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Interval Total Colorings of Graphs with a Spanning Star

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Abstract

An interval total t–coloring of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by $i, i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. In this paper we prove that if $G = (V, E)$ is a graph containing the vertex u with $d_G(u) = |V| - 1$, $k(G) = \max_{v \in V(v \neq u)} d_G(v) < |V| - 1$ and G admits an interval total t–coloring then $t \leq |V| + 2k(G)$. We also show that this upper bound is sharp. Further we determine all possible values of t for which the wheels have an interval total t–coloring.

1. Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G - by $\Delta(G)$. A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of G. If α is a total coloring of a graph G then $\alpha(v)$ and $\alpha(e)$ denote the color of a vertex $v \in V(G)$ and the color of an edge $e \in E(G)$ in the coloring α. For a total coloring α of a graph G and for any $v \in V(G)$ define the set $S[v, \alpha]$ as follows:

$$S[v, \alpha] \equiv \{\alpha(v)\} \cup \{\alpha(e) \mid e \text{ is incident to } v\}$$

Let $\lfloor a \rfloor$ ($\lceil a \rceil$) denote the greatest (the least) integer $\leq a$ ($\geq a$). For two integers $a \leq b$ the set $\{a, a+1, \ldots, b\}$ is denoted by $[a, b]$.

An interval total t–coloring [1, 2] of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors.

For $t \geq 1$ let $\mathcal{T}_t$ denote the set of graphs which have an interval total t–coloring, and assume: $\mathcal{T} \equiv \bigcup_{t \geq 1} \mathcal{T}_t$. For a graph $G \in \mathcal{T}$ the least and the greatest values of t, for which $G \in \mathcal{T}_t$, are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively.

Terms and concepts that we do not define here can be found in [3, 4].

2. An Upper Bound for $W_\tau(G)$

In this section we derive an upper bound for $W_\tau(G)$ depending on degrees and number of vertices of the graph G with a spanning star, that is the vertex with degree $|V(G)|-1$. Further, we construct graphs for which this upper bound is sharp.

Theorem 1: Let G be a graph containing the vertex u with $d_G(u) = |V(G)|-1$, $k(G) = \max_{v \in V(G)(v \neq u)} d_G(v) < |V(G)|-1$ and $G \in \mathcal{T}$. Then $W_\tau(G) \leq |V(G)| + 2k(G)$.

Proof: Let α be an interval total $W_\tau(G)$–coloring of the graph G.

Consider the vertex u. We show that $1 \leq \min S[u,\alpha] \leq k(G)+1$.

Suppose, to the contrary, that $\min S[u,\alpha] \geq k(G)+2$. Since $d_G(v) \leq k(G)$ for any $v \in V(G)(v \neq u)$, then $\min S[v,\alpha] \geq 2$ for any $v \in V(G)(v \neq u)$, which is a contradiction.

Now we have $1 \leq \min S[u,\alpha] \leq k(G)+1$, hence, $|V(G)| \leq \max S[u,\alpha] \leq |V(G)| + k(G)$. This implies that $\max S[v,\alpha] \leq |V(G)| + 2k(G)$ for any $v \in V(G)(v \neq u)$.

Let k be an even integer, n be a positive integer such that $k \leq \frac{n-1}{2}$ and $n-1 \equiv 0 (mod\, k)$. Define the graph $G_{k,n}$ as follows (see Fig. 1):

$$V(G_{k,n}) = \{u\} \cup \{v_j^i | 1 \leq i \leq \frac{n-1}{k}, 1 \leq j \leq k\};$$

$$E(G_{k,n}) = \{(u, v_j^i) | 1 \leq i \leq \frac{n-1}{k}, 1 \leq j \leq k\} \cup \{(v_r^i, v_s^i) | 1 \leq i \leq \frac{n-1}{k}, 1 \leq r < s \leq k\}$$

Clearly, $|V(G_{k,n})| = n$, $d_{G_{k,n}}(u) = n-1$ and $d_{G_{k,n}}(v_j^i) = k$, $i = 1, \ldots, \frac{n-1}{k}, j = 1, \ldots, k$.

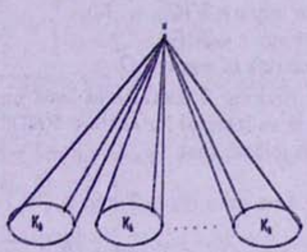

Figure 1. The graph $G_{k,n}$.

Theorem 2: Let k be an even integer, n be a positive integer such that $k \leq \frac{n-1}{2}$ and $n-1 \equiv 0 (mod\, k)$. Then $G_{k,n} \in \mathcal{T}$ and $W_\tau(G_{k,n}) = n + 2k$.

Proof: For the proof of the theorem we construct an interval total $(n+2k)$–coloring of the graph $G_{k,n}$.

Define a total coloring α of the graph $G_{k,n}$ in the following way:

1) $\alpha(u) = n$ and $\alpha(v_j^1) = 2j-1$, $j = 1, \ldots, k$;

2) for $i = 2, \ldots, \frac{n-1}{k} - 1$, $j = 1, \ldots, k$ $\alpha\left(v_j^i\right) = ki + 2j$;

3) $\alpha\left(v_j^{\frac{n-1}{k}}\right) = n + 2j$, $j = 1, \ldots, k$;

4) for $i = 1, \dots, \frac{n-1}{k} - 1$, $j = 1, \dots, k$, $\alpha\left(\left(u, v_j^i\right)\right) = ki + j$;

5) $\alpha\left(\left(u, v_j^{\frac{n-1}{k}}\right)\right) = n + j$, $j = 1, \dots, k$;

6) for $r = 1, \dots, k$, $s = 1, \dots, k$, $r \neq s$, $\alpha((v_r^1, v_s^1)) = r + s - 1$;

7) for $i = 2, \dots, \frac{n-1}{k} - 1$, $r = 1, \dots, k$, $s = 1, \dots, k$, $r \neq s$, $\alpha((v_r^i, v_s^i)) = ki + r + s$;

8) for $r = 1, \dots, k$, $s = 1, \dots, k$, $r \neq s$, $\alpha\left(\left(v_r^{\frac{n-1}{k}}, v_s^{\frac{n-1}{k}}\right)\right) = n + r + s$.

It is not difficult to check that α is an interval total $(n+2k)$−coloring of the graph $G_{k,n}$.

In the next section we show that there are graphs G containing the vertex u with $d_G(u) = |V(G)| - 1$, $k(G) = \max_{v \in V(G)(v \neq u)} d_G(v) < |V(G)| - 1$ and $G \in \mathcal{T}$, but $W_\tau(G) < |V(G)| + 2k(G)$.

3. Interval Total Colorings of Wheels

The wheel $W_n (n \geq 4)$ is defined as follows:

$$V(W_n) = \{u, v_1, v_2, \dots, v_{n-1}\} \quad \text{and}$$

$$E(W_n) = \{(u, v_i) \mid 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-2\} \cup \{(v_1, v_{n-1})\}.$$

Lemma 1: Let α be an interval total t−coloring of a graph G then a total coloring β, where

1) $\beta(v) = t + 1 - \alpha(v)$ for any $v \in V(G)$,
2) $\beta(e) = t + 1 - \alpha(e)$ for any $e \in E(G)$,

is also an interval total t−coloring of a graph G.

Proof: Clearly, a total coloring β contains at least one vertex or edge with color i, $i = 1, 2, \dots, t$. Since $S[v, \alpha]$ is an interval for any $v \in V(G)$, then $S[v, \alpha] = [a, b]$. From the definition of the coloring β it follows that $S[v, \beta] = [t + 1 - b, t + 1 - a]$ for any $v \in V(G)$.

Lemma 2: For any $n \geq 4$ we have $W_n \in \mathcal{T}$ and $w_\tau(W_n) = \begin{cases} n+2, & \text{if } n = 4, \\ n, & \text{if } n \geq 5. \end{cases}$

Proof: Clearly, $W_4 = K_4$ hence $W_4 \in \mathcal{T}$ and $w_\tau(W_4) = w_\tau(K_4) = 6$ [2].
Assume that $n \geq 5$.
For the proof of the lemma we construct an interval total n−coloring of the graph W_n.
Case 1: n is even.
Define a total coloring α of the graph W_n as follows:
1) $\alpha(u) = n, \alpha(v_1) = 2$ and for $i = 2, \dots, \frac{n}{2} - 1$ $\alpha(v_i) = 2i + 1$;
2) $\alpha(v_{\frac{n}{2}}) = n - 2, \alpha(v_{\frac{n}{2}+1}) = n - 4$ and for $j = \frac{n}{2} + 2, \dots, n-1$ $\alpha(v_j) = 2(n - j + 1)$;
3) for $k = 1, 2, \dots, \frac{n}{2}$ $\alpha((u, v_k)) = 2k - 1$;
4) for $l = \frac{n}{2} + 1, \dots, n-1$ $\alpha((u, v_l)) = 2(n - l)$;
5) for $p = 1, \dots, \frac{n}{2} - 1$ $\alpha((v_p, v_{p+1})) = 2(p+1)$ and $\alpha\left(\left(v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\right)\right) = n - 3$;
6) for $q = \frac{n}{2} + 1, \dots, n-2$ $\alpha((v_q, v_{q+1})) = 2(n - q) + 1$ and $\alpha((v_1, v_{n-1})) = 3$.
Case 2: n is odd.
Define a total coloring β of the graph W_n as follows:
1) $\beta(u) = n, \beta(v_1) = 2$ and for $i = 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$ $\beta(v_i) = 2i + 1$;
2) $\beta(v_{\lfloor \frac{n}{2} \rfloor}) = n - 4, \beta(v_{\lceil \frac{n}{2} \rceil}) = n - 2$ and for $j = \lceil \frac{n}{2} \rceil + 1, \dots, n-1$ $\beta(v_j) = 2(n - j + 1)$;
3) for $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ $\beta((u, v_k)) = 2k - 1$;

4) for $l = \lceil \frac{n}{2} \rceil, \dots, n-1$ $\beta((u, v_l)) = 2(n-l)$;
5) for $p = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$ $\beta((v_p, v_{p+1})) = 2(p+1)$ and $\beta\left(\left(v_{\lfloor \frac{n}{2} \rfloor}, v_{\lceil \frac{n}{2} \rceil}\right)\right) = n-3$;
6) for $q = \lceil \frac{n}{2} \rceil, \dots, n-2$ $\beta((v_q, v_{q+1})) = 2(n-q)+1$ and $\beta((v_1, v_{n-1})) = 3$.
It is not difficult to check that α is an interval total n-coloring of the graph W_n, when n is even, and β is an interval total n-coloring of the graph W_n, when n is odd. Hence $W_n \in \mathcal{T}$. On the other hand, clearly, $w_\tau(W_n) \geq \chi''(W_n) = \Delta(W_n) + 1 = n$, therefore $w_\tau(W_n) = n$.

Lemma 3: For any $n \geq 5$ we have $W_n \in \mathcal{T}_{n+1} \cap \mathcal{T}_{n+2}$. **Proof:** First we show that $W_n \in \mathcal{T}_{n+2}$, for any $n \geq 5$.
Define a total coloring α of the graph W_n as follows:
1) $\alpha(u) = 1, \alpha(v_1) = 3, \alpha(v_{\lceil \frac{n}{2} \rceil}) = n-1$ and for $i = 2, \dots, \lceil \frac{n}{2} \rceil - 1$ $\alpha(v_i) = 2(i+1)$;
2) for $j = \lceil \frac{n}{2} \rceil + 1, \dots, n-1$ $\alpha(v_j) = 2(n-j)+3$;
3) for $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ $\alpha((u, v_k)) = 2k$;
4) for $l = \lfloor \frac{n}{2} \rfloor + 1, \dots, n-1$ $\alpha((u, v_l)) = 2(n-l)+1$;
5) for $p = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ $\alpha((v_p, v_{p+1})) = 2p+3$
6) for $q = \lfloor \frac{n-1}{2} \rfloor + 1, \dots, n-2$ $\alpha((v_q, v_{q+1})) = 2(n-q+1)$ and $\alpha((v_1, v_{n-1})) = 4$.
It is easily seen that α is an interval total $(n+2)$-coloring of the graph W_n.
Now we show that $W_n \in \mathcal{T}_{n+1}$, for any $n \geq 5$.
Define a total coloring β of the graph W_n as follows:
1) for $\forall v \in V(W_n)$ $\beta(v) = \alpha(v)$;
2) for $\forall e \in E(W_n)$

$$\beta(e) = \begin{cases} \alpha(e), \text{ if } \alpha(e) \neq n+2, \\ n-2, \text{ otherwise.} \end{cases}$$

It is easily seen that β is an interval total $(n+1)$-coloring of the graph W_n.

Lemma 4: For any $n \geq 4$ we have $W_\tau(W_n) \geq n+3$.

Proof: Clearly, for the proof of the lemma it suffices to construct an interval total $(n+3)$-coloring of the graph W_n, for $n \geq 4$.
Case 1: n is even.
Define a total coloring α of the graph W_n in the following way:
1) for $i = 1, 2, \dots, \frac{n}{2}+1$ $\alpha(v_i) = 2i-1$;
2) for $j = \frac{n}{2}+2, \dots, n-1$ $\alpha(v_j) = 2(n-j+1)$;
3) for $k = 1, 2, \dots, \frac{n}{2}$ $\alpha((v_k, v_{k+1})) = 2k$;
4) for $l = \frac{n}{2}+1, \dots, n-2$ $\alpha((v_l, v_{l+1})) = 2(n-l)+1$ and $\alpha((v_1, v_{n-1})) = 3$;
5) for $p = 2, \dots, \frac{n}{2}$ $\alpha((u, v_p)) = 2p+1$ and $\alpha((u, v_1)) = 4$;
6) for $q = \frac{n}{2}+1, \dots, n-1$ $\alpha((u, v_q)) = 2(n-q+2)$ and $\alpha(u) = n+3$.
Case 2: n is odd.
Define a total coloring β of the graph W_n in the following way:
1) for $i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ $\beta(v_i) = 2i-1, \beta((v_i, v_{i+1})) = 2i$;
2) for $j = \lceil \frac{n}{2} \rceil, \dots, n-1$ $\beta(v_j) = 2(n-j+1)$;
3) for $k = \lceil \frac{n}{2} \rceil, \dots, n-2$ $\beta((v_k, v_{k+1})) = 2(n-k)+1$ and $\beta((v_1, v_{n-1})) = 3$;
4) for $p = 2, 3, \dots, \lceil \frac{n}{2} \rceil$ $\beta((u, v_p)) = 2p+1$ and $\beta((u, v_1)) = 4$;
5) for $q = \lceil \frac{n}{2} \rceil + 1, \dots, n-1$ $\beta((u, v_q)) = 2(n-q+2)$ and $\beta(u) = n+3$.
It is not difficult to check that α is an interval total $(n+3)$-coloring of the graph W_n, when n is even, and β is an interval total $(n+3)$-coloring of the graph W_n, when n is odd.
Remark 1: Note that $W_\tau(W_n) = n+3$, for $4 \leq n \leq 8$.

Lemma 5: For any $n \geq 9$ we have $W_\tau(W_n) \geq n+4$.

Proof: Clearly, for the proof of the lemma it suffices to construct an interval total $(n+4)$−coloring of the graph W_n, for $n \geq 9$.

Case 1: n is even.

Define a total coloring α of the graph W_n in the following way:

1) $\alpha(u) = 7, \alpha(v_1) = 1, \alpha(v_2) = 6, \alpha(v_3) = 8$ and for $i = 4, \dots, \frac{n}{2} - 2$ $\alpha(v_i) = 2i+1$;

2) $\alpha(v_{\frac{n}{2}-1}) = n+2, \alpha(v_{\frac{n}{2}}) = n+4$ and for $j = \frac{n}{2}+1, \dots, n-2$ $\alpha(v_j) = 2(n-j), \alpha(v_{n-1}) = 3$;

3) $\alpha((u, v_1)) = 3, \alpha((u, v_2)) = 5$ and for $k = 3, \dots, \frac{n}{2} - 1$ $\alpha((u, v_k)) = 2k+3$;

4) for $l = \frac{n}{2}, \dots, n-1$ $\alpha((u, v_l)) = 2(n-l+1)$;

5) $\alpha((v_1, v_2)) = 4, \alpha((v_2, v_3)) = 7$ and for $p = 3, \dots, \frac{n}{2} - 2$ $\alpha((v_p, v_{p+1})) = 2(p+2)$;

6) for $q = \frac{n}{2} - 1, \dots, n-2$ $\alpha((v_q, v_{q+1})) = 2(n-q)+1$ and $\alpha((v_1, v_{n-1})) = 2$.

Case 2: n is odd.

Define a total coloring β of the graph W_n in the following way:

1) $\beta(u) = 7, \beta(v_1) = 1, \beta(v_2) = 6, \beta(v_3) = 8$ and for $i = 4, \dots, \lfloor \frac{n}{2} \rfloor - 1$ $\beta(v_i) = 2i+1$;

2) $\beta(v_{\lfloor \frac{n}{2} \rfloor}) = n+4, \beta(v_{\lceil \frac{n}{2} \rceil}) = n+2$ and for $j = \lceil \frac{n}{2} \rceil + 1, \dots, n-2$ $\beta(v_j) = 2(n-j), \beta(v_{n-1}) = 3$;

3) $\beta((u, v_1)) = 3, \beta((u, v_2)) = 5$ and for $k = 3, \dots, \lfloor \frac{n}{2} \rfloor$ $\beta((u, v_k)) = 2k+3$;

4) for $l = \lceil \frac{n}{2} \rceil, \dots, n-1$ $\beta((u, v_l)) = 2(n-l+1)$;

5) $\beta((v_1, v_2)) = 4, \beta((v_2, v_3)) = 7$ and for $p = 3, \dots, \lfloor \frac{n}{2} \rfloor$ $\beta((v_p, v_{p+1})) = 2(p+2)$;

6) for $q = \lceil \frac{n}{2} \rceil, \dots, n-2$ $\beta((v_q, v_{q+1})) = 2(n-q)+1$ and $\beta((v_1, v_{n-1})) = 2$.

It is easy to check that α is an interval total $(n+4)$−coloring of the graph W_n, when n is even, and β is an interval total $(n+4)$−coloring of the graph W_n, when n is odd.

Lemma 6: For any $n \geq 4$ we have $W_\tau(W_n) \leq n+4$.

Proof: From the theorem 1 we have that $W_\tau(W_n) \leq n+6$, for any $n \geq 4$.

First we prove that $W_n \notin \mathcal{T}_{n+5}$.

Suppose, to the contrary, that α is an interval total $(n+5)$−coloring of the graph W_n, for $n \geq 4$.

Consider the vertex u. Clearly, $1 \leq \min S[u, \alpha] \leq 6$, hence $n \leq \max S[u, \alpha] \leq n+5$.

Lemma 3 implies that the following three cases are possible:

1) $S[u, \alpha] = [6, n+5]$;

2) $S[u, \alpha] = [5, n+4]$;

3) $S[u, \alpha] = [4, n+3]$.

Case 1: $S[u, \alpha] = [6, n+5]$ or $S[u, \alpha] = [5, n+4]$.

Clearly, $\alpha((u, v_i)) \geq 5, i = 1, \dots, n-1$. This implies that $\min S[v_i, \alpha] \geq 2, i = 1, \dots, n-1$, which is a contradiction.

Case 2: $S[u, \alpha] = [4, n+3]$.

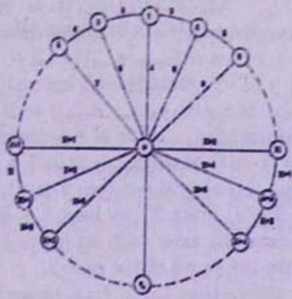

Figure 2.

First we show that $\alpha(u) \neq 4$. Suppose that $\alpha(u) = 4$. This implies that $\alpha((u, v_i)) \geq 5$, $i = 1, \dots, n-1$, which is a contradiction.

Let $e = (u, v_1)$ and $\alpha(e) = 4$. Note that $\alpha(v_1) = 1$.

Without loss of generality, we may assume that $\alpha((v_1, v_2)) = 2, \alpha((v_1, v_{n-1})) = 3, \alpha((u, v_2)) = 5, \alpha((u, v_{n-1})) = 6$, and there is a vertex v_k such that either $\alpha(v_k) = n+5$, or $\alpha((v_k, v_{k+1})) = n+5$ (see Fig. 2).

Let us consider the simple paths

$$P_1 = (v_1, (v_1, v_2), v_2, \dots, v_k, (v_k, v_{k+1}), v_{k+1})$$

and

$$P_2 = (v_{n-1}, (v_{n-1}, v_{n-2}), v_{n-2}, \dots, v_{k+1}, (v_{k+1}, v_k), v_k),$$

where $1 \leq k \leq n-2$.

Let us show that for $i = 2, \dots, k$.

1) $\alpha(v_i) = 2i-1, \alpha((v_i, v_{i+1})) = 2i, \alpha((u, v_i)) = 2i+1$,

2) $\alpha(v_{n+1-i}) = 2i, \alpha((v_{n-i}, v_{n+1-i})) = 2i+1, \alpha((u, v_{n+1-i})) = 2(i+1)$,

We use induction by i. For $i = 2$ it suffices to prove that $\alpha(v_2) = 3, \alpha((v_2, v_3)) = 4, \alpha(v_{n-1}) = 4,\ \alpha((v_{n-2}, v_{n-1})) = 5$.

Consider the vertex v_2. Since $\alpha((v_1, v_2)) = 2$ and $\alpha((u, v_2)) = 5$ then $\min S[v_2, \alpha] = 2$ and $\max S[v_2, \alpha] = 5$, therefore $\{3, 4\} \subseteq S[v_2, \alpha]$. If we suppose that $\alpha(v_2) = 4$ then $\alpha((v_2, v_3)) = 3$ and $\max S[v_3, \alpha] < 7$, which contradicts $\max S[v_3, \alpha] \geq 7$. From this we have $\alpha((u, v_3)) = 7$ (see Fig. 2).

Now we consider the vertex v_{n-1}. Since $\alpha((v_1, v_{n-1})) = 3$ and $\alpha((u, v_{n-1})) = 6$ then $\min S[v_{n-1}, \alpha] = 3$ and $\max S[v_{n-1}, \alpha] = 6$, therefore $\{4, 5\} \subseteq S[v_{n-1}, \alpha]$. If we suppose that $\alpha(v_{n-1}) = 5$ then $\alpha((v_{n-2}, v_{n-1})) = 4$ and $\max S[v_{n-2}, \alpha] < 8$, which contradicts $\max S[v_{n-2}, \alpha] \geq 8$ (see Fig. 2).

Suppose that the statements 1) and 2) are true for all $i', 1 \leq i' \leq i$. We prove that the statements 1) and 2) are true for the case $i+1$, that is $\alpha(v_{i+1}) = 2i+1, \alpha((v_{i+1}, v_{i+2})) = 2i+2, \alpha((u, v_{i+1})) = 2i+3$ and $\alpha(v_{n-i}) = 2i+2, \alpha((v_{n-i-1}, v_{n-i})) = 2i+3, \alpha((u, v_{n-i})) = 2i+4$. From the induction hypothesis we have:

1') $\alpha(v_j) = 2j-1, \alpha((v_j, v_{j+1})) = 2j, \alpha((u.v_j)) = 2j+1$,

2') $\alpha(v_{n+1-j}) = 2j, \alpha((v_{n-j}, v_{n+1-j})) = 2j+1, \alpha((u, v_{n+1-j})) = 2(j+1)$,

for $j = 2, \dots, i$.

1′) and 2′) implies that $\alpha((u, v_{i+1})) = 2i + 3$ and $\alpha((u, v_{n-i})) = 2i + 4$.

Consider the vertex v_{i+1}. Since $\alpha((v_i, v_{i+1})) = 2i$ and $\alpha((u, v_{i+1})) = 2i + 3$ then $\min S[v_{i+1}, \alpha] = 2i$ and $\max S[v_{i+1}, \alpha] = 2i + 3$, therefore $\{2i + 1, 2i + 2\} \subseteq S[v_{i+1}, \alpha]$. If we suppose that $\alpha(v_{i+1}) = 2i + 2$ then $\alpha((v_{i+1}, v_{i+2})) = 2i + 1$ and $\max S[v_{i+2}, \alpha] < 2i + 5$, which contradicts $\max S[v_{i+2}, \alpha] \geq 2i + 5$. From this we have $\alpha((u, v_{i+2})) = 2i + 5$ (see Fig. 2). Next we consider the vertex v_{n-i}. Since $\alpha((v_{n+1-i}, v_{n-i})) = 2i + 1$ and $\alpha((u, v_{n-i})) = 2i + 4$ then $\min S[v_{n-i}, \alpha] = 2i + 1$ and $\max S[v_{n-i}, \alpha] = 2i + 4$, therefore $\{2i + 2, 2i + 3\} \subseteq S[v_{n-i}, \alpha]$. If we suppose that $\alpha(v_{n-i}) = 2i + 3$ then $\alpha((v_{n-i-1}, v_{n-i})) = 2i + 2$ and $\max S[v_{n-i-1}, \alpha] < 2i + 6$, which contradicts $\max S[v_{n-i-1}, \alpha] \geq 2i + 6$ (see Fig. 2).

From 1′) we have $k \geq \frac{n}{2} + 2$. From 2′) we have $k \leq \frac{n}{2} - 1$.

It is easy to see that does not exist such an index k, which satisfy the aforementioned inequalities. This completes the prove of the case 2.

Analogously it can be shown that $W_n \notin \mathcal{T}_{n+6}$, hence $W_\tau(W_n) \leq n + 4$, for any $n \geq 4$.

From lemmas 2, 3, 4, 5, 6 and remark 1 we have the following result:

Theorem 3: For $n \geq 4$ we have

(1) $W_n \in \mathcal{T}$,

(2) $w_\tau(W_n) = \begin{cases} n + 2, & \text{if } n = 4, \\ n, & \text{if } n \geq 5. \end{cases}$

(3) $W_\tau(W_n) = \begin{cases} n + 3, & \text{if } 4 \leq n \leq 8, \\ n + 4, & \text{if } n \geq 9, \end{cases}$

(4) if $w_\tau(W_n) \leq t \leq W_\tau(W_n)$ then $W_n \in \mathcal{T}_t$.

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Կմախքային աստղով գրաֆների միջակայքային լիակատար ներկումներ

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Ամփոփում

G գրաֆի լիակատար ներկումը $1,2,\ldots,t$ գույներով կանվանենք միջակայքային լիակատար t –ներկում, եթե ամեն մի i գույնով, $i = 1,2,\ldots,t$, ներկված է առնվազն մեկ գագաթ, կամ կող, և յուրաքանչյուր v գագաթին կից կողերը, և այդ գագաթը ներկված է $d_G(v)+1$ հաջորդական գույներով, որտեղ $d_G(v)$ -ով նշանակված v գագաթի աստիճանը G գրաֆում: Այս աշխատանքում ապացուցված է, որ եթե $G = (V,E)$ -ն, որը պարունակում է այնպիսի u գագաթ, որ $d_G(u) = |V|-1$, $k(G) = \max_{v\in V(v\neq u)} d_G(v) < |V|-1$ և G գրաֆն ունի միջակայքային լիակատար t- ներկում, ապա then $t \leq |V| + 2k(G)$: Նաև ցույց է տրված, որ այս վերին գնահատականը հասանելի է: Այնուհետև, գտնվել են t- ի բոլոր հնարավոր արժեքները, որոնց համար անիվները ունեն միջակայքային լիակատար t- ներկում: