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Dual Laplace - Stieltjes Transformations of Critical Risks in Case of Negative Insurance Payments

Artak Martirosyan

Institute for Informatics and Automation Problems of NAS of RA
e-mail: artakm81@inbox.ru

Abstract

The present paper is devoted to critical risks of collective insurance models with negative insurance payments (connected with contracts with usual life rent). Limit theorems arisen in critical situations are represented and the dual Laplass-Stiltjes transformations are found for critical risks arisen in collective insurance risks models with negative insurance payments. The specifications of the considered collective risk model and the adaptive control strategy for multiperiodic insurance risk model introduced by Malinovskii is illustrated.

1. Introduction

We examine the critical risks and its Dual Laplace-Stieltjes transformations (LST) of the long term collective risk model of an insurance company conducting purely rent operations (see [1]).

The insurance industry is subject to intensive regulation. Supervision authorities watch compliance with the regulatory principles designed to balance the solvency and equity requirements. According to the regulatory principles, each insurer must report his financial position yearly or even more often. if required. The insurance process is viewed therefore as a series of successive insurance years. Each year starts with a manager's control intervention which fine-tunes tariffs. reserves. ruin probability and other operational characteristics of the probability mechanism of insurance. Its influence remains in force throughout the whole insurance year. i.e., until the next report and subsequent control intervention. The insurance regulation and supervision would be blind without a comprehensive model. or a set of models, describing the probability mechanism of insurance within an operating period. Suitable is the Lundberg's collective risk model which considers the net result of the risk business of an insurer from the position of a "remote observer. and is often named the main achievement of the 20-th century risk theory. The ruin probability in case of the negative insurance payments, at the first time was studied by Saxen ([2]. [3]) and Arfwedson ([4]).

The financial risk and danger of ruin exist for any company. Risk estimation is a basis of decision-making. In actuarial science the important place is occupied with methods of construction of models of risk.

The background of the present paper is a general multiperiodic controlled risk model introduced in Malinovskii in 2003 (see [5]. [6]). The trajectory of a general multiperiodic insurance process with annual accounting and subsequent annual control may be diagrammed as

$$\omega_0 \underbrace{\xrightarrow{\gamma_0} u_0 \xrightarrow{\pi_1} \omega_1}_{1\ st\ year} \ldots \xrightarrow{\pi_{k-1}} \omega_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} u_{k-1} \xrightarrow{\pi_k} \omega_k}_{k\ th\ year} \ldots .$$

According to this diagram (for $k = 1, 2, ...$). at the end of $(k-1)$-th year the state variable ω_{k-1} is observed. It describes the insurer's position at that moment. Then, at the beginning of k- th year the control rule γ_{k-1} is applied to choose the control variable u_{k-1}. Thereupon the k- th year probability mechanism of insurance unfolds; the transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k- th year.

The annual probability mechanisms of insurance are modeled in this paper by means of the Poisson process and process with heavy tail distribution functions (DF).

The insurance company provides its clients with regular premiums which decrease the reserve at rate $c < 0$. Without loss of generality, we assume that $c = -1$. The events of client deaths or contract interruptions follow at random moments $\{t_i\}_{i=1}^{\infty}$. Each such event increases the reserve of the company by the amount of the unpaid parts of rents $\{X_i\}_{i=1}^{\infty}$ which are positive, independent and identically distributed random variables (RV) with DF F, $(F(x) = 0,\ x < 0)$ and with the mathematical expectation $a > 0$. We assume that the moments $\{t_i\}_{i=1}^{\infty}$ form a Poisson point process of intensity λ. The insurance amount $S(u) = \sum_{0 \le t_i \le u} X_i$, which the company receives in the time interval $(0, u]$, is a generalized Poisson process with the intensity λ and jump DF F. i. e.

$$P\{S(u) \le x\} = \sum_{n=0}^{\infty} \frac{e^{-\lambda u}(\lambda u)^n}{n!} F^{n*}(x),\ x \ge 0,$$

where F^{n*} is the n - th convolution of the DF F ($F^{0*}(x) = 1$, when $x \ge 0$, and $F^{0*}(x) = 0$, when $x < 0$).

The average sum of "payments" ("profits") (see [1]. [7]) $\rho_1 = \lambda a$ in unit of time (the insurance premium [8], [9]) we call loading of the company [10].

When ρ_1 is fixed, usually, at a risk estimation the central limit theorem is used. In duration of time loading, and therefore the risk can reach their critical values. The critical value of loading ρ_1 is the speed (c=-1) of reduction of reserves of company. It is called the gross insurance premium in unit of time (see [8]).

The reserve of the company at the moment t is $x - t + S(t)$. where x is the initial capital. can become negative at some time moment giving rise to a situation of ruin for the company. Event $\{x - t + S(t) < 0\}$ is called "ruin" (see [11]). but we will call it "ruin situation".

Let T_x is the first moment of the ruin situation. i.e. $T_x = \inf\{t : x - t + S(t) < 0,\ 0 \le t < \infty\}$ and $T_x = \infty$. if $x - t + S(t) \ge 0$ for all $t \ge 0$ (see [1]). Denote $r(t) = \sup_{0 \le u \le t} \zeta(u)$ and $r = \sup_{0 \le u < \infty} \zeta(u)$, where $\zeta(u) = u - S(u)$. For company not to be in ruin situation in the time interval $(0, t]$. it is necessary that $\zeta(u) \ge x$. $u \in (0, t]$. The form of the trajectory $\zeta(u)$ is represented on Fig. 1.

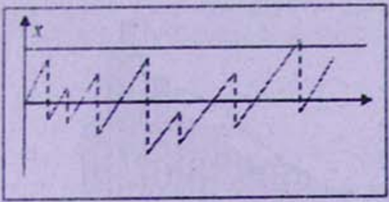

Fig. 1 Process $u - S(u)$.

The specialist badly familiar with the theory of risks and elementary probability theory may think that at the moment T_z the insurance company will really be ruined and process $\zeta(t)$ should be stopped in this point. But it is not so, since T_z is a random process and it can accept any positive values with some probability. Even if it is known that $P\{T_z = t_o, t_o = const\} = 1$ it does not mean that the company will be ruined in the moment t_o. Besides, the term "ruin" in the risk theory carries technical sense (see [12]) and does not mean the valid ruin of the company[1]. The probability of ruin is a measure. i.e. some analytical indicator measuring risk of insurance operations (see [12]).

The absence of the percent factor in collective risk models is the cause of arising of possible ruin situations. Here risk depends on the intensity of insurance events and on the jump of "insurance payments". In duration of time intensity of insurance cases or jumps of "insurance payments" can increase. In such situations applied methods of an estimation of risk are unsuitable. Even in case of normal Gaussian approximation there are also some disadvantages. The normal Gaussian approximation can be used only in case of existence of the second order moment of the RV. Besides transformations of centralization and normalization are saving the asymmetry (i.e. if the initial distribution of the RV is asymmetric (see Fig. 2), then after the transformations of centralization and normalization the asymmetry will be saved (see Fig. 3)). Therefore there will be great errors in that approximation.

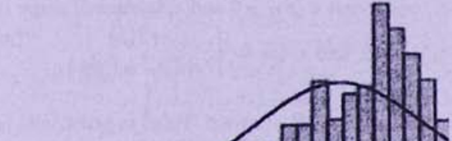

Fig. 2 Histogram of initial data.

[1]The term "ruin" in the risk theory has arisen historically and precisely this event would be named for example. "deficiency". Insufficiency of a reserve does not mean ruin of the insurance company in sense of stopping of its operations or bankruptcies; the term "ruin" of the theory of risk should be understood as technical. If the balance of reserve fund is negative. it yet does not mean negativity of balance of the company as a whole as the company can have other sources of repayment of deficiency (for example, own means. loans. etc.). On the other hand. even at positive balance the company can experience financial difficulties if the part of actives in which the reserve fund is laid out. has low liquidity (the real estate objects, precious metals and so forth). Thus. it is not necessary to mix ruins probability with probability of illiquidity (see [12]).

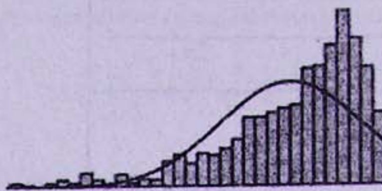

Fig. 3 Histogram of transformed data.

In such situations the problems are arising for approximation by means of non symmetric distributions. The distributions of that type are for example the stable distributions and the distributions belonging to the domain of its attraction, which have the fat-tailed nature and the finite moment of the order $\gamma \in (0.2]$. DFs of those laws have heavy tails, i.e. are satisfing to the following generalized condition of great risks:

$$1 - F(x) \sim \frac{A(\gamma - 1)}{\Gamma(2-\gamma)} x^{-\gamma} L(x), x \to \infty, A > 0, 1 < \gamma \le 2. \qquad (1)$$

where $\Gamma(\bullet)$ - is the Euler's Gamma function and $L(x) > 0$ is the slowly varying function (SVF) in infinity[2].

2. The Statement of the Problem.

Processes of risks are $r(t)$ and r. the DFs of which we denote by $W(t,x)$ and $W(x)$ correspondingly. These probabilities are the risks of the insurance company.

By saying "critical risk" we understand the following existing nondegenerate limit DF $\lim\limits_{\rho\to 0} W\left(\frac{x}{\theta(\rho)}\right) = W_o(x)$, $\lim\limits_{(\rho,t)} W\left(t, \frac{x}{\theta(\rho)}\right) = W_\tau(x)$ and $\lim\limits_{\rho\to 0} W\left(t, \frac{x}{\theta(\rho)}\right) = W_t(x)$, $(t = const)$. $x \ge 0$, where (ρ, t) means that $\rho = |1-\rho_1| \to 0$ and $t \to +\infty$ jointly so that $t\alpha(\rho) \to \tau$, $0 < \tau < \infty$ at normalization coefficient $\theta(\rho) \underset{\rho\to 0}{\longrightarrow} 0$ and existence of some $\alpha(\rho) \underset{\rho\to 0}{\longrightarrow} 0$. It is proved (see [7]), that such $\alpha(\rho)$ exists and $\alpha(\rho) = \begin{cases} \rho\theta(\rho), & \theta(\rho) = o(\omega) \, or \theta(\rho) \sim \omega, \\ B\theta(\rho)^\gamma L\left(\frac{1}{\theta(\rho)}\right), & \omega = o(\theta(\rho)). \end{cases}$
where $B = \lambda A$.

In [7] the existence of DFs $W_o(x)$. $W_\tau(x)$ and $W_t(x)$ in precritical $(\rho_1 \uparrow 1)$ and postcritical $(\rho_1 \downarrow 1)$ situations is proved under the condition (1) of great risks.

The importance of the application DFs with heavy tails at the analysis of rare catastrophic events is also noted in [13]. [14]. We will also notice that the known Lundbergs-Kramer theorem of the classical theory of risk concerns with cases when $1 - F(x) \sim e^{-x}$. $x \to +\infty$ and is inapplicable to the case of payments of the large size. Then the "Lundbergs coefficient" is not even defined (see [15]).

It is known (see [1]) that $W(t.x) = 1 - \int\limits_x^t \frac{x}{y} d_y P\{S(y) \le y - x\}$, $0 < x \le t$. and $W(x) = \lim\limits_{t\to\infty} W(t.x) = 1 - e^{-\omega x}$. $x > 0$. where ω is the greatest nonnegative root of the equation $\nu(s) = s - \lambda(1 - \psi(s)) = 0$ ($\omega = 0$ for $0 \le \rho_1 \le 1$. and $\omega > 0$ for $\rho_1 > 1$ (see [1]).

[2]Function $L(x) > 0$ is called SVF in infinity. if $L(tx) \sim L(t)$, $t \to \infty$, $\forall x > 0$.

When $\rho_1 > 1$ is fixed, we have $\omega = \rho_1\omega - \lambda(\psi(\omega) - 1 + a\omega)$ and taking into account the inequality (see [16]) $\left|e^{\beta} - 1 - \beta\right| \le |\beta|^2/2$, $Re\beta \le 0$, we'll obtain

$$\rho\omega = \lambda \int_0^{+\infty} \left(e^{-\omega x} - 1 + \omega x\right) dF(x) \le \lambda\omega^2/2.$$

So $\omega \ge 2\rho/\lambda$, and therefore $1 - W(x) = e^{-\omega x} \le e^{-2\rho x/\lambda}$, which loses its meaning when $\rho \to 0$.

In this paper the dual LSTs are obtained for $W_\tau(x)$ under the following conditions.

1) The LST $\psi(s) = \int_0^\infty e^{-sx} dF(x)$, $Re(s) \ge 0$ admits asymptotical representation of the form

$$\psi(s) - 1 + as \sim As^\gamma L(1/s),\, s \downarrow 0, A > 0, 1 < \gamma \le 2, \tag{2}$$

where measurable function $L(x) > 0$ is a SVF at infinity. It is true to note (see [11]) that when $1 < \gamma < 2$, then condition (2) is equivalent to condition (1).

2) Assume that $\rho \to 0$ (condition of the critical loading).

Denote $L_2^{(\alpha)}(t) = \frac{1}{t^\alpha M^{(\alpha)}(t)}$, where $M^{(\alpha)}(t)$ is the inverse function of $\frac{t^\alpha}{L(t)}$.[3] $0 < \alpha \le 2$. It is proved (see [7]), that $\omega \sim \left(\frac{\rho}{B}\right)^{\frac{1}{\gamma-1}} L_2^{(\gamma-1)}\left(\frac{B}{\rho}\right)$ when $\rho_1 \downarrow 1$. When $\rho_1 \uparrow 1$ we use the same notation $\omega \sim \left(\frac{\rho}{B}\right)^{\frac{1}{\gamma-1}} L_2^{(\gamma-1)}\left(\frac{B}{\rho}\right)$ for quantities of the rate $\left(\frac{\rho}{B}\right)^{\frac{1}{\gamma-1}} L_2^{(\gamma-1)}\left(\frac{B}{\rho}\right)$ and assume that $\frac{\rho}{B} \to 0$, $\rho_1 \uparrow 1$.

Let the functions $\Diamond = \nabla(s)$ and $\Diamond = \Delta(s)$ respectively be the unique solutions of equations $z^\gamma - z = s$ and $z^\gamma + z = s$, $s \ge 0$, satisfying the conditions $\nabla(0) = 1$ and $\Delta(0) = 0$[4] (see [7]). Let $\alpha(\rho) \underset{\rho \to 0}{\longrightarrow} 0$,

$$\varepsilon(\rho) = \begin{cases} \omega, & \alpha(\rho) = o(\rho\omega), \quad \rho_1 \downarrow 1, \\ \alpha(\rho)\rho^{-1}, & \alpha(\rho) = o(\rho\omega), \quad \rho_1 \uparrow 1, \\ \omega, & \alpha(\rho) \sim \rho\omega, \quad \rho_1 \uparrow\downarrow 1, \\ \left(\alpha(\rho) B^{-1}\right)^{1/\gamma} L_2^{(\gamma)}\left(B(\alpha(\rho))^{-1}\right), & \rho\omega = o(\alpha(\rho)), \quad \rho_1 \uparrow\downarrow 1, \end{cases}$$

and

$$A(s) = A(s, \alpha(\rho)) = \begin{cases} 1, & \alpha(\rho) = o(\rho\omega), \quad \rho_1 \downarrow 1, \\ s, & \alpha(\rho) = o(\rho\omega), \quad \rho_1 \uparrow 1, \\ \nabla(s), & \alpha(\rho) \sim \rho\omega, \quad \rho_1 \downarrow 1, \\ \Delta(s), & \alpha(\rho) \sim \rho\omega, \quad \rho_1 \uparrow 1, \\ s^{1/\gamma}, & \rho\omega = o(\alpha(\rho)), \quad \rho_1 \uparrow\downarrow 1. \end{cases}$$

Let us recall some known results from [7].

Theorem 1: *If the condition* (2) *is fulfilled and* $\theta(\rho) \underset{\rho_1 \downarrow 1}{\longrightarrow} 0$, *then the limit* $\lim_{\rho_1 \downarrow 1} P\{\theta(\rho)\tau \le x\} = 1 - e^{-x}$, $x > 0$ *exists if and only if* $\theta(\rho) \sim \omega$.

In [7] the following new representation of non - ruin probability is found

$$W(t,x) = \begin{cases} 1 - \int_x^t d_y U(x,y), & 0 < x \le t \\ 1, & x \ge t \end{cases}, x > 0, t > 0. \tag{3}$$

[3]For example in case of modeling by distribution of type of Pareto $F(x) = \frac{x^\gamma}{1+x^\gamma}$, we have $L(x) = \frac{x^\gamma}{1+x^\gamma}$ and $y = M^{(\alpha)}(x)$ is the solution of equation $y^{\alpha-\gamma} + y^\alpha = x$.

[4]Equation $z^\gamma + z = s$ and its solution $\Delta(s)$ are introduced by Danielian [17]. Then, they were also considered in [18] and [19].

where $\int_0^\infty e^{-sy} d_y U(x,y) = e^{-x\omega(s)}$ and $z = \omega(s)$ is the unique root of the equation $\nu(z) = s$, $Re s \geq 0$. By means of (3) the following theorem is proved.

Theorem 2: *Let the condition (2) be fulfilled, $\theta(\rho) \sim \varepsilon(\rho)$ as $\rho \to 0$ and $t \to \infty$, so that $t\alpha(\rho) \to \tau$, $0 < \tau \leq \infty$. Then there exists the limit DF $W_\tau(x)$, $x > 0$ and*

1) $W_\tau(x) = 1 - e^{-x}$, in case of $\theta(\rho) = o(\omega)$, when $\rho_1 \downarrow 1$;

2) $W_\tau(x) = \begin{cases} 0 & x < \tau \\ 1 & x \geq \tau \end{cases}$, in case of $\theta(\rho) = o(\omega)$, when $\rho_1 \uparrow 1$.

3) $W_\tau(x) = 1 - \int_0^\tau f(v,x)\,dv$, in case of $\theta(\rho) \sim \omega$ and $\omega = o(\theta(\rho))$ when $\rho_1 \to 1$, where $\int_0^\infty e^{-sv} f(v,x)\,dv = e^{-xA(s,\rho\theta(\rho))}$, $s \geq 0$ and (see [7])

$$f(x,t) = \frac{1}{\gamma\pi x} \sum_{n=0}^{\infty} \frac{(-1)^n (t-\delta(x))^n}{n!} \Gamma\left(\frac{n+1}{\gamma}\right) x^{-\frac{n+1}{\gamma}} \sin\frac{n+1}{\gamma}\pi,\ 1 < \gamma \leq 2,$$

with $\delta(\tau) = \begin{cases} \tau\, sign(1-\rho_1), & \theta(\rho) \sim \omega \text{ or } \theta(\rho) = o(\omega), \rho \to 0, \\ 0. & \omega = o(\theta(\rho)), \rho \to 0 \end{cases}$.

In all other cases $W_\tau(x) = 1$, $x > 0$.

Remark 1: *$W_t(x) = 1$, $x > 0$, for any fixed t and any $\theta(\rho) \underset{\rho\to 0}{\longrightarrow} 0$.*

Corollary 1: *Under the conditions of Theorem 2, when $\gamma = 2$ and $\rho_1 \to 1$, $W_\tau(x) = 1 - F_x(2\tau)$ if $\omega = o(\theta(\rho))$ and $W_\tau(x) = 1 - e^{sign(1-\rho_1)\frac{x}{2} - \frac{\tau}{4}} f_{x/\sqrt{2}}(\tau) * e^{-\frac{\tau}{4}}$ if $\theta(\rho) \sim \omega$, where $F_x(u)$ is DF with density $f_x(u) = \frac{x}{\sqrt{2\pi u^3}} \exp\left\{-\frac{x^2}{2u}\right\}$, $u > 0$.*

3. Dual Laplace - Stieltjes Transformations of Critical Risks

Let $p(x,\alpha,\varphi)$ be a stable density with characteristic function $\exp\left\{-|s|^\alpha \exp\left\{\pm\frac{\pi i\varphi}{2}\right\}\right\}$, $0 < \alpha \leq 2$ (see [11]). Denote $\Phi_\tau(x) = \tau^{-\frac{1}{\gamma}} \int_{-\infty}^{x} p\left(u\tau^{-\frac{1}{\gamma}}, \gamma, \gamma-2\right) du$, $x \in R^1$, $\tau \in R^+$ and $E_y(x) = \begin{cases} 1, & x > y, \\ 0, & x \leq y. \end{cases}$ In case of generalized Poisson process $S(t)$ distributed on the whole real axis, in [7] the following theorem is proved for random process $\zeta(t) = S(t) - ct$.

Theorem 3: $\int_{-\infty}^{+\infty} e^{isx} dF(x) - 1 - ais \sim -AC_\gamma |s|^\gamma L(1/|s|)$, $s \to 0$, *where $A > 0$, $1 < \gamma \leq 2$ and $C_\gamma = \exp\left\{\pm\frac{\pi i(2-\gamma)}{2}\right\}$. If $\theta(\rho) \underset{\rho\to 0}{\longrightarrow} 0$ and $t \to +\infty$ so that $t\alpha(\rho) \to \tau$, $\tau \in [0, +\infty]$, then there exists the limit* $\lim P\{\theta(\rho)\zeta(t) \leq x\} = \Phi(\tau, x)$, $x \in R^1$, *where $\Phi(0,x) = E_0(x)$, $\Phi(+\infty, x) \equiv 0$, $\Phi(\tau,x) = E_{-\delta(\tau)}(x)$ when $\theta(\rho) = o(\omega)$ and $\Phi(\tau,x) = \Phi_\tau(x - \delta(\tau))$ when $\theta(\rho) \sim \omega$ or $\omega = o(\theta(\rho))$, and $0 < \tau < +\infty$.*

Denote $\beta(s) = s^{1/\gamma}$ when $\delta(\tau) = 0$ and $\beta(s) = \Diamond(s)$ when $\delta(\tau) \neq 0$.

The dual Laplace - Stieltjes transformations of critical risks are represented in the following theorem.

Theorem 4: *Function*

$$N(s,v,\mu,\tau) = \exp\left\{-\int_0^\infty (e^{-sx} - 1)\, d_x \left[\int_0^\infty e^{-vu\mu} \frac{1-\Phi(\tau u, x)}{u} du\right]\right\},\ 0 < \nu < \infty,\ s \geq 0,$$

$0 < \tau < +\infty$, $\mu > 0$ *has a representation*

$$N(s,v,\mu,\tau) = \frac{\beta(\mu v/\tau)}{s + \beta(\mu v/\tau)} \tag{4}$$

and when $\mu = 1$ *it is a dual LST for DF* $U(x.t) = 1 - \int_0^t \phi(\tau u.x)du$, *where*

$$\phi(x.\tau) = \begin{cases} \frac{1}{x^\gamma} p\left(\frac{\tau}{x^\gamma}, \frac{1}{\gamma}, -\frac{1}{\gamma}\right), & \delta(\tau) = 0, \\ f(\tau.x), & \delta(\tau) \neq 0. \end{cases}$$

Proof: Consider the nontrivial cases of function $N(s, v, \mu, \tau)$ arising when $\theta(\rho) \sim \omega$ and $\omega = o(\theta(\rho))$. In those cases DF $\Phi(\tau, x)$ has density $h(x.\tau.\gamma - 2) = \tau^{-1/\gamma} p\left((x - \delta(\tau))\tau^{-1/\gamma}, \gamma, \gamma - 2\right)$. Taking into account that $p(-x, \alpha, \varphi) = p(x, \alpha, -\varphi)$ (see [11], [20]), we get $h(x, \tau, \gamma - 2) = \tau^{-1/\gamma} p\left(-(x - \delta(\tau))\tau^{-1/\gamma}, \gamma, 2 - \gamma\right)$ and denoting $k = n + 1$ in the representation

$$p(x, \alpha, \varphi) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \left[\frac{\Gamma(k\alpha^{-1}+1)}{k!}(-x)^k \sin\left(\frac{\pi k(\varphi-\alpha)}{2\alpha}\right)\right], 1 < \alpha < 2.$$

(see [11], [20]) we obtain

$$h(x, \tau, 2-\gamma) = \tau^{-\frac{1}{\gamma}} p\left((\delta(\tau) - x)\tau^{-\frac{1}{\gamma}}, \gamma, 2-\gamma\right) =$$
$$\frac{1}{\pi\gamma} \sum_{n=0}^{\infty} \left[\frac{(x-\delta(\tau))^n}{n!} \Gamma\left(\frac{n+1}{\gamma}\right) \tau^{-\frac{n+1}{\gamma}} \sin \frac{\pi(n+1)(\gamma-1)}{\gamma}\right].$$

By means of $\sin \frac{\pi(n+1)(\gamma-1)}{\gamma} = (-1)^n \sin \frac{\pi(n+1)}{\gamma}$, we get

$$h(x, \tau, 2-\gamma) = \frac{1}{\pi\gamma} \sum_{n=0}^{\infty} \left[\frac{(-1)^n (x-\delta(\tau))^n}{n!} \Gamma\left(\frac{n+1}{\gamma}\right) \tau^{-\frac{n+1}{\gamma}} \sin \frac{\pi(n+1)}{\gamma}\right] = \frac{\tau}{x} f(\tau.x).$$

In case of $\delta(\tau) = 0$ comparing with (see [11]) $p(x, \alpha, \varphi) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \left[\frac{\Gamma(k\alpha+1)}{k!}(-x^{-\alpha})^k \sin \frac{\pi k(\varphi-\alpha)}{2}\right]$, $0 < \alpha < 1$, we obtain

$$h(x, \tau, 2-\gamma) = \frac{\tau}{x^{\gamma+1}} \frac{1}{\pi \frac{\tau}{x^\gamma}} \sum_{k=1}^{\infty} \left[\frac{1}{k!} \Gamma\left(\frac{k}{\gamma} + 1\right) \left(-\left(\frac{\tau}{x^\gamma}\right)^{-\frac{1}{\gamma}}\right)^k \sin\left(-\frac{\pi k}{\gamma}\right)\right] = \frac{\tau}{x^{\gamma+1}} p\left(\frac{\tau}{x^\gamma}, \frac{1}{\gamma}, -\frac{1}{\gamma}\right).$$

So we have $h(x, \tau, 2-\gamma) = \frac{\tau}{x}\phi(x, \tau)$. Changing the integration order in definition of function $N(s, v, \mu, \tau)$, we get

$$N(s, v, \mu, \tau) = \exp\left\{\int_0^{\infty} e^{-vu\mu} \left[\int_0^{\infty} (e^{-sx} - 1)\, h(x.\tau u.\gamma - 2)\, dx\right] u^{-1} du\right\}.$$

Again we change the integration order and make a change of the integration variable $z = \tau u$. Taking into account relations obtained for $h(x.\tau.2-\gamma)$ we find the equality

$$N(s.v.\mu.\tau) = \exp\left\{\int_0^{\infty} \frac{e^{-sx}-1}{x} \left[\int_0^{\infty} e^{-\frac{v\mu z}{\tau}} \phi(x, z)\, dz\right] dx\right\}.$$

Density $\phi(x.z)$ has LST $\exp\{-x\beta(s)\}$, where $\beta(s) = \diamondsuit(s)$, or $\beta(s) = s^{1/\gamma}$. Then $N(s.v.\mu.\tau) = \exp\left\{\int_0^{\infty} \frac{e^{-sx}-1}{x} \exp\left\{-x\beta\left(\frac{v\mu}{\tau}\right)\right\} dx\right\}$. Taking into account that $\int_0^{\infty} \frac{1-e^{-sx}}{x} e^{-\mu x} dx = \ln\left(1 + \frac{s}{\mu}\right)$, we obtain (4). The last proposition of Theorem 4 follows from the following equality

$$v \int_0^{\infty} e^{-ru} \left\{ \int_0^{\infty} e^{-sx} d_x U(x.t) \right\} dt = \int_0^{+\infty} e^{-sx} d_x \left[v \int_0^{+\infty} e^{-vt} U(x.t)\, dt \right] = \frac{\beta(v/\tau)}{s+\beta(v/\tau)}.$$

where it is taken into account, that $v \int_0^{+\infty} e^{-vt} U(x,t)\, dt = 1 - e^{-x\beta(v/\tau)}$ and $\int_0^{+\infty} e^{-vt} dt = v^{-1}$.
The theorem is proved.

Remark 2: *By means of relation* $\tau\phi(x,\tau) = xh(x.\tau.\gamma-2)$ *the limit DF* $W_\tau(x)$ *takes the form* $W_\tau(x) = 1 - \int_0^{T} \frac{x}{v} h(x,v,\gamma-2)\, dv$. *Hence in limit the form of ruin probability is preserved.*

Remark 3: *From the formula* $P\{T_x \leq t\} = 1 - W(x.t)$ *the analog result for RV* T_x *follows.*

4. Conclusion

The compound form of the dual LST of the limit law $W_\tau(x)$ is represented in simple form, which can be applied in solution of problems of finding of the critical risks, of asymptotic estimations and of obtaining of inequality type estimations. Those estimates can be used in adaptive control strategy for multiperiodic insurance risk model investigation.

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Կրիտիկական ռիսկերի կրկնակի Լապլաս-Ստիլտեսի ձևափոխությունները բացասական ապահովագրական վճարների դեպքում

Ա. Մարտիրոսյան

Ամփոփում

Սույն աշխատանքը նվիրված է բացասական վճարներով (սովորական կյանքի ռենտաների պայմանագրերով) ապահովագրական մոդելի կրիտիկական ռիսկերին: Ներկայացված են կրիտիկական իրավիճակներում առաջացող սահմանային թեորեմներ և գտնված են բացասական վճարներով կոլեկտիվ ապահովագրության ռիսկի մոդելում առաջացող կրիտիկական ռիսկերի կրկնակի Լապլաս-Ստիլտեսի ձևափոխությունները: Լուսաբանված են դիտարկված կոլեկտիվ ռիսկի մոդելի և Մալինովսկու կողմից բազմապարբերական ապահովագրական ռիսկի մոդելի համար ներդրված ադապտիվ կառավարման գործընթացի մանրամասները: