Mathematical Problems of Computer Science 33, 35–40, 2010.

A Note on Short Paths in Oriented Graphs

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Abstract

Let G be an oriented graph of order $p \geq 3$ and minimum semi-degrees at least $[p/2] - k$ for a positive integer k. For a subset C of vertices G, we obtain sufficient conditions implying that for any pair of distinct vertices $x, y \in V(G) - C$ there is a path from x to y of length less than a given integer which does not contain the vertices of C.

1. Introduction and Notation

In this note we shall consider finite directed graphs (digraphs) without loops and multiple arcs.We use the standard terminology and notation on digraphs as given in [1]. We still provide most of the necessary definitions for the convenience of the reader. An oriented graph (orgraph) is a digraph without cycles of length two. The vertex set and the arc set of an orgraph G are denoted by $V(G)$ and $E(G)$, respectively. If xy is an arc of G, then we say that x dominates y and y is dominated by x. For two subsets A. $B \subseteq V(D)$ with $A \cap B = \emptyset$. we define $E(A \to B)$ as the set $\{xy \in E(D)/x \in A, y \in B\}$ and if every vertex of A dominates every vertex of B, then we say that A dominates B. denoted by $A \to B$. If $A = \{x\}$ then we write x instead of $\{x\}$. The outset of x is the set $O(x) = \{y \in V(G)/xy \in E(G)\}$ and $I(x) = \{y \in V(G)/yx \in E(G)\}$ is the inset of x. Similarly, if $A \subseteq V(G)$ then $O(x, A) = \{y \in A/xy \in E(G)\}$ and $I(x, A) = \{y \in A/yx \in E(G)\}$. Let $\overline{A} = V(G) - A$. The out-degree of vertex x is $od(x) = |O(x)|$ and $id(x) = |I(x)|$ is the in-degree of x. We define $od^*(x) = V(G) - od(x) - 1$, $id^*(x) = V(G) - od(x) - 1$, $od(x, A) = |O(x, A)|$ and $id(x, A) = |I(x, A)$. The suborgraph of G induced by a subset A of $V(G)$ is denoted by $\langle A \rangle$.

For integers a and b, $(a \leq b)$, let $[a, b]$ denote the set of all integers between a and b and for any number c , $[c]$ denotes the integer part of a.

The path consisting of the distinct vertices $x_1, x_2, \ldots, x_n$ ($n \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, n-1]$, is denoted $x_1 x_2 \cdots x_n$ and is called an (x_1, x_n)-path. The length of a path is the number of its arcs. Let x and y are two distinct vertices of orgraph G. Denote by $d(x, y)$ the length of a shortest (x, y)-path in G, if exists. If $A \subseteq V(G)$ and $x \in V(G)$ then for a positive integer s let $O^s(x, A) = \{y \in A/d(x, y) = s\}$ and $I^s(x, A) = \{y \in A/d(y, x) = s\}$.

Note that the problems considered in the note and conserning short paths in orgraphs arise, in particular, at routing stage of VLSI desgins [2].

2. Preliminary Results

We begin with a simple lemma which is used in proofs of Theorems 1-5.

Lemma 1. *If G is an orgraph of order $p \geq 3$, then G contains vertices x and y (u and v, respectively) with $od(x) \leq (p-1)/2$ and $od^*(y) \geq [p/2]$ ($id(u) \leq (p-1)/2$ and $id^*(v) \geq [p/2]$, respectively).*

In the following throughout this note, G will denote an orgraph on p vertices, with the minimum semi-degrees at least $n-k$ where $n = [p/2]$ and $k \geq 1$.

Observe that for each vertex $x \in V(G)$,
if $p = 2n+1$ then $od^*(x) \leq n+k$ and $id^*(x) \leq n+k$,
if $p = 2n$ then $od^*(x) \leq n+k-1$ and $id^*(x) \leq n+k-1$.
Now let us prove the following

Lemma 2. *Let $C \subset V(G)$, $|C| = m$ and $x \in V(G) - C$. Then for each integer s, $2 \leq s \leq p-m-1$, the following holds*

$$\min\{|\bigcup_{i=1}^{s} O^i(x,\overline{C})|;\quad |\bigcup_{i=1}^{s} I^i(x,\overline{C})|\} \geq a(m,k,s).$$

where

$$a(m,k,s) = \frac{(2^s-1)(n-k-m)-2^{s-1}+3}{2^{s-1}}.$$

Proof. Without loss of generality, we only prove that $\{|\cup_{i=1}^{s} O^i(x,\overline{C})| \geq a(m,k,s)$. The proof is by induction on s. We have that $|O^1(x,\overline{C})| \geq n-k-m$, and, by Lemma 1, there is a vertex $z \in \cup_{i=1}^{s-1} O^i(x,\overline{C})$ such that

$$od(z, \bigcup_{i=1}^{s-1} O^i(x,\overline{C}) \leq \left[\frac{|\cup_{i=1}^{s-1} O^i(x,\overline{C})|-1}{2}\right].$$

Hence it is easy to see that

$$|O^2(x,\overline{C})| \geq n-k-m-\frac{|O^1(x,\overline{C})|-1}{2}$$

and if $s \geq 3$ then

$$|O^s(x,\overline{C})| \geq n-k-m-1-\frac{|\cup_{i=1}^{s-1} O^i(x,\overline{C})|-1}{2}. \tag{1}$$

Therefore

$$|O^1(x,\overline{C})|+|O^2(x,\overline{C})| \geq |O^1(x,\overline{C})|+n-k-m-\frac{|O^1(x,\overline{C})|-1}{2} \geq \frac{3(n-k-m)+1}{2}, \tag{2}$$

and for $s = 2$ Lemma 2 is proved. So we may proceed to the induction step, assuming that Lemma 2 is true for $s-1 \geq 2$. By the inductive assumption we have

$$|\bigcup_{i=1}^{s-1} O^i(x,\overline{C})| \geq \frac{(2^{s-1}-1)(n-k-m)-2^{s-2}+3}{2^{s-2}}.$$

Hence from (1) it is not difficult to see that

$$|\bigcup_{i=1}^{s} O^i(x,\overline{C})| = |\bigcup_{i=1}^{s-1} O^i(x,\overline{C})| + |O^s(x,\overline{C})| \geq n-k-m-1+\frac{|\cup_{i=1}^{s-1} O^i(x,\overline{C}))|+1}{2} \geq$$

$$\geq n-k-m-1+\frac{(2^{s-1}-1)(n-k-m)+3}{2^{s-1}} \geq \frac{(2^s-1)(n-k-m)-2^{s-1}+3}{2^{s-1}}.$$

This completes the proof of Lemma 2. ■

3. Main Results

Theorem 1. *If $n \geq 2k+1$ then $d(x,y) \leq k+2$ for all two distinct vertices $x, y \in V(G)$.*

Proof. Suppose, on the contrary, that there are two distinct vertices x and y such that $d(x,y) \geq k+3$. We consider the following two cases. ■

Cace 1. $k = 2q-1$ *and* $q \geq 1$.

It is easy to see that for any i, j, $(1 \leq i, j \leq q)$ we have $O^i(x) \cap I^j(y) = \emptyset$ and

$$E(\{x\} \cup \bigcup_{i=i}^{q} O^i(x) \rightarrow \{y\} \cup \bigcup_{i=i}^{q} I^i(x)) = \emptyset. \tag{3}$$

First assume, that $q = 1$, i.e. $k = 1$. Since $|O^1(x)| \geq n-1$ and $|I^1(x)| \geq n-1$, then, by Lemma 1 and by (3) for some vertex $z \in O^1(x)$ we have

$$n+1 \geq od^*(x) \geq n+1+\left[\frac{n-1}{2}\right],$$

i.e. $n \leq 2$, which contradicts $n \geq 2k+1$.

Now assume that $q = 2$, i.e. $k = 3$. Using (2) and (3) with Lemma 1, we coclude that there is a vertex $z \in \{x\} \cup O^1(x) \cup O^2(x)$ such that

$$n+k \geq od^*(z) \geq \frac{3n-3k+1}{2}+1+\frac{3n-3k+1}{4}.$$

Therefore $n \leq 6$, which contradicts $n \geq 2k+1 = 7$.

Finally assume that $q \geq 3$. Then, by Lemma 1 and 2 and by (3), there is a vertex $z \in \{x\} \cup \bigcup_{i=1}^{q} O^i(x)$ such that

$$n+k \geq od^*(z) \geq \frac{(2^q-1)(n-k)+3}{2^{q-1}}+\frac{(2^q-1)(n-k)-2^{q-1}+3}{2^q}.$$

From this we obtain that

$$n \leq 2k+\frac{2^{q-1}+3k-9}{2^{q+1}-3} < 2k+1,$$

which is a contradiction.

Cace 2. $k = 2q-2$ *and* $q \geq 2$.

For each pair i, j $(1 \leq i \leq q, 1 \leq j \leq q-1)$ we have $O^i(x) \cap I^j(x) = \emptyset$ and

$$E\left(\{x\} \cup \bigcup_{i=1}^{q} O^i(x) \to \{y\} \cup \bigcup_{i=1}^{q-1} I^i(y)\right) = \emptyset. \tag{4}$$

Assume that $q = 2$, i.e. $k = 2$. Using (2) and (4) with Lemma 1, we conclude that there is a vertex $z \in I^1(y)$ such that

$$n + k \geq id^*(x) \geq \frac{3n - 3k + 1}{2} + \frac{n - k - 1}{2} + 2.$$

Hence $n \leq 3k - 2 = 4$, which contradicts that $n \geq 5$.

Now assume that $q \geq 3$. Then, by Lemma 1 and 2 and by (4), there is a vertex $z \in \{y\} \cup \bigcup_{i=1}^{q-1} I^i(y)$ such that

$$n + k \geq id^*(z) \geq \frac{(2^q - 1)(n - k) + 3}{2^{q-1}} + \frac{(2^{q-1} - 1)(n - k) - 2^{q-2} + 3}{2^{q-1}}.$$

It follows easily that

$$n \leq 2k + \frac{2k - 2^{q-2} - 6}{2^q - 2} < 2k + 1.$$

This contradicts that $n \geq 2k + 1$. The proof of Theorem 1 is completed.

Remark 1. Let D be an orgraph with vertex set $V(D) = \{x, y, u, v, w\}$ and arc set $E(D) = \{xu, yx, yu, uv, vw, wy, wx\}$. It is easy to see that $d(x, y) = 4$, i.e. for $k = 1$ and $n = 5$ the Theorem 1 is not true.

Theorem 2. *Let either $p = 2n$ and $n > 2k + 1,5m - 0,25 + (3k + 1.5m - 9,75)/(2^{s+1} - 3)$ or $p = 2n + 1$ and $n > 2k + 1,5m + 0,25 + (3k + 1,5m - 8,25)/(2^{s+1} - 3)$, where the integer $s \geq 2$. If $C \subset V(G)$ and $|C| = m$, then for each two distinct vertices x and y of $\overline{C}$ in suborgraph $\langle C \rangle$ there is an (x, y)-path of the length at most $2s + 1$.*

Proof. The proof is by contradiction. Suppose that there exist two distinct verticex $x, y \in \overline{C}$ such that each (x, y)-path in suborgraph $\langle \overline{C} \rangle$ has length at least $2s + 2$. It is not difficult to see that $xy \notin E(G)$, $O^i(x, \overline{C}) \cap I^j(x, \overline{C}) = \emptyset$ for each pair $i. j$ $(1 \leq i, j \leq s)$ and

$$E(\{x\} \cup \bigcup_{i=1}^{s} O^i(x, \overline{C}) \to \{y\} \cup \bigcup_{i=1}^{s} I^i(x, \overline{C}) = \emptyset.$$

Hence, by lemma 1, there is a vertex $z \in \{y\} \cup \bigcup_{i=1}^{s} I^i(y, \overline{C})$ such that

$$id^*(z) \geq |\{x\} \cup \bigcup_{i=1}^{s} O^i(x, \overline{C})| + \left[\frac{|\bigcup_{i=1}^{s} O^i(x, \overline{C}) \cup \{y\}|}{2}\right].$$

Therefore, it follows from Lemma 2 that

$$id^*(z) \geq \frac{3(2^s - 1)(n - k - m) - 2^{s-1} + 9}{2^s}. \tag{5}$$

Assume that $p = 2n + 1$. Since $id^*(z) \leq n + k$, then from (5) it follows that

$$2^s(n + k) \geq 3(2^s - 1)(n - k - m) - 2^{s-1} + 9$$

and
$$n \le \frac{(3.2^s-3)(k+m)+2^{s-1}(2k+1)-9}{2.2^s-3}.$$

Therefore
$$n \le 2k+1,5m+0,25+\frac{3k+1,5m-8,25}{2^{s+1}-3},$$

which is a contradiction.

Now assume that $p=2n$. Then $id^*(z)\le n-k-1$ and we obtain a contradiction as in case $p=2n+1$ by the similar way. This completes the proof of Theorem 2. ■

Theorem 3. *Let either $p=2n$ and $n>2k+1.5m-0,25+(2k+m-5,5)/(2^s-2)$ or $p=2n+1$ and $n>2k+1,5m+0,25+(2k+m-5,5)/(2^s-2)$, where the integer $s\ge 3$. If $C\subset V(G)$ and $|C|=m$, then for each two distinct vertices x and y of $\overline{C}$ in suborgraph $\langle V\overline{C}\rangle$ there is an (x,y)-path of length at most $2s$.*

Theorem 3 can be proved by similar arguments used in Theorem 2.

Theorem 4. *Let either $p=2n+1$ and $n\ge 5k+3m-2$ or $p=2n$ and $n\ge 5k+3m-4$. If $C\subset V(G)$ and $|C|=m\ge 0$, then for each two distinct vertices $x,y\in\overline{C}$ in suborgraph $\langle\overline{C}\rangle$ there is an (x,y)-path of length at most 3.*

Proof. Suppose, to the contrary, that there are two distinct vertices $x,y\in\overline{C}$ such that in $\langle\overline{C}\rangle$, $d(x,y)\ge 4$. Then $xy\notin E(G)$ and
$$O^1(x,\overline{C})\cap I^1(y,\overline{C})=E(O^1(x,\overline{C})\to I^1(y,\overline{C}))=\emptyset.$$

Since $|O^1(x,\overline{C})|$ and $|I^1(y,\overline{C})|\ge n-k-m$, then, by lemma 1, there is a vertex $z\in I^1(y,\overline{C})$, such that
$$id^*(z)\ge n-k-m+2+\frac{n-k-m-1}{2}\ge\frac{3n-3k-3m+3}{2}.$$

Hence, if $p=2n+1$ then $n+k\ge(3n-3k-3m+3)/2$ and $n\le 5k+3m-3$, and if $p=2n$ then $n+k-1\ge(3n-3k-3m+3)/2$ and $n\le 5k+3m-5$. Thus in each case we have a contradiction and Theorem 4 is proved.

The following remark shows that Theorem 4 is false for $p=2n+1$ and $n=5k+3m-3$. ■

Remark 2. For integers $k\ge 1$, $m\ge 0$ and $n=5k+3m-3$ let H be an orgraph with $2n+1$ vertices whose vertex set $V(H)$ can be partitioned into sets A, B, C, D and $\{x,y\}$ such that

1. $|A|=|B|=4k+2m-3$, $|D|=2k+m-1$, $|C|=m$,

2. The suborgraphs $\langle A\rangle$ and $\langle B\rangle$ are regular tournaments and the suborgraphs $\langle C\rangle$ and $\langle D\rangle$ either are regular tournaments or almost regular tournaments,

3. The orgraph H also satisfies the following conditions $\{x\}\to A\cup C$, $B\cup C\to\{y\}$, $A\to D\cup C$, $D\cup C\to B$, $\{y\}\to\{x\}\cup A\cup D$, $B\cup D\to\{x\}$ and $B\to A$. Moreover if $z\in D$ then $od(z,C)\ge[|C|/2]$ and $id(z,C)\ge[|C|/2]$ and if $z\in C$ then $od(z,D)\ge[|D|/2]$ and $id(z,D)\ge[|D|/2]$.

It is not difficult to see that the semi-degrees of each vertex of H are at least $n-k$ and in $\langle\overline{C}\rangle$ all (x,y)-paths have length at least 4, i.e. for $p=2n+1$ and $n=5k+3m-3$ Theorem 4 is not true.

Note that for any integers $p = 2n$ and $n = 5k + 3m - 5$ we can constract also an orgraph on $2n$ vertices with minimum semi-degrees at least $n - k$ for which Theorem 4 is false.

Using a proof analogous to that of Theorem 4 we can show the following:

Theorem 5. *Let either* $p = 2n$ *and* $n \geq 3k + 2m - 2$ *or* $p = 2n + 1$ *and* $n \geq 3k + 2m - 1$. *If* $C \subset V(G)$ *and* $|C| = m$, *then for each two distinct vertices* $x, y \in V(G) - C$ *in suborgraph* $\langle \overline{C} \rangle$ *there is an* (x, y)*-path of length at most 4.*

Remark 3. For any integers $p = 2n$ and $n = 3k + 2m - 3$ ($p = 2n + 1$ and $n = 3k + 2m - 2$, respectively) there is an orgraph on p vertices with minimum semi-degrees at least $n - k$ for which Theorem 5 is false if $m + 2k \geq 9$ ($m + 2k \geq 8$, respectively).

References

[1] J. Bang-Jensen and G. Gutin, Digraphs. Theory. Algorithms and Applications, Springer, 2000.

[2] J. D. Ullman, Computational Aspects of VLSI,Computer Science Press, 1984.

Ուղղորդված գրաֆներում կարճ ճանապարհների մասին

Ս. Դարբինյան և Ի. Կարապետյան

Ամփոփում

Դիցուք G-ն p- գագաթանի ($p \geq 3$) ուղղորդված գրաֆ է, որի գագաթների կիսաաստիճանները փոքր չեն $[p/2] - k$ թվից (որտեղ $k \geq 1$ և ամբողջ թիվ է), և դիցուք C-ն G գրաֆի գագաթների որև> ենթաբազմություն է: Ներկա աշխատանքում ստացված են բավարար պայմաններ, որոնց դեպքում G գրաֆի ցանկացած երկու տարբեր x և y գագաթների համար գոյություն ունի C բազմության գագաթներով չանցնող և տրված թվից փոքր երկարությամբ x -ից դուրս եկող և y-ը մտնող կողմնորոշված ճանապարհ: