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Upper Bounds for the Maximum Span in Interval Total Colorings of Graphs

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Abstract

An interval total t-coloring of a graph G is a total coloring of G with colors $1, 2, \dots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \dots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. In this paper we prove that if a connected graph G with n vertices admits an interval total t-coloring, then $t \leq 2n - 1$. Furthermore, we show that if G is a connected r-regular graph with n vertices which has an interval total t-coloring and $n \geq 2r + 2$, then this upper bound can be improved to $2n - 3$. We also give some other upper bounds for the maximum span in interval total colorings of graphs.

1. Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all have degree a and the vertices in the other part all have degree b. An edge-coloring of a graph G with colors $1, 2, \dots, t$ is called an interval t-coloring if for each $i \in \{1, 2, \dots, t\}$ there is at least one edge of G colored by i, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable, if there is $t \geq 1$ for which G has an interval t-coloring. The set of all interval colorable graphs is denoted by $\mathcal{N}$. For a graph $G \in \mathcal{N}$, the greatest value of t (the maximum span) for which G has an interval t-coloring is denoted by $W(G)$.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [2]. In [2, 3] they proved the following theorem.

Theorem 1: *If G is a connected triangle-free graph and $G \in \mathcal{N}$, then*

$$W(G) \leq |V(G)| - 1.$$

In particular, from this result it follows that if G is a connected bipartite graph and $G \in \mathcal{N}$, then $W(G) \leq |V(G)| - 1$. It is worth to notice that for some families of bipartite graphs this upper bound can be improved. For example. in [1] Asratian and Casselgren proved the following

Theorem 2: *If G is a connected (a,b)-biregular bipartite graph with $|V(G)| \geq 2(a+b)$ and $G \in \mathcal{N}$, then*

$$W(G) \leq |V(G)| - 3.$$

For general graphs. Kamalian proved the following

Theorem 3:[7] *If G is a connected graph and $G \in \mathcal{N}$, then*

$$W(G) \leq 2|V(G)| - 3.$$

The upper bound of Theorem 3 was improved in [6].

Theorem 4: [6]*If G is a connected graph with $|V(G)| \geq 3$ and $G \in \mathcal{N}$. then*

$$W(G) \leq 2|V(G)| - 4.$$

On the other hand, in [12] Petrosyan proved the following theorem.

Theorem 5: *For any $\varepsilon > 0$, there is a graph G such that $G \in \mathcal{N}$ and*

$$W(G) \geq (2-\varepsilon)|V(G)|.$$

For planar graphs. the upper bound of Theorem 3 was improved in [4].

Theorem 6:[4] *If G is a connected planar graph and $G \in \mathcal{N}$, then*

$$W(G) \leq \frac{11}{6}|V(G)|.$$

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices. edges. and no incident vertices and edges obtain the same color. The concept of total coloring was introduced by Vizing [13] and independently by Behzad [5]. An interval total t-coloring [8, 9] of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$. and the edges incident to each vertex v together with v are colored by $d_G(v)+1$ consecutive colors. A graph G is interval total colorable. if there is $t \geq 1$ for which G has an interval total t-coloring. The set of all interval total colorable graphs is denoted by $\mathcal{T}$. For a graph $G \in \mathcal{T}$. the greatest value of t (the maximum span) for which G has an interval total t-coloring is denoted by $W_\tau(G)$. Terms and concepts that we do not define can be found in [14. 15].

In this paper we derive some upper bounds for the maximum span in interval total colorings of graphs.

2. Main Results

Theorem 7: *If G is a connected graph and $G \in \mathcal{T}$, then*

$$W_\tau(G) \leq 2|V(G)| - 1.$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and α be an interval total $W_\tau(G)$-coloring of the graph G. Define an auxiliary graph H as follows:

$$V(H) = U \cup W.$$

where

$$U = \{u_1, u_2, \ldots, u_n\},$$

$$W = \{w_1, w_2, \ldots, w_n\}$$

and

$$E(H) = \{u_iw_j, u_jw_i \mid v_iv_j \in E(G), 1 \le i \le n, 1 \le j \le n\} \cup \{u_iw_i \mid 1 \le i \le n\}.$$

Clearly, H is a connected bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring β of the graph H in the following way:

(1) $\beta(u_iw_j) = \beta(u_jw_i) = \alpha(v_iv_j)$ for every edge $v_iv_j \in E(G)$.

(2) $\beta(u_iw_i) = \alpha(v_i)$ for $i = 1, 2, \ldots, n$.

It is easy to see that β is an interval $W_\tau(G)$-coloring of the graph H. Since H is a connected bipartite graph and $H \in \mathcal{N}$, by Theorem 1, we have

$$W_\tau(G) \le |V(H)| - 1 = 2|V(G)| - 1.$$

thus

$$W_\tau(G) \le 2|V(G)| - 1.$$

The theorem is proved.

Note that the bound in Theorem 7 is sharp for simple paths P_n (see [10]) and complete graphs K_n (see [9, 11]), since $W_\tau(P_n) = W_\tau(K_n) = 2n - 1$ for any $n \in \mathbf{N}$.

Theorem 8: *If G is a connected r-regular graph with $|V(G)| \ge 2r + 2$ and $G \in \mathcal{T}$, then*

$$W_\tau(G) \le 2|V(G)| - 3.$$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and α be an interval total $W_\tau(G)$-coloring of the graph G. Define an auxiliary graph H as follows:

$$V(H) = U \cup W,$$

where

$$U = \{u_1, u_2, \ldots, u_n\},$$

$$W = \{w_1, w_2, \ldots, w_n\}$$

and

$$E(H) = \{u_iw_j, u_jw_i \mid v_iv_j \in E(G), 1 \le i \le n, 1 \le j \le n\} \cup \{u_iw_i \mid 1 \le i \le n\}.$$

Clearly, H is a connected $(r + 1)$-regular bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring β of the graph H in the following way:

(1) $\beta(u_iw_j) = \beta(u_jw_i) = \alpha(v_iv_j)$ for every edge $v_iv_j \in E(G)$.

(2) $\beta(u_iw_i) = \alpha(v_i)$ for $i = 1, 2, \ldots, n$.

It is easy to see that β is an interval $W_\tau(G)$-coloring of the graph H. Since H is a connected $(r+1)$-regular bipartite graph with $|V(H)| \geq 2(2r+2)$ and $H \in \mathcal{N}$, by Theorem 2, we have

$$W_\tau(G) \leq |V(H)| - 3 = 2|V(G)| - 3.$$

thus

$$W_\tau(G) \leq 2|V(G)| - 3.$$

The theorem is proved.

Next, we derive some upper bounds for $W_\tau(G)$ depending on degrees and diameter of a connected graph G.

Theorem 9: *Let G be a connected graph and $G \in \mathcal{T}$. Then*

$$W_\tau(G) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v),$$

where $\mathbf{P}$ *is the set of all shortest paths in the graph* G.

Proof: Consider an interval total $W_\tau(G)$-coloring α of G. We distinguish the following four possible cases:

1) there are vertices $v, v' \in V(G)$ such that $\alpha(v) = 1$, $\alpha(v') = W_\tau(G)$;

2) there is a vertex v and there is an edge e' such that $\alpha(v) = 1$, $\alpha(e') = W_\tau(G)$;

3) there is an edge e and there is a vertex v' such that $\alpha(e) = 1$, $\alpha(v') = W_\tau(G)$;

4) there are edges $e, e' \in E(G)$ such that $\alpha(e) = 1$, $\alpha(e') = W_\tau(G)$.

Case 1: there are vertices $v, v' \in V(G)$ such that $\alpha(v) = 1$, $\alpha(v') = W_\tau(G)$.

Let P_1 be a shortest path joining v with v', where

$$P_1 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}),$$

$$v_1 = v, \quad v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1)$$

$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2).$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i).$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k).$$

$$W_\tau(G) = \alpha(v') = \alpha(v_{k+1}) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq$$

$$1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

Case 2: there is a vertex v and there is an edge e' such that $\alpha(v) = 1$, $\alpha(e') = W_\tau(G)$.

Let $e' = v'w$ and P_2 be a shortest path joining v with v', where

$$P_2 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}),$$
$$v_1 = v, \quad v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(e') = \alpha(v_{k+1}w) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq 1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P\in \mathbf{P}} \sum_{v\in V(P)} d_G(v).$$

Case 3: there is an edge e and there is a vertex v' such that $\alpha(e) = 1$, $\alpha(v') = W_\tau(G)$.

Let $e = uv$ and P_3 be a shortest path joining v with v', where

$$P_3 = (v_1, e_1, v_2, e_2, \ldots, v_i, e_i, v_{i+1}, \ldots, v_k, e_k, v_{k+1}),$$
$$v_1 = v, \quad v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \leq 1 + d_G(v_1),$$
$$\alpha(e_2) \leq \alpha(e_1) + d_G(v_2),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_i).$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(v') = \alpha(v_{k+1}) \leq \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \leq$$

$$1 + \sum_{i=1}^{k+1} d_G(v_i) \leq 1 + \max_{P\in \mathbf{P}} \sum_{v\in V(P)} d_G(v)$$

Case 4: there are edges $e, e' \in E(G)$ such that $\alpha(e) = 1$, $\alpha(e') = W_\tau(G)$.

Let $e = uv$ and $e' = v'w$. Without loss of generality we may assume that a shortest path P_4 joining e and e' joins v and v', where

$$P_4 = (v_1, e_1, v_2, e_2, \dots, v_i, e_i, v_{i+1}, \dots, v_k, e_k, v_{k+1}),$$

$$v_1 = v, \ \ v_{k+1} = v'.$$

Note that

$$\alpha(e_1) \le 1 + d_G(v_1),$$
$$\alpha(e_2) \le \alpha(e_1) + d_G(v_2),$$
$$\dots\dots\dots\dots\dots$$
$$\alpha(e_i) \le \alpha(e_{i-1}) + d_G(v_i),$$
$$\dots\dots\dots\dots\dots$$
$$\alpha(e_k) \le \alpha(e_{k-1}) + d_G(v_k),$$
$$W_\tau(G) = \alpha(e') = \alpha(v'w) \le \alpha(e_k) + d_G(v_{k+1}).$$

By summing these inequalities, we obtain

$$W_\tau(G) \le 1 + \sum_{i=1}^{k+1} d_G(v_i) \le 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} d_G(v).$$

The theorem is proved.

Corollary 1: *Let G be a connected graph and $G \in \mathcal{T}$. Then*

$$W_\tau(G) \le 1 + (diam(G) + 1)\Delta(G),$$

where $diam(G)$ is the diameter of G.

Note that the bound in Theorem 9 is sharp for trees (see [10]) and the bound in Corollary 1 is sharp for complete graphs K_n (see [9, 11]).

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Գրաֆների միջակայքային լիակատար ներկումներում մասնակցող գույների առավելագույն հնարավոր թվի վերին գնահատականներ

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Ամփոփում

G գրաֆի լիակատար ներկումը $i = 1, 2, \dots, t$ գույներով կանվանենք միջակայքային լիակատար t–ներկում, եթե ամեն մի i գույնով, $i = 1, 2, \dots, t$ ներկված է առնվազն մեկ գագաթ կամ կող և յուրաքանչյուր v գագաթին կից կողերը և այդ գագաթը ներկված են $d_G(v) + 1$ հաջորդական գույներով, որտեղ $d_G(v)$-ով նշանակված է v գագաթի աստիճանը G գրաֆում: Այս աշխատանքում ապացուցվում է, որ եթե n գագաթանի G կապակցված գրաֆը ունի միջակայքային լիակատար t–ներկում, ապա $t \leq 2n - 1$: Ավելին, ցույց է տրվում, որ եթե n գագաթանի G կապակցված r–համասեռ գրաֆը ունի միջակայքային լիակատար t–ներկում և $n \geq 2r + 2$, ապա $t \leq 2n - 3$: Նաև աշխատանքում տրվում են գրաֆների միջակայքային լիակատար ներկումներում մասնակցող գույների առավելագույն հնարավոր թվի այլ գնահատականներ: