\mathbf{s}}(x, s), x \in \mathcal{X}, s \in \mathcal{S}\}$ defined by

$$Q_{\mathbf{x},\mathbf{s}}(x, s) = \frac{1}{N} N(x, s \mid \mathbf{x}, \mathbf{s}), \qquad x \in \mathcal{X}, \quad s \in \mathcal{S}.$$

The conditional type of $\mathbf{x}$ for given $\mathbf{s}$ is the conditional distribution $Q_{\mathbf{x}|\mathbf{s}} = \{Q_{\mathbf{x}|\mathbf{s}}(x|s), x \in \mathcal{X}, s \in \mathcal{S}\}$ defined by

$$Q_{\mathbf{x}|\mathbf{s}}(x|s) = \frac{Q_{\mathbf{x},\mathbf{s}}(x, s)}{P_{\mathbf{s}}(s)} = \frac{N(x, s|\mathbf{x}, \mathbf{s})}{N(s \mid \mathbf{s})}, \qquad x \in \mathcal{X}, \quad s \in \mathcal{S}.$$

Let X and S are some random variables defined by probability distributions $P = \{P(s), s \in \mathcal{S}\}$ and $Q = \{Q(x|s), x \in \mathcal{X}, s \in \mathcal{S}\}$. The conditional entropy of X for given S is:

$$H_{P,Q}(X \mid S) = -\sum_{x,s} P(s) Q(x|s) \log Q(x|s).$$

The conditional divergences (Kullback-Leibler information) of the distribution $P \circ Q = \{P(s)Q(x|s), x \in \mathcal{X}, s \in \mathcal{S}\}$ with respect to the distribution $P \circ G_l = \{P(s) G_l(x|s), x \in \mathcal{X}, s \in \mathcal{S}\}$ is

$$D(P \circ Q || P \circ G_m | P) = D(Q || G_m | P) = \sum_{x,s} P(s) Q(x|s) \log \frac{Q(x|s)}{G_m(x|s)}, \;\; l = \overline{1, m}.$$

We denote by $\mathcal{P}^N(\mathcal{S})$ the set of all types on $\mathcal{S}$ for given N, by $\mathcal{P}(\mathcal{S})$ – the set of all possible probability distributions P on $\mathcal{S}$ and by $\mathcal{Q}^N(\mathcal{X}|\mathbf{s})$ the set of all possible conditional types on $\mathcal{X}$ for given $\mathbf{s}$. Let $\mathcal{T}^N_{P_{\mathbf{s}},Q}(X \mid \mathbf{s})$ is the family of vectors $\mathbf{x}$ of the conditional type Q for given $\mathbf{s}$ of the type $P_{\mathbf{s}}$. We need the following inequalities [16]:

$$| \mathcal{Q}^N(\mathcal{X}|\mathbf{s}) | \leq (N+1)^{|\mathcal{X}||\mathcal{S}|}. \tag{6}$$

for any type $P \in \mathcal{P}^N$

$$(N+1)^{-|\mathcal{X}||\mathcal{S}|} \exp\{N H_{P_{\mathbf{s}},Q}(X|S)\} \leq | \mathcal{T}^N_{P_{\mathbf{s}},Q}(X \mid \mathbf{s}) | \leq \exp\{N H_{P_{\mathbf{s}},Q}(X|S)\}. \tag{7}$$

For construction of the necessary LAO test for preliminarily given positive numbers $E_{1|1},...,E_{M|M}$, we define the following subsets of distributions:

$$\mathcal{R}_m(P) \triangleq \{Q: \ D(Q||G_m|P) \leq E_{m|m}\}, \ \ m=\overline{1,M}. \tag{8a}$$

$$\mathcal{R}_{M+1}(P) \triangleq \{Q: \ D(Q||G_m|P) > E_{m|m}, \ \ m=\overline{1,M}\}, \tag{8b}$$

and the following values of reliabilities:

$$E^*_{m|m} = E^*_{m|m}(E_{m|m}) \triangleq E_{m|m}, \ \ m=\overline{1,M}, \tag{9a}$$

$$E^*_{l|m} = E^*_{l|m}(E_{l|l}) \triangleq \inf_{P\in\mathcal{P}(S)} \inf_{Q\in\mathcal{R}_l(P)} D(Q||G_m|P), \ \ m=\overline{1,M}, \ m\neq l, \ l=\overline{1,M}. \tag{9b}$$

$$E^*_{M+1|m} = E^*_{M+1|m}(E_{1|1},E_{2|2},...,E_{M|M}) \triangleq \inf_{P\in\mathcal{P}(S)} \inf_{Q\in\mathcal{R}_{M+1}(P)} D(Q||G_m|P), \ \ m=\overline{1,M}. \tag{9c}$$

The main result of the paper is formulated in

Theorem 1: *If all distributions G_m, $m=\overline{1,M}$, are different in the sense that $D(G_l||G_m)>0$, $l<m$, and the positive numbers $E_{1|1},E_{2|2},...,E_{M|M}$ are such that the following inequalities hold*

$$E_{1|1} < \min_{P\in\mathcal{P}(S)} \min_{m=\overline{2,M}} D(G_m||G_1), \tag{10.a}$$

$$E_{m|m} < \min\left[\min_{l=\overline{1,m-1}} E^*_{l|m}(E_{l|l}), \ \min_{P\in\mathcal{P}(S)} \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \ \ m=\overline{2,M-1}, \tag{10.b}$$

$$E_{M|M} < \min_{l=\overline{1,M-1}} E^*_{l|M}(E_{l|l}), \tag{10.c}$$

then there exists a LAO sequence of tests, all elements of the reliability matrix of which $\mathbf{E}^*=\{E^*_{l|m}\}$ are positive and are defined in (9).

When one of the inequalities (10.a), *or* (10.b) *is violated, then at least one element of the matrix $\mathbf{E}^*$ is equal to* 0.

3. Proof of the Theorem 1

We begin by proof of prove the positive statement of the theorem. We consider sequence of test φ^*, which is defined by division of the sample space $\mathcal{X}^N$ on the following $M+1$ subsets for every $\mathbf{s}\in\mathcal{S}^N$

$$B^{(N)}_m(\mathbf{s}) = \bigcup_{Q\in\mathcal{R}^{(N)}_m(P_\mathbf{s})} \mathcal{T}^N_{P_\mathbf{s},Q}(X|\mathbf{s}) \ \ \ m=\overline{1,M+1}. \tag{11}$$

Let us prove that the collection of sets in (11) determines a test, namely each $\mathbf{x}$ belongs to one and only to one subset $B^{(N)}_m(\mathbf{s})$:

$$B^{(N)}_m(\mathbf{s})\bigcap B^{(N)}_r(\mathbf{s}) = \emptyset, \ \ m\neq r, \ \text{ and } \ \bigcup_{m=1}^{M+1} B^{(N)}_m(\mathbf{s}) = \mathcal{X}^N.$$

Really, for $m=\overline{1,M}$, $r=\overline{2,M}$, for each $m<r$, let us consider arbitrary $\mathbf{x}\in B^{(N)}_m(\mathbf{s})$. It follows that there are $\mathcal{T}^N_{P_\mathbf{s},Q}(X|\mathbf{s})\subset B^{(N)}_m(\mathbf{s})$, such that $D(Q_\mathbf{x}||G_m|P_\mathbf{s})\leq E_{m|m}$. Because $m<r$, from conditions (10) it follows that $E_{r|r}<E^*_{m|r}(E_{m|m})$. From definition (9b) and

inequality $D(Q_x||G_m|P_s) \leq E_{m|m}$ we obtain that $E_{r|r} < E^*_{m|r}(E_{m|m}) < D(Q_x||G_r|P_s)$. Hence $Q_x \notin \mathcal{R}_r$ and from (11) it follows that $\mathbf{x} \notin \mathcal{B}_r^{(N)}(\mathbf{s})$.

We can verify that $\mathcal{B}_{M+1}^{(N)}(\mathbf{s}) \cap \mathcal{B}_m^{(N)}(\mathbf{s}) = \emptyset$, $m = \overline{1,M}$, because if $\mathbf{x} \in \mathcal{B}_{M+1}^{(N)}(\mathbf{s})$ then for type Q_x the following inequality is true $D(Q_x||G_m|P_s) > E_{m|m}$, $m = \overline{1,M}$. According to definition of $\mathcal{B}_m^{(N)}(\mathbf{s})$, $m = \overline{1,M}$, we see that $\mathbf{x} \notin \mathcal{B}_m^{(N)}(\mathbf{s})$.

When H_m is true, the sample $\mathbf{x} = (x_1, x_2, ..., x_N)$ from $\mathcal{T}^N_{P_s,Q}(X|\mathbf{s})$ has the following probability

$$G_m^N(\mathbf{x} \mid \mathbf{s}) = \prod_{n=1}^{N} G_m(x_n|s_n) = \prod_{x,s} G_m(x|s)^{N(x,s|\mathbf{x},\mathbf{s})} = \prod_{x,s} G_m(x|s)^{NP_s(s)Q(x|s)}$$

$$= \exp\{N\sum_{x,s}(-P_s(s)Q(x|s)\log\frac{Q(x|s)}{G_m(x|s)} + P_s(s)Q(x|s)\log Q(x|s))\} \tag{12}$$

$$= \exp\{-N[D(Q \parallel G_m|P_s) + H_{P_s,Q}(X \mid S)]\}.$$

Now, for $m = \overline{1,M}$, using (3), (6), (8), (10), (11) and (12) we can estimate $\alpha_{m|m}^{(N)}(\varphi^*)$ as follows:

$$\alpha_{m|m}^{(N)}(\varphi^*) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N(\overline{\mathcal{B}_m^{(N)}(\mathbf{s})}|\mathbf{s}) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N\left(\bigcup_{Q:D(Q||G_m|P_s)>E_{m|m}} \mathcal{T}^N_{P_s,Q}(X|\mathbf{s})\middle|\mathbf{s}\right)$$

$$\leq \max_{\mathbf{s}\in\mathcal{S}^N}(N+1)^{|\mathcal{X}||\mathcal{S}|} \sup_{Q:D(Q||G_m|P_s)>E_{m|m}} G_m(\mathcal{T}^N_{P_s,Q}(X|\mathbf{s})|\mathbf{s})$$

$$\leq (N+1)^{|\mathcal{X}||\mathcal{S}|} \sup_{P_s\in\mathcal{P}^N(\mathcal{S})} \sup_{Q:D(Q||G_m|P_s)>E_{m|m}} \exp\{-ND(Q||G_m|P_s)\}$$

$$\leq \exp\{-N[\inf_{P\in\mathcal{P}(\mathcal{S})} \inf_{Q:D(Q||G_m|P)>E_{m|m}} D(Q||G_m|P) - o_N(1)]\} \leq \exp\{-N[E_{m|m} - o_N(1)]\},$$

where $o_N(1) \to 0$ with $N \to \infty$. From here (9.a) follows.

We can obtain similar estimates for $l = \overline{1,M}$, $m = \overline{1,M}$, $l \neq m$. According to (2), (7), (8), (11) and (12) we have

$$\alpha_{l|m}(\varphi_N^*) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N(\mathcal{B}_l^{(N)}(\mathbf{s})|\mathbf{s}) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N\left(\bigcup_{Q:D(Q||G_l|P_s)\leq E_{l|l}} \mathcal{T}^N_{P_s,Q}(X|\mathbf{s})\middle|\mathbf{s}\right)$$

$$\leq \max_{\mathbf{s}\in\mathcal{S}^N}(N+1)^{|\mathcal{X}||\mathcal{S}|} \sup_{Q:D(Q||G_m|P_s)\leq E_{l|l}} G_m^N(\mathcal{T}^N_{P_s,Q}(X|\mathbf{s})|\mathbf{s})$$

$$\leq (N+1)^{|\mathcal{X}||\mathcal{S}|} \sup_{P_s\in\mathcal{P}^N(\mathcal{S})} \sup_{Q:D(Q||G_m|P_s)\leq E_{l|l}} \exp\{-ND(Q||G_m|P_s)\} \tag{13}$$

$$= \exp\{-N(\inf_{P_s\in\mathcal{P}^N(\mathcal{S})} \inf_{Q:D(Q||G_m|P_s)\leq E_{l|l}} D(Q||G_m|P_s) - o_N(1))\}.$$

Now let us prove the inverse inequalities for $l = \overline{1,M}$, $m = \overline{1,M}$, $l \neq m$. From (2), (7), (11) and (12) we obtain:

$$\alpha_{l|m}(\varphi_N^*) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N(\mathcal{B}_l^{(N)}(\mathbf{s})|\mathbf{s}) = \max_{\mathbf{s}\in\mathcal{S}^N} G_m^N\left(\bigcup_{Q:D(Q||G_l|P_s)\leq E_{l|l}} \mathcal{T}^N_{P_s,Q}(X|\mathbf{s})\middle|\mathbf{s}\right).$$

$$\geq \max_{s\in S^N} \sup_{Q:D(Q||G_l|P_s)\leq E_{l|l}} G_m^N(T_{P_s,Q}(X|s)|s)$$

$$\geq \sup_{P_s\in\mathcal{P}^N(S)} (N+1)^{-|X||S|} \sup_{Q:D(Q||G_l|P_s)\leq E_{l|l}} \exp\{-ND(Q||G_m|P_s)\} \quad (14)$$

$$= \exp\{-N(\inf_{P_s\in\mathcal{P}^N(S)} \inf_{Q:D(Q||G_l|P_s)\leq E_{l|l}} D(Q||G_m|P_s) + o_N(1))\}.$$

According to the definition (4) the reliability $E_{m|l}(\varphi^*)$ of the test sequence φ^* is the limit superieure $\overline{\lim_{N\to\infty}} \left\{-\frac{1}{N}\log \alpha_{l|m}(\varphi_N^*)\right\}$, taking into account (13), (14) and the continuity of the functional $D(Q||G_l|P_s)$ we obtain that $\lim_{N\to\infty} \left\{-\frac{1}{N}\log \alpha_{l|m}(\varphi_N^*)\right\}$ exists and in correspondence with (9b) equals to $E_{l|m}^*$. Thus $E_{l|m}(\varphi^*) = E_{l|m}^*$, $m=\overline{1,M}$, $l=\overline{1,M}$, $l\neq m$.

Similarly we can obtain the upper and lower bounds for $\alpha_{M+1|m}^N(\varphi^*)$, $m=\overline{1,M}$. Applying the analogical resonnement we get (9.c). The proof of the first part of the theorem will be accomplished if we demonstrate that the sequence of tests φ^* is LAO, that is for given $E_{1|1},\dots,E_{M|M}$ and every other sequence of tests φ^{**} for all $m\in\overline{1,M+1}$, $l=\overline{1,M}$, $E_{m|l}(\varphi^{**})\leq E_{m|l}^*$.

Suppose the contrary, that there exists sequence of tests φ^{**} defined by the sets $\mathcal{D}_1^{(N)},\dots,\mathcal{D}_{M+1}^{(N)}$ and

$$E_{m|l}(\varphi^{**}) > E_{m|l}(\varphi^*), \text{ for some } m=\overline{1,M+1},\ l=\overline{1,M},\ m\neq l. \quad (15)$$

For N large enough this condition is equivalent to the inequality

$$\alpha_{m|l}(\varphi_N^{**}) \leq \alpha_{m|l}(\varphi_N^*). \quad (16)$$

Let us examine the sets $D_m^{(N)}\cap B_m^{(N)}$, $m=\overline{1,M}$. This intersection cannot be empty, because in that case

$$\alpha_{m|m}(\varphi_N^{**}) = \max_{s\in S^N} G_m^N\left(\overline{\mathcal{D}}_m^{(N)}\right) \geq G_m^N\left(B_m^{(N)}(s)|s\right)$$

$$\geq \max_{s\in S^N} \max_{Q:D(Q||G_m|P_s)\leq E_{m|m}} G_m^N\left(T_{P,Q}^N(X|S)|s\right) \geq \exp\{-N(E_{m|m}+o_N(1))\}.$$

Let us show that $B_m^{(N)}\cap\mathcal{D}_l^{(N)}=\emptyset$, $m=\overline{1,M}$, $l=\overline{1,M+1}$. Suppose the contrary. If there exists Q such that $D(Q||G_m)\leq E_{m|m}$ and $T_{P,Q}^N(X|S)|s)\subset\mathcal{D}_l^N(s)$, then

$$\alpha_{l|m}(\varphi_N^{**}) = \max_{s\in S^N} G_m^N\left(\overline{\mathcal{D}}_m^{(N)}(s)\right) \geq G_m^N\left(B_m^{(N)}(s)|s\right) > G_m^N\left(T_Q^N(X)\right)$$

$$\geq \exp\{-N(E_{m|m}+o_N(1))\}.$$

When $\emptyset\neq\mathcal{D}_l^{(N)}(s)\cap T_{P,Q}^N(X|S)|s)\neq T_{P,Q}^N(X|S)|s)$, we also obtain that

$$\alpha_{l|m}(\varphi_N^{**}) > G_l^N\left(\mathcal{D}_l^{(N)}\bigcap T_{P,Q}^N(X|S)|s)\right)$$

$$\geq \exp\{-N(E_{m|m}+o_N(1))\}.$$

From here it follows that $E_{l|m}(\varphi^{**})\leq E_{m|m}$, which in turn according to (5) provides that $E_{l|m}(\varphi^{**})=E_{m|m}$. From (5) it also follows that $E_{m|m}\leq E_{l|m}(\varphi^*)$, $l=\overline{1,M+1}$, hence we have $E_{l|m}(\varphi^{**}) < E_{l|m}(\varphi^{*})$, for all $l = \overline{1, M+1}$, $m = \overline{1, M}$ which contradicts to (15). Hence we conclude that $\mathcal{D}_m^{(N)} \cap \mathcal{B}_m^{(N)} = \mathcal{B}_m^{(N)}$ for $m = \overline{1, M}$.

Let us consider intersections $\mathcal{D}_m^{(N)}(\mathbf{s}) \cap \mathcal{B}_{M+1}^{(N)}(\mathbf{s})$, $m = \overline{1, M}$. These intersections are empty too. If $\mathcal{D}_m^{(N)} \cap \mathcal{B}_{M+1}^{(N)}(\mathbf{s}) \neq \emptyset$, and because $\mathcal{D}_m^{(N)}(\mathbf{s}) \cap \mathcal{B}_m^{(N)}(\mathbf{s}) = \mathcal{B}_m^{(N)}(\mathbf{s})$, $\mathcal{D}_m^{(N)}(\mathbf{s}) \cap \mathcal{D}_{M+1}^{(N)}(\mathbf{s}) = \emptyset$, $\mathcal{B}_m^{(N)}(\mathbf{s}) \cap \mathcal{B}_{M+1}^{(N)}(\mathbf{s}) = \emptyset$ it follows that $|\mathcal{D}_m^{(N)}(\mathbf{s})| > |\mathcal{B}_m^{(N)}(\mathbf{s})|$. Thus

$$\alpha_{m|l}(\varphi_{(N)}^{**}) = G_l(\mathcal{D}_m^{(N)}(\mathbf{s})) \geq G_l(\mathcal{B}_m^{(N)}) = \alpha_{m|l}(\varphi_{(N)}^{*}),$$

which contradicts to (16). Thus we conclude that $\mathcal{D}_m^N = \mathcal{B}_m^N$, $m = \overline{1, M+1}$, which means that $\varphi^{**} \equiv \varphi^{*}$ and φ^{*} is LAO test.

For the proof of the second part of Theorem 1 it is enough to remark that if one of the conditions (10) is violated, then from (8) and (9) it follows that at least one of the elements $E_{m|l}$ is equal to 0.

4. Multiple Hypothesis Testing with Rejection of Decision for Invariant Object

For the case of invariant object, the result is the following:

$$\mathcal{R}_m \triangleq \{Q: \ D(Q||G_m) \leq E_{m|m}\}, \ \ m = \overline{1, M}. \tag{17a}$$

$$\mathcal{R}_{M+1} \triangleq \{Q: \ D(Q||G_m) > E_{m|m}, \ \ m = \overline{1, M}\}, \tag{17b}$$

$$E^*_{m|m} = E^*_{m|m}(E_{m|m}) \triangleq E_{m|m}, \ \ m = \overline{1, M}, \tag{18a}$$

$$E^*_{l|m} = E^*_{l|m}(E_{l|l}) \triangleq \inf_{Q \in \mathcal{R}_l} D(Q||G_m), \ \ l, m = \overline{1, M}, \ \ m \neq l, \tag{18b}$$

$$E^*_{M+1|m} = E^*_{M+1|m}(E_{1|1}, E_{2|2}, \ldots, E_{M|M}) \triangleq \inf_{Q \in \mathcal{R}_{M+1}} D(Q||G_m), \ \ m = \overline{1, M}. \tag{18c}$$

The following theorem is the consequence of Theorem 1:

Theorem 2: *If all distributions G_m, $m = \overline{1, M}$, are different in the sense that $D(G_l||G_m) > 0$, $l < m$, and the positive numbers $E_{1|1}, E_{2|2}, \ldots, E_{M|M}$ are such that the following inequalities hold*

$$E_{1|1} < \min_{l = \overline{2, M}} D(G_l||G_1), \tag{19.a}$$

$$E_{m|m} < \min \left[\min_{l = \overline{1, m-1}} E^*_{l|m}(E_{l|l}), \ \min_{l = \overline{m+1, M}} D(G_l||G_m) \right], \ \ m = \overline{2, M-1}, \tag{19.b}$$

$$E_{M|M} < \min_{l = \overline{1, M-1}} E^*_{l|M}(E_{l|l}), \tag{19.c}$$

then there exists a LAO sequence of tests, all elements of the reliability matrix of which $\mathbf{E}^* = \{E^*_{l|m}\}$ are positive and are defined in (18).

When one of the inequalities (19.a) or (19.b) is violated, then at least one element of the matrix $\mathbf{E}^*$ is equal to 0.

Corollary (Generalization of Stein's Lemma): *When $\alpha_{m|m}(\varphi_N) = \varepsilon_m$, $0 < \varepsilon_m < 1$, $m = \overline{1, M}$, $N = 1, 2, \ldots$ then:*

$$\lim_{N \to \infty} \frac{1}{N} \alpha_{l|m} \left(\alpha_{m|m}(\varphi_N) = \varepsilon_m \right) = -D(G_l||G_m).$$

$$m \neq l, \ m, l = \overline{1, M}.$$

$$\lim_{N \to \infty} \frac{1}{N} \alpha_{M+1|m} \left(\alpha_{m|m}(\varphi_N) = \varepsilon_m, \ m = \overline{1, M} \right) = 0, \ m = \overline{1, M}.$$

Remark: It can be shown by analogy with [22] that

$$E^*_{M+1|m} = E_{m|m}, \quad m = \overline{1, M}.$$

That is the elements of the last column are equal to the diagonal elements of the same row and are minimal in this row.

5. Conclusion

The problem of reliabilities investigation for testing with rejection of decision of many hypotheses is solved. In [20] Ahlswede and Haroutunian formulated an ensemble of problems on multiple hypotheses testing and on identification of hypotheses for many objects. The problem of many hypotheses testing for the model consisting of three or more independent objects was investigated in [21], [22]also in series of other works. The problem with rejection of decision can be examined for testing hypotheses concerning two or more varying objects.

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Կամայականորեն փոփոխվող օբյեկտի նկատմամբ որոշումից հրաժարմամբ բազմակի վարկածների LAO ստուգումը

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Ամփոփում

Ուսումնասիրվել է կամայականորեն փոփոխվող օբյեկտի վերաբերյալ որոշումից հրաժարումով բազմակի վարկածների վիճակագրականորեն ստուգման գործընթացը, երբ վարկածների միջև ընտրությունը կատարվում է ըստ անկախ դիտարկումների արդյունքների: Ցույց է տրվել որոշումների ընդունման օպտիմալ ընթացակարգը: Կամայականորեն փոփոխվող օբյեկտի վերաբերյալ, որի վիճակները հայտնի են վիճակագրին հետազոտվել է հնարավոր սխալների զույգերի ցուցիչների (հուսալիությունների) փոխկապվածությունների մատրիցը: Որպես հեևանք, արտածվել են անփոփոխ օբյեկտի վերաբերյալ հրաժարման որոշումով բազմակի վարկածների LAO ստուգման սխալների հուսալիությունների զույգերի փոխկապվածությունները արտահայտող ֆունկցիաները: