Mathematical Problems of Computer Science 33, 172–182, 2010.

Estimate for the Arithmetical Cost of an Algebraic Multigrid Preconditioner

Amir Ranjbar

Yerevan State University,
Islamic Azad University of Minoodasht (Iran)
ranjbar@minoodasht.ac.ir

Abstract

A multigrid preconditioner for the matrix arising in finite element approximation of model elliptic boundary value problem is proposed. Hierarchical triangular grids with bisection form the basis of multigrid construction. The main purpose of the paper is to evaluate the arithmetical cost of a preconditioner step.

1. Introduction

The algebraic multigrid preconditioning method is an efficient tool for numerical solution of large linear systems with finite element matrices. An approach to construct algebraic multigrid preconditioners of optimal order of computational complexity, so-called *Algebraic Multigrid/Substructuring Method* (AM/S-method), has been suggested and developed in [1,2]. A series of comprehensive results for elliptic boundary value problems have been obtained using that method [3-6].

When constructing multigrid preconditioners, hierarchical grids based on the bisection of mesh cells are usually used. However, other types of mesh refinement are also of great interest. We aim to study an effect of different rates of mesh refinement on the properties of AM/S-preconditioners. With this purpose, in the paper we construct and evaluate computational complexity of multigrid preconditioner for hierarchical triangular grids based on bisection.

Let us formulate a model elliptic boundary value problem for which we will construct and compare AM/S-preconditioners.

Suppose Δ is an equilateral unit triangle with boundary Γ in the plane of variables $x=(x_1,x_2)$. Denote by $H_0^1(\Delta)$ the subspace of the Sobolev space $H^1(\Delta)$ that consists of the functions vanishing on Γ. Consider a weak formulation of Dirichlet boundary value problem for Poisson equation: *for a given function $f \in L_2(\Delta)$ find the function $u \in H_0^1(\Delta)$ such that*

$$\int_{\Delta} \nabla u \nabla w \, dx = \int_{\Delta} f w \, dx \qquad \forall w \in H_0^1(\Delta) \tag{1.1}$$

(here and below $dx \equiv dx_1 dx_2$)

2. Hierarchical triangular grids based on bisection

Let τ_0 be an initial uniform triangulation of the domain Δ, formed by equilateral triangles with side length h_0 (Fig.1). Clearly, that $h_0 = 1/d_0$, where d_0 is an integer. We will consider the triangulation τ_0 as the *coarsest* one. Constructing the hierarchical sequence of grids is based on the refining procedure which is performed by subdividing triangular cells of the previous triangulation into four congruent ones (Fig.2). In fact, the refinement is carried out by bisection of the sides of triangular cells.

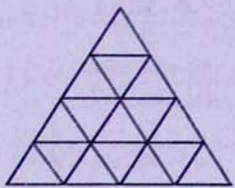

Fig. 1 Initial triangulation τ_0 of the domain Δ ($h_0 = 1/4$).

Using refining procedure just described we obtain a sequence of triangulations τ_k, $k = 0, 1, 2, \ldots, p$, where $p > 1$ is an integer. Note, that τ_p is the *finest* triangulation. Let the triangulation τ_k correspond to the k th *level* of the refinement. With any triangulation τ_k we associate the grid ω_k whose nodes are the vertices of triangles (*triangular elements*) which form the triangulation. In doing so, ω_0 is the *coarsest* grid while ω_p is the *finest* one.

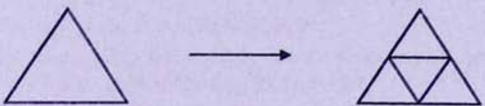

Fig. 2 Refining procedure (bisection).

For all values $k = 0, 1, \ldots, p$ we introduce the following notation:

N_k is the set of nodes of the grid ω_k which belong to Δ (don't belong to Γ);

n_k is the number of nodes in the set N_k;

G_k is the space of grid functions defined on the set N_k;

V_k is the space of functions continuous in $\overline{\Delta}$, linear in each triangle of the triangulation τ_k and vanishing on Γ.

The one-to-one correspondence holds between piecewise-linear functions from V_k and grid functions from G_k. Namely, a function $\hat{u} \in V_k$ is put in correspondence with a grid function $u \in G_k$ the i th component of which equals to the value of the function $\hat{u}$ at the i th node of the set N_k. In other words, the function $\hat{u} \in V_k$ is *a prolongation* of the grid function $u \in G_k$ in function space V_k. Let us denote this operation by symbol *prol*. Thus, $\hat{u} = prol(u \in G_k : V_k)$.

Let h_k be the step size of the uniform triangular grid ω_k, $k = 0, 1, \ldots, p$. Obviously, that $h_k = h_0 / 2^k$. Define integers $d_k \equiv 1/h_k$, $k = 0, 1, \ldots, p$. It is easily seen that

$$d_k = 2^k d_0, \qquad k = 0, 1, \ldots, p. \tag{2.1}$$

The following auxiliary statement holds true. It will be used below, when calculating the arithmetical cost of a preconditioning step.

Lemma 2.1. *For all values* $k = 1,2,\dots,p$ *the estimations*

$$4 < \frac{n_k}{n_{k-1}} \le 4 + 8\cdot 2^{-k} \tag{2.2}$$

are valid.

Proof. By simple considerations we get the expression

$$n_k = \frac{d_k^2 - 3d_k + 2}{2}, \qquad k = 0,1,\dots,p. \tag{2.3}$$

Using this, we have
$$\frac{n_k}{n_{k-1}} = \frac{d_k^2 - 3d_k + 2}{d_{k-1}^2 - 3d_{k-1} + 2}, \qquad k = 1,2,\dots,p.$$

Since $d_{k-1} = \frac{1}{2}d_k$, then
$$\frac{n_k}{n_{k-1}} = 4\,\frac{d_k^2 - 3d_k + 2}{d_k^2 - 6d_k + 8} = 4\,\frac{d_k - 1}{d_k - 4}.$$

Taking into account (2.1), we get

$$\frac{n_k}{n_{k-1}} = 4\,\frac{2^k d_0 - 1}{2^k d_0 - 4}, \qquad k = 1,2,\dots,p. \tag{2.4}$$

We can assume that $d_0 \ge 4$ (in this case $n_0 \ge 3$). Therefore,

$$1 < \frac{2^k d_0 - 1}{2^k d_0 - 4} \le 1 + 2\cdot 2^{-k}, \qquad k = 1,2,\dots,p. \tag{2.5}$$

Hence, from (2.4) and (2.5) we obtain the inequalities (2.2). □

By construction we have $N_k \supset N_{k-1}$, $k = 1,2,\dots,p$. Therefore, at the k th level the partitioning

$$N_k = N_k^{(1)} \cup N_k^{(2)} \tag{2.6}$$

can be defined, where

$$N_k^{(1)} = N_k \setminus N_{k-1}, \quad N_k^{(2)} = N_{k-1} \tag{2.7}$$

(see Fig.3). Nodes from the set $N_k^{(2)}$ we will call *old nodes* of the k th level while the nodes from $N_k^{(1)}$ will be referred to as *new nodes* of that level. Let $n_k^{(i)}$ $(i = 1,2)$ be the number of nodes in the set $N_k^{(i)}$. Then

$$n_k^{(1)} = n_k - n_{k-1}, \quad n_k^{(2)} = n_{k-1}. \tag{2.8}$$

The following ordering of the nodes will be used: *the nodes from* $N_k^{(1)}$ *(new nodes) are numbered first in some order and then the nodes from* $N_k^{(2)}$ *(old nodes).*

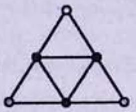

Fig. 3 Partitioning the nodes (o - old nodes, ● - new nodes).

In accordance with the rule for numbering the nodes, any grid function $u \in G_k$ may be represented in the form

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} \}\, N_k^{(1)} \\ \}\, N_k^{(2)} \end{matrix}, \quad u_i \in G_k^{(i)},\ i = 1,2,$$

where $G_k^{(i)}$ is the space of grid functions defined on the set $N_k^{(i)}$.

Let $k \geq 1$. Consider a triangular element $e \in \tau_{k-1}$. At the next level of refining the grid the element e is subdivided into four triangular elements of the k th level. As a result, on the k th level the element $e \in \tau_{k-1}$ turns into a *superelement* E (see Fig.4).

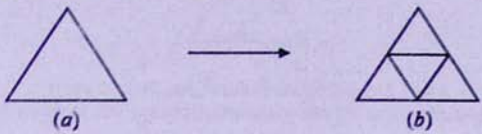

Fig. 4 (a) An element $e \in \tau_{k-1}$ and (b) corresponding superelement $E \in T_k$.

For all values $k = 1,2,\dots p$ let T_k be the set of superelements of the k th level.

3. Finite element matrices on hierarchical grids

Let us consider the finest level of refining the grid. The finite element problem, corresponding to the problem (1.1) is formulated as follows (see [7,8], for instance): *find a function* $\hat{v} \in V_p$ *such that*

$$\int_\Delta \nabla\hat{v}\nabla\hat{w}dx = \int_\Delta f\hat{w}dx \qquad \forall \hat{w} \in V_p. \tag{3.1}$$

The problem so defined leads to the system of grid equations

$$Av = g, \tag{3.2}$$

where the symmetric positive definite matrix A of order $n \equiv n_p$ is determined by the relation

$$w^T Au = \int_\Delta \nabla\hat{u}\nabla\hat{w}dx \qquad \forall u, w \in G_p \tag{3.3}$$

(here we have $\hat{u} = prol(u:V_p)$ and $\hat{w} = prol(w:V_p)$).

For our multigrid construction, simultaneously with the finite element matrix A let us define finite element matrices $A^{(k)}$, $k = 0,1,\dots,p$ with the help of the relations

$$w^T A^{(k)}u = \int_\Delta \nabla\hat{u}\nabla\hat{w}dx \qquad \forall u, w \in G_k. \tag{3.4}$$

According to (3.3), that $A^{(p)} = A$. Thus, we have a sequence of finite element matrices

$$A = A^{(p)}, A^{(p-1)}, \dots, A^{(1)}, A^{(0)} \tag{3.5}$$

associated with hierarchical grids ω_k, $0 \leq k \leq p$. The matrices $A^{(k)}$ are referred to as *stiffness matrices* (see [8], for instance).

For the values $k = 1,2,\dots,p$, according to the partitioning (2.6) of the nodes in the set N_k the matrices $A^{(k)}$ admit block representations

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \begin{matrix} \}N_k^{(1)} \\ \}N_k^{(2)} \end{matrix} \tag{3.6}$$

with $n_k^{(i)} \times n_k^{(j)}$ submatrices $A_{ij}^{(k)}$ $(i, j = 1,2)$. The block $A_{22}^{(k)}$ in (3.6) is a nonsingular diagonal matrix.

4. Two-grid preconditioners

Let us first formulate an auxiliary statement which forms the basis of further considerations. Consider an equilateral triangle e with vertices numbered 1,2 and 3 (Fig.5).

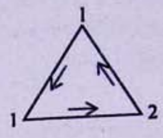

Fig. 5 Numbering the nodes of equilateral triangle.

Suppose u and v are some functions defined on the set of nodes of that triangle. Insert a bilinear form

$$\varphi_e(u,v) \equiv (u_2-u_1)(v_2-v_1)+(u_3-u_2)(v_3-v_2)+(u_1-u_3)(v_1-v_3), \tag{4.1}$$

where u_i and v_i are the values of the functions u and v, respectively, at the i th vertex. The following statement may be proved by straightforward calculations (see also [2,3]).

Lemma 4.1. *For any functions u and v linear on e the following equality holds*

$$\int_e \nabla u \nabla v\, dx = \frac{\sqrt{3}}{6}\varphi_e(u,v). \tag{4.2}$$

Consider a refinement level k, $0 \le k \le p$. The finite element matrix $A^{(k)}$ has been defined with the help of the relation (3.4). Let us write that relation in the form

$$w^T A^{(k)} u = \sum_{e\in\tau_k}\int_e \nabla\hat{u}\nabla\hat{w}\, dx.$$

From here, using Lemma 4.1, we find that the matrix $A^{(k)}$ satisfies the relation

$$w^T A^{(k)} u = \frac{\sqrt{3}}{6}\sum_{e\in\tau_k}\varphi_e(\hat{u},\hat{w}) \qquad \forall u,w\in G_k. \tag{4.3}$$

This relation will be essentially used in constructing the preconditioners.

With each superelement $E\in T_k$, $1\le k\le p$, the nodes of which are numbered as shown in Fig. 6, we associate a bilinear form

$$\begin{aligned}\Phi_E(u,v) \;\equiv\; & (u_1-u_5)(v_1-v_5)+(u_1-u_6)(v_1-v_6)+(u_2-u_6)(v_2-v_6)+\\ & (u_2-u_4)(v_2-v_4)+(u_3-u_4)(v_3-v_4)+(u_3-u_5)(v_3-v_5),\end{aligned} \tag{4.4}$$

where u_i and v_i are values of the functions u and v, respectively, at the i th node.

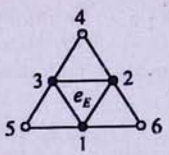

Fig. 6 Numbering the nodes of the superelement $E\in T_k$.

Let us address now to the relation (4.3), for $1\le k\le p$. If we group the elements of the k th level to form the superelements then the matrix $A^{(k)}$ satisfies the relation

$$w^T A^{(k)} u = \frac{\sqrt{3}}{6}\sum_{E\in T_k}\left[\Phi_E(\hat{u},\hat{w})+2\varphi_{e_E}(\hat{u},\hat{w})\right] \qquad \forall u,w\in G_k, \tag{4.5}$$

where $e_E\in\tau_k$ is a triangular element whose vertices are the midpoints of the superelement E (Fig.6).

Let us define a symmetric positive definite matrix $B^{(k)}$ of the order n_k by means of the relation

$$w^T B^{(k)} u = \frac{\sqrt{3}}{6} \sum_{E \in T_k} \Phi_E(\hat{u}, \hat{w}) \qquad \forall u, w \in G_k . \tag{4.6}$$

Comparing relations (4.5) and (4.6), we note that for each superelement E the bilinear form associated with inner triangular element $e_E \in \tau_k$ has been eliminated. By virtue of that the upper left block in the block representation

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \begin{matrix} \} N_k^{(1)} \\ \} N_k^{(2)} \end{matrix} \tag{4.7}$$

of the matrix $B^{(k)}$, i.e. the block $B_{11}^{(k)}$ becomes a diagonal matrix while the blocks $A_{12}^{(k)}, A_{21}^{(k)}$ and $A_{22}^{(k)}$ are identical with those of the block representation (3.6) of the matrix $A^{(k)}$. We will consider the matrix $B^{(k)}$ as *a preconditioner* for the matrix $A^{(k)}$, $1 \le k \le p$.

Let

$$S_{22}^{(k)} = A_{22}^{(k)} - A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \tag{4.8}$$

be the Schur complement of the matrix $B^{(k)}$, represented in the block form (4.7). Then the matrix $B^{(k)}$ takes the form

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & S_{22}^{(k)} + A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \end{bmatrix}. \tag{4.9}$$

The following important result holds true (see [2]).

Theorem 4.1. *For all values* $k = 1,2,\dots,p$ *the equality*

$$S_{22}^{(k)} = \frac{1}{2} A^{(k-1)} \tag{4.10}$$

is valid.

Based on Theorem 4.1, the matrix $B^{(k)}$, where $1 \le k \le p$, will be referred to as *two-grid preconditioner* for the matrix $A^{(k)}$. As follows from (4.10), the block representation (4.9) of the matrix $B^{(k)}$ can be written as

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & \frac{1}{2} A^{(k-1)} + A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \end{bmatrix}. \tag{4.11}$$

Throughout this paper we will denote by $sp(A)$ the spectrum of a matrix A. The following statement can be proved using the technique of transition onto the superelement level developed in [1,2].

Theorem 4.2. *For all values* $k = 1,2,\dots,p$ *we have*

$$sp(B^{(k)^{-1}} A^{(k)}) \subseteq [1,5] . \tag{4.12}$$

As a simple consequence of Theorem 4.2, we obtain the following estimates for the spectral condition number of matrices $B^{(k)^{-1}} A^{(k)}$, i.e.

$$\kappa(B^{(k)^{-1}} A^{(k)}) \equiv \frac{\lambda_{\max}^{(k)}}{\lambda_{\min}^{(k)}} \le 5, \qquad k = 1,2,\dots,p. \tag{4.13}$$

5. Multigrid preconditioner

Thus, we have constructed the sequence of finite element matrices

$$A^{(0)}, A^{(1)}, \dots, A^{(p)} \equiv A \tag{5.1}$$

and the corresponding sequence of two-grid preconditioners

$$B^{(1)}, B^{(2)}, \dots, B^{(p)} . \tag{5.2}$$

Now construct a multigrid preconditioner for the matrix A using the inner Chebyshev iterative procedures. The idea and main principles of the approach have been developed in [1,2,9].

In (4.11) we have obtained two by two block representation of the two-grid preconditioner $B^{(k)}$, $1 \le k \le p$. Let us choose an integer $s \ge 1$ and for values $k = 1,2,\dots,p$ successively define matrices

$$M^{(k)} = \begin{bmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & \frac{1}{2} R^{(k-1)} + A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \end{bmatrix}, \tag{5.3}$$

where

$$\begin{cases} R^{(0)} = A^{(0)}, & \textit{for} \quad k = 1, \\ R^{(k-1)} = A^{(k-1)} \left[I^{(k-1)} - \prod_{j=1}^{s} \left(I^{(k-1)} - \theta_j^{(k-1)} M^{(k-1)^{-1}} A^{(k-1)} \right) \right]^{-1}, & \textit{for} \quad 2 \le k \le p \end{cases} \tag{5.4}$$

(here $I^{(k-1)}$ is the identity matrix of order n_{k-1}). In (5.4) we have

$$\theta_j^{(k-1)} = \frac{2}{(\beta_{k-1} + \alpha_{k-1}) + (\beta_{k-1} - \alpha_{k-1}) t_j^{(s)}} \quad , \quad i = 1, 2, \dots, s ,$$

where $t_j^{(s)}$ are the roots of the Chebyshev polynomial of the first kind of degree s and $[\alpha_{k-1}, \beta_{k-1}]$ is the segment containing the spectrum of the matrix $M^{(k-1)^{-1}} A^{(k-1)}$, that is

$$sp(M^{(k-1)^{-1}} A^{(k-1)}) \subseteq [\alpha_{k-1}, \beta_{k-1}].$$

The formulae for the bounds of spectra of matrices $M^{(k)^{-1}} A^{(k)}$, $k = 1,2,\dots,p$ are determined as follows. Since $M^{(1)} = B^{(1)}$ then by Theorem 4.2

$$sp(M^{(1)^{-1}} A^{(1)}) \subseteq [\alpha_1, \beta_1] , \tag{5.5}$$

where $\alpha_1 = 1$, $\beta_1 = 5$. For some $k \ge 2$, suppose

$$sp(M^{(k-1)^{-1}} A^{(k-1)}) \subseteq [\alpha_{k-1}, \beta_{k-1}], \tag{5.6}$$

where $0 < \alpha_{k-1} < \beta_{k-1}$. From the theory of Chebyshev iterative methods (see [10], for instance), we have

$$sp(M^{(k)^{-1}} B^{(k)}) \subseteq [1 - \gamma_{k-1}^{(s)}, 1 + \gamma_{k-1}^{(s)}] , \tag{5.7}$$

where

$$\gamma_{k-1}^{(s)} = \frac{2 q_{k-1}^{s}}{1 + q_{k-1}^{2s}} \ , \ q_{k-1} = \frac{\sqrt{c_{k-1}} - 1}{\sqrt{c_{k-1}} + 1} \ , \ c_{k-1} = \frac{\beta_{k-1}}{\alpha_{k-1}} . \tag{5.8}$$

Then, proceeding from the equality

$$M^{(k)^{-1}} A^{(k)} = (M^{(k)^{-1}} B^{(k)})(B^{(k)^{-1}} A^{(k)})$$

and using (5.7) and (4.12) we conclude that

$$sp(M^{(k)^{-1}} A^{(k)}) \subseteq [\alpha_k, \beta_k] , \tag{5.9}$$

where

$$\alpha_k = 1 - \gamma_{k-1}^{(s)}, \ \beta_k = 5(1 + \gamma_{k-1}^{(s)}) . \tag{5.10}$$

Thus, the sequence of matrices

$$M^{(1)}, M^{(2)}, \dots, M^{(p)} \equiv M \tag{5.11}$$

has been constructed. The matrix M is considered as *a multigrid preconditioner* for the matrix A from (3.2) (see also (5.1)).

Above we introduced the quantities

$$c_k = \frac{\beta_k}{\alpha_k}, \quad k = 1,2,\dots,p \tag{5.12}$$

into our consideration (see (5.8)). According to (5.9), these quantities are the upper bounds for the spectral condition number of the matrices $M^{(k)^{-1}}A^{(k)}$, that is

$$\kappa(M^{(k)^{-1}}A^{(k)}) \le c_k, \quad k = 1,2,\dots,p. \tag{5.13}$$

As follows from (5.5), (5.8) and (5.10), these quantities can be obtained by recurrent procedure

$$\begin{cases} c_1 = 5, \\ c_k = 5\left[\dfrac{(\sqrt{c_{k-1}}+1)^s + (\sqrt{c_{k-1}}-1)^s}{(\sqrt{c_{k-1}}+1)^s - (\sqrt{c_{k-1}}-1)^s}\right]^2, \quad k = 2,3,\dots,p. \end{cases} \tag{5.14}$$

An analysis carried out in [3] shows that if the condition

$$s^2 > 5 \tag{5.15}$$

is met then the equation

$$x = 5\left[\frac{(\sqrt{x}+1)^s + (\sqrt{x}-1)^s}{(\sqrt{x}+1)^s - (\sqrt{x}-1)^s}\right]^2 \tag{5.16}$$

has a unique positive root c_*. Moreover, $c_* > 5$. The latter means that the sequence of quantities c_k, $k = 1,2,\dots,p$ defined in (5.12) increases monotonically and is bounded from above under unlimited growth of p, that is

$$c_1 < c_2 < \dots < c_p < c_*. \tag{5.17}$$

According to the condition (5.15), we may take only the values $s \ge 3$. On the other hand, the more is the integer s the more value of arithmetical operations we need to to solve a system with preconditioner M. On the base of detailed analysis carried out below, we come at the following conclusion. In order to the number of arithmetical operations required for solving a system with matrix M be proportional to n (i.e., the dimension of the finest grid system (3.2)), we must take $s \le 3$.

Thus, $s = 3$. In this case the recurrent procedure (5.14) takes the following form

$$\begin{cases} c_1 = 5, \\ c_k = 5c_{k-1}\left(\dfrac{c_{k-1}+3}{3c_{k-1}+1}\right)^2, \quad k = 2,3,\dots,p. \end{cases} \tag{5.18}$$

The quantity c_* from (5.17) is determined as the positive root of the equation

$$x = 5x\left(\frac{x+3}{3x+1}\right)^2.$$

Calculating, we find

$$c_* = 3 + 2\sqrt{5} \le 7.48. \tag{5.19}$$

So, we arrive at the following assertion.

Theorem 5.1. *If $s = 3$, then the estimate*

$$\kappa(M^{-1}A) \le 3 + 2\sqrt{5} \tag{5.20}$$

for the spectral condition number of matrix $M^{-1}A$ is valid.

In an iterative method with the matrix M as a multigrid preconditioner, we need to solve linear systems with matrices $M^{(k)}$. For the time being, suppose that s is a positive integer.

Consider a linear system

$$M^{(k)}v = g, \tag{5.21}$$

where

$$v=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad g=\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \quad v_i, g_i \in G_k^{(i)}, \quad i=1,2.$$

Proceeding from the two by two block structure (5.3) of the matrix $M^{(k)}$, we find the following algorithm.

Procedure MG PREC/M$^{(k)}$

1. calculate

$$z_2 = 2(g_2 - A_{21}^{(k)} B_{11}^{(k)^{-1}} g_1); \tag{5.22}$$

2. solve the system

$$R^{(k-1)} v_2 = z_2; \tag{5.23}$$

for $2 \le k \le p$: finding the solution of the system (5.23) is equivalent to performing s steps of the Chebyshev iterative process

$$M^{(k-1)} \frac{v_2^{(j)} - v_2^{(j-1)}}{\theta_j^{(k-1)}} = -A^{(k-1)} v_2^{(j-1)} + z_2, \quad j=1,2,\ldots,s, \tag{5.24}$$
$$v_2^{(0)} = 0 \;\; ; \;\; v_2 = v_2^{(s)};$$

for $k=1$: the system

$$A^{(0)} v_2 = z_2 \tag{5.25}$$

is solved;

3. calculate

$$v_1 = B_{11}^{(k)^{-1}} (g_1 - A_{12}^{(k)} v_2). \tag{5.26}$$

end

Let us give some comments to our Procedure **MG PREC/M$^{(k)}$**. Remind that the matrix $B_{11}^{(k)}$ is non-singular diagonal matrix. Thereby, the inversion of the block $B_{11}^{(k)}$ in (5.22) and (5.26) presents no difficulties. Then, the system of grid equations (5.25) on the coarsest level is assumed to be solved by a direct method requiring $O(1)$ arithmetical operations (since the system (5.25) has low dimension).

In conclusion, let us estimate the computational complexity of a preconditioning step. Let A_{ops} be the number of arithmetical operations required for solving a system with matrix M (so-called *the arithmetical cost of the preconditioner* M). Insert the following notation:

$A_{ops}^{(k)}$ – the number of arithmetical operations required for solving a system (5.21) with the matrix $M^{(k)}, k=1,2,\ldots,p$ (in accordance with notation (5.11), $A_{ops}^{(p)} = A_{ops}$);

$A_{ops}^{(0)}$ – the number of arithmetical operations required for solving a system (5.25) with the matrix $A^{(0)}$ (we suppose that $A_{ops}^{(0)} = O(1)$).

Basing on the Procedure **MG PREC/M$^{(k)}$**, by direct calculations we find the upper bounds for the quantities $A_{ops}^{(k)}$, that is

$$A_{ops}^{(k)} \le 6n_k + (7+16s)n_{k-1} + sA_{ops}^{(k-1)}, \qquad 2 \le k \le p, \tag{5.27}$$
$$A_{ops}^{(1)} \le 6n_1 + 7n_0 + A_{ops}^{(0)}. \tag{5.28}$$

In Lemma 2.1 we have estimated the relationship between the quantities n_k and n_{k-1}. Using the left-hand side of double-sided inequality (2.2), from (5.27) and (5.28) we obtain estimates

$$A_{ops}^{(k)} \le \left(\frac{31}{4} + 4s\right) n_k + sA_{ops}^{(k-1)}, \qquad 2 \le k \le p, \tag{5.29}$$

$$A^{(1)}_{ops} \le \frac{31}{4} n_1 + A^{(0)}_{ops}. \qquad (5.30)$$

Consecutive using the recurrent relations (5.29) yields

$$A^{(p)}_{ops} \le \left(\frac{31}{4} + 4s\right)\left[n_p + sn_{p-1} + \cdots + s^{p-2} n_2\right] + s^{p-1} A^{(1)}_{ops}.$$

Then, having the inequality (5.30), we get

$$A^{(p)}_{ops} \le \left(\frac{31}{4} + 4s\right)\left[n_p + sn_{p-1} + \cdots + s^{p-1} n_1\right] + s^{p-1} A^{(0)}_{ops}.$$

Let us apply once more the left-hand side of double-sided inequality (2.2). We obtain

$$A^{(p)}_{ops} \le \left(\frac{31}{4} + 4s\right)\left[1 + \left(\frac{s}{4}\right) + \left(\frac{s}{4}\right)^2 + \cdots + \left(\frac{s}{4}\right)^{p-1}\right] n_p + s^{p-1} A^{(0)}_{ops}.$$

In order to the sum

$$1 + \left(\frac{s}{4}\right) + \left(\frac{s}{4}\right)^2 + \cdots + \left(\frac{s}{4}\right)^{p-1}$$

be bounded from above, under unlimited growth of p, we must take $s \le 3$. In this case

$$A^{(p)}_{ops} \le \frac{\left(\frac{31}{4} + 4s\right)}{1 - \frac{s}{4}} n_p + s^{p-1} A^{(0)}_{ops}.$$

For $s = 3$ (see (5.15)) we obtain the estimate

$$A^{(p)}_{ops} \le 79 n_p + 3^{p-1} A^{(0)}_{ops}. \qquad (5.31)$$

Above, when proving Lemma 2.1, we used the expressions (2.3) for the quantities n_k. Particularly,

$$n_p = \frac{d_p^2 - 3d_p + 2}{2}.$$

Since $d_p = 2^p d_0$ (see (2.1)), we have

$$n_p = \frac{d_0^2 2^{2p} - 3d_0 2^p + 2}{2}.$$

We assumed that $d_0 \ge 4$ (see the proof of Lemma 2.1). Consequently, $n_p \ge 8 \cdot 2^{2p} - 6 \cdot 2^p + 1$.

It can be readily seen that $8 \cdot 2^{2p} - 6 \cdot 2^p + 1 \ge 21 \cdot 3^{p-1}$

for $p \ge 1$. Thus, $n_p \ge 21 \cdot 3^{p-1}$.

Taking into account the last inequality, from (5.31) we obtain the estimate

$$A^{(p)}_{ops} \le \left(79 + \frac{1}{21} A^{(0)}_{ops}\right) n_p,$$

or

$$A_{ops} \le \left(79 + \frac{1}{21} A^{(0)}_{ops}\right) n \qquad (5.32)$$

(remind, that $A^{(p)}_{ops} \equiv A_{ops}, n_p \equiv n$).

Thus, the multigrid preconditioner constructed can be considered to belong to the class of optimal preconditioners, since it is spectrally equivalent to the initial stiffness matrix and its arithmetical cost is proportional to the dimension of finest-grid problem.

6. Conclusions

In this paper we have shown that AM/S-method allows to construct efficient preconditioners for elliptic mesh operators. Two points of the paper should be emphasized. The first point is the estimate (5.20) for the condition number of preconditioned stiffness matrix. The second one is the estimate (5.31) of the arithmetical cost of the multigrid preconditioner (see also (5.32)). In subsequent papers we will construct AM/S-preconditioners for different rates of mesh refinement and compare the results with those of the present paper.

References

1. Yu.A. Kuznetsov, "Algebraic multigrid domain decomposition methods", *Sov. J. Numer. Anal. Math. Modelling*, vol.4, No.5, pp. 351-379, 1989.
2. Yu.R. Hakopian and Yu.A. Kuznetsov, "Algebraic multigrid/substructuring preconditioners on triangular grids", *Sov. J. Numer. Anal. Math. Modelling*, vol.6, No.6, pp. 453-483, 1991.
3. O. Axelsson, Yu.R. Hakopian and Yu.A. Kuznetsov, "Multilevel preconditioning for perturbed finite element matrices", *IMA J. Numer. Anal.*, vol.17, pp. 125-149, 1997.
4. Yu.R. Hakopian, "Algebraic multilevel/substructuring preconditioner in finite element method with piecewise quadratic approximation", "*Mathematical Problems of Computer Science*", vol.21, pp. 164-180, 2000.
5. Yu.R. Hakopian, "Algebraic multilevel preconditioners for third order finite element approximation", *Algebra, Geometry & their Applications (Seminar Proceedings)*. Yerevan State University, Armenia, vol.1, pp. 20-39, 2001.
6. Yu.R. Hakopian and A.S. Harutyunyan, "Two-level preconditioners for serendipity finite element matrices", *Linear Algebra Appl.*, vol.13, No.10, pp. 847-864, 2006.
7. G. Strang and G. Fix, "An Analysis of the Finite Element Method", Prentice-Hall Englewood Cliffs, N.J., 1973.
8. O.C. Zenkiewich and K. Morgan, "Finite Elements and Approximation", Wiley, NY, 1983.
9. O. Axelsson and P.S.Vassilevski, "Algebraic multilevel preconditioning methods",I, *Numer. Math.*, vol. 56, pp. 157-177, 1989.
10. O. Axelsson. Iterative Solution Methods, Cambridge University Press, 1994.

Հանրահաշվական բազմացանցային վերապայմանավորիչի թվաբանական գնի գնահատականը

Ա. Ռանջբար

Ամփոփում

Աշխատանքում կառուցվում է բազմացանցային վերապայմանավորիչ մոդելային էլիպտական եզրային խնդրի վերջավոր տարրային մոտարկման արդյունքում ստացվող մատրիցի համար: Բազմացանցային կառուցվածքի հիմքում ընկած են հիերարխիական եռանկյուն ցանցեր: Հոդվածի հիմնական նպատակն է գնահատել վերապայմանավորման քայլի թվաբանական գինը: