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The Optimal Permissible Placement by the Height of the Transitive Oriented Tree Containing One Vertex of Branching

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Abstract

In this paper we consider the optimal permissible placement by the height of the following type of transitive oriented graphs (the height of the vertex is the number of the arcs passing through the vertex): the graph consists of chains of quantity $s \geq 2$ branching after the end of the chain. The problem was solved by the algorithm of $s \log s$ complexity.

1. Introduction

1.1. Necessary information

Let us consider $G = (V, E)$ graph. $F: V \to \{1, 2, ..., |V|\}$ one-by-one transformation is called the numeration (placement, arrangement) of graph G. F(v) is called position or number of vertex $v \in V$.
Let us consider $G = (V, E)$ graph and its F numeration. Let us define the length of edge of the graph
$length_F(e) = |F(v) - F(u)|, e = (u, v) \in E$
In other words, to numerate graph G means to put its vertices on the line according to the number of the vertices so, that the subsequent points are placed equidistant. If $F(u) < F(p) < F(v)$ or $F(v) < F(p) < F(v)$, we will say that the edge $e = (u, v)$ is passing through vertex $p \in V$. For the given vertex $v \in V$, $h_F(v)$ will be the number of edges passing through it and will be called the height of the vertex v in numeration F. Let us define the following definitions for F numeration of graph G:
height $H(F, G) = \max_{v \in V} \{h_F(v)\}$
length $L(F, G) = \sum_{e \in E} length_F(e)$,
width $W(F, G) = \max_{e \in E} \{length_F(e)\}$.
And
length $L(G) = \min_F L(F, G)$,
width $W(G) = \min_F W(F, G)$,

height $H(G) = \min_F H(F, G)$ for graph G.

The aforementioned definitions are applied for oriented graphs as well.

The numeration for oriented graph G is called permissible, if $F(u) - F(v)$ for arbitrary $(u,v) \in E$ arc. The numeration of an oriented graph is permissible, if its vertices are placed on the line, such that all arcs are stretched in the same direction, from left to right. Note that for a permissible numeration

$length_F(e) = F(v) - F(u), e = (u,v) \in E$.

It is clear, that if an oriented graph contains a contour, a permissible numeration will not exist.

Let us consider a contourless oriented graph G. An oriented graph G is called transitive if $\forall x,y,z \in V$, $(x,y) \in E, (y,z) \in E \Rightarrow (x,z) \in E$.

1.2. Minimal height numeration problem for oriented graphs

For the given oriented graph $G = (V, E)$ find a F_0 permissible numeration, such that its height equals to the oriented graph's height $H(F_0, G) = H(G)$. This problem is NP-complete. Minimal length and width numeration problems are also NP-complete[4-6]. The solution algorithms for some special cases of these problems are of polynomial complexity[3,7]. Let us discuss the following special cases of minimal height numeration problem:

2. The optimal permissible placement by the height of the transitive oriented tree containing one vertex of branching

Let us consider the optimal permissible placement by the height of the transitive oriented graphs of the following type: the transitive oriented graph consists of a transitive oriented chain and branching from the end of it transitive oriented chains of quantity $s \geq 2$.

Let us assume the initial chain by L, the branched chains assume by X_i. The whole graph let's assume by $(L, X_1, ..., X_s)$.

Problem

For the given transitive oriented graph $(L, X_1, ..., X_s)$ find the optimal permissible placement by the height.

Algorithm

At first place the vertices of chain L, then place the chains X_i by the ascending order of the quantities of their vertices.

Theorem. *The algorithm generates optimal placement of an oriented graph $(L, X_1, ..., X_s)$.*

Proof. Let's assume by LX_i the transitive chain consisting of the chain L and the chain X_i. The graphic of the chain's height has the form of reversed parabola and arrives at his maximum point in the middle of the chain. From the middle to borders the height is descending.

Definition: *We'll call by the maximum vertex of X_i in LX_i the vertex, on which the number of passing arcs of chain LX_i is maximum.*

To prove the theorem let us give the following lemma:

Lemma. *If in arbitrary placement of $(L, X_1, ..., X_s)$ graph all the vertices of X_i $(1 \leq i \leq s)$ move to the maximum vertex of X_i in LX_i, the height of placement will not increase.*

Proof. Let us assume by $|L| = p$, $|X_i| = m$.

Let us assume by x_k the maximum vertex of X_i in LX_i, by x_j the j-th vertex of X_i. Let us $j > k$ and $x_k, ..., x_j$ vertices are already placed together, move the vertex x_{j-1} near the vertex x_j. After it the height of vertex x_{j+1} will not be greater, than the height of vertex x_k, because more arcs from LX_i will pass above the vertex x_k, than above the vertex x_{j+1} (as the vertex x_k is the maximum vertex of X_i in LX_i and from the remaining chains the same arcs will pass above the vertices x_k and x_{j+1}).

The height of the vertices placed between the previous positions of the vertices x_j and x_{j+1} is $(p+j)(m-j)$, after the moving is $(p+j+1)(m-j-1)$. Since $p+k \geq \frac{m+p}{2}, k \geq \frac{m-p}{2}, j>k$, so $j > \frac{(m-p)}{2}$, therefore we'll have $(p+j+1)(m-j-1)-(p+j)(m-j) = m-p-2j-1<0$. Thus the height did not increase.

In case $j<k$ let's assume that the vertices $x_{j+1},...x_k$ are placed together, move the vertex x_j before the vertex x_{j+1}. Note, that for $p>m$ this case is impossible, because x_k is the first vertex of chain X_i, i.e. in $p>m$ case always the $j>k$ case takes place. After the movement the height of x_j will not be greater than the height of x_k, as in the previous case. The height of the vertices placed between the previous positions of x_j and x_{j+1} were $(p+j)(m-j)$, after the movement their height is $(p+j-1)(m-j+1)$.

Such as the maximum vertex of X_i in LX_i is placed in the middle of the LX_i chain, so $p+k = \left]\frac{m+p}{2}\right[, k = \left]\frac{m-p}{2}\right[, j>k \quad ,\Rightarrow \quad j<\frac{m-p}{2}$. Therefore $(p+j-1)(m-j+1)-(p+j)(m-j) = -m+p+2j+1 \leq 0$. Thus the height of the placement did not increase after the movement.

Now let's return to the proof of the theorem.

It follows from the Lemma, that we can arrange the vertices of the graph chain by chain, that is to group the vertices of X_i-s. Let the chain X_{j+1} be placed after the chain X_j and $|X_j| > |X_{j+1}|$. Let's assume by $X_j = Y, X_{j+1} = X, |X_{j+1}| = m, |X_j| = n$. Assume by $h(x_{max})$ the quantity of the arcs of LX, passing through the maximum vertex of X in LX, assume by $h(y_{max})$ the quantity of arcs of LY, passing thought the maximum vertex of Y in LY. Move the vertices of X chain before the vertices of chain Y. After the movement the height of the vertices of Y will not increase. Let's show that the height of the vertices of X will not be greater, than the height of the maximum vertex of Y in the previous placement, i.e let's show that $h(x_{max}) + |L||Y| \leq h(y_{max}) + |L||X|$.

Let us consider two cases:

1. $|L| \geq |X|$
2. $|L| < |X|$

Let us consider the case $|L| \geq |X|$. We have $h(x_{max}) = |L|(|X|-1) = |L||X| - |L|$ and $h(y_{max}) \geq h(y_1) = |L|(|Y|-1) = |L||Y| - |L|$, where the $h(y_1)$ is the number of arcs of chain LY, passing through the first vertex of chain Y.

Thus $h(x_{max}) + |L||Y| = |L||X| - |L| + |L||Y| = h(y_1) + |L||X| \leq h(y_{max}) + |L||X|$.

Let us consider the case $|L| < |X|$.

$$h(x_{max}) = \left]\frac{p+m}{2}\right[\left(\left[\frac{p+m}{2}\right]-1\right) = \begin{cases} \frac{p+m}{2}\left(\frac{p+m}{2}-1\right), \text{ when p+m is odd} \\ \left(\frac{p+m}{2}-\frac{1}{2}\right)^2, \text{ otherwise.} \end{cases} =$$

$$=\begin{cases} \frac{(p+m)^2}{4}-\frac{p+m}{2}, & \text{when } p+m \text{ is odd} \\ \frac{(p+m)^2}{4}-\frac{p+m}{2}+\frac{1}{4}, & \text{otherwise.} \end{cases},$$

$$h(y_{max})-h(x_{max})=\frac{(p+n)^2}{4}-\frac{(p+m)^2}{4}-\frac{(p+n)}{2}+\frac{(p+m)}{2}+c$$

Where $|c|=\begin{cases} 0, & \text{when the m and n have the same oddness} \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$

We'll show, that $h(y_{max})-h(x_{max})-p(n-m)\geq 0$, i.e.

$p^2+2pn+n^2-p^2-2pm-m^2-2p-2n+2p+2m+4pn+4pm+4c=(m+n-2p)(n-m)\geq 4c$

$|4c|\leq 1$. Taking into account that $n>m>p$, the inequality really takes place. For the determination of the ascending order of the quantities of vertices of graph's chains $s\log s$ operations will be necessary. Thus the complexity of the algorithm is $s\log s$.

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Մեկ ճյուղավորման գագաթ պարունակող տրանզիտիվ օրիենտացված ծառի օպտիմալ թույլատրելի տեղադրումն ըստ բարձրության

Ա. Խաչատուրյան

Ամփոփում

Աշխատանքում շարադրված են հետևյալ տիպի տրանզիտիվ օրիենտացված գրաֆների օպտիմալ թույլատրելի համարակալումն ըստ բարձրության (գագաթի բարձրություն ասելով հասկանում ենք գագաթի վրայով անցնող աղեղների քանակը)՝ գրաֆը կազմված է շղթայից և շղթայի վերջից ճյուղավորվող $s\geq 2$ քանակությամբ շղթաներից: Խնդիրը լուծվել է $s\log s$ բարդության ալգորիթմով, որտեղ s-ը դիտարկված գրաֆի ճյուղավորվող շղթաների քանակն է :