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On Reliable Identification of Two Independent Markov Chain Distributions

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Abstract

In this paper the problem of logarithmically asymptotically optimal (LAO) identification of distributions for two independent simple homogeneous stationary Markov chains with finite number of states is studied. The problem of identification under reliability requirement of distributions for one Markov chain was studied by Haroutunian and Navaei.

1. Introduction

In [1] Ahlswede and Haroutunian formulated some problems on multiple hypotheses testing for many objects and on identification of hypotheses under reliability requirement. The problem of many hypotheses testing on distributions of a finite state Markov chain is studied in [4] via large deviations techniques and also identification of distributions for one Markov chain is studied in [6]. In this paper we solve the problem of distributions identification of simple homogeneous stationary finite Markov chains for two independent objects. We take known the definitions and results on many hypotheses LAO testing for the case of Markov chains and identification of distribution subject to the reliability criterion presented in [2], [5], [6].

2. Problem of LAO Identification of Distribution for Two Independent Markov Chains and Formulation of Results

We expand the concept of identification for two independent homogenious stationary finite Markov chains. Let X_1 and X_2 be independent random variables (RV) taking values in the same finite set $\mathcal{X}$ with one of L PDs, they are characteristics of corresponding independent objects. The random vector (X_1, X_2) assume values $(x^1, x^2) \in \mathcal{X} \times \mathcal{X}$.

Let $(\mathbf{x}_1, \mathbf{x}_2) = ((x_0^1, x_0^2), \ldots, (x_n^1, x_n^2), \ldots, (x_N^1, x_N^2)), x_n^1, x_n^2 \in \mathcal{X},\ n = \overline{1, N}$, be a sequence of results of $N+1$ independent observations of two simple homogeneous stationary Markov chain with finite number I of states. The statistic must define unknown PDs of the objects on the base of observed data. This selection is denoted by Φ_N. The objects are independent, so the test Φ_N may be considered as the pair of the tests ϕ_N^1 and ϕ_N^2 for the respective separate objects. We will denote the whole compound test sequence by Φ. The test ϕ_N^i is defined by a partition of the space $\mathcal{X}^{N+1}$ on the L sets and to every trajectory $\mathbf{x}$ the test ϕ_N puts in correspondence one from L hypotheses. So the space $\mathcal{X}^{N+1}$ will be divided into L parts,

$$\mathcal{G}_{l,i}^N = \{x_i,\ \phi_N(x_i) = l\},\ l = \overline{1, L}, i = 1, 2.$$

We denote by

$$\alpha_{l_1,l_2|m_1,m_2}(\Phi_N) = Q_{m_1} \circ P_{m_1}(\mathcal{G}_{l_1,1}^N) Q_{m_2} \circ P_{m_2}(\mathcal{G}_{l_2,2}^N),$$

the probability of the erroneous acceptance by the test Φ_N of the hypotheses pair (H_{l_1}, H_{l_2}) provided that (H_{m_1}, H_{m_2}) is true, where $(m_1, m_2) \neq (l_1, l_2)$, $m_i, l_i = \overline{1, L}$, $i = 1, 2$. The probability to reject a true pair of hypotheses (H_{m_1}, H_{m_2}) is

$$\alpha_{m_1,m_2|m_1,m_2}^N(\Phi_N) \triangleq \sum_{(l_1,l_2) \neq (m_1,m_2)} \alpha_{l_1,l_2|m_1,m_2}^N(\Phi_N). \tag{1}$$

We also study corresponding error probability exponents $E_{l_1,l_2|m_1,m_2}(\Phi_N)$ of the sequence of tests Φ, called reliabilities:

$$E_{l_1,l_2|m_1,m_2}(\Phi) \triangleq \overline{\lim_{N \to \infty}} - \frac{1}{N} \log \alpha_{l_1,l_2|m_1,m_2}(\Phi_N), \quad m_i, l_i = \overline{1, L}, \quad i = 1, 2. \tag{2}$$

We denote by $E(\Phi^i)$ the reliability matrices of the sequences of tests Φ^i, $i = 1, 2$, for each of the objects. From (1) and (2) we see that as in [1], [2]:

$$E_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2) \neq (m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi). \tag{3}$$

We need analogical to [2] the following:

Lemma:If elements $E_{l|m}(\Phi^i)$, $m, l = \overline{1, L}$, $i = 1, 2$, positive, then the following equalities are valid for $\Phi = (\phi^1, \phi^2)$:

$$E_{l_1,l_2|m_1,m_2}(\Phi) = E_{l_1|m_1}(\phi^1) + E_{l_2|m_2}(\phi^2), \quad m_1 \neq l_1, \quad m_2 \neq l_2, \tag{4}$$

$$E_{l_1,l_2|m_1,m_2}(\Phi) = E_{l_i|m_i}(\phi^i), \quad m_{3-i} = l_{3-i} \quad m_i \neq l_i, \quad i = 1, 2. \tag{5}$$

The relation (4) is valid also if the reliability $E_{l|m}(\Phi^i) = 0$ for several m, l and i. The test sequence Φ^* we call LAO for two objects if for given positive values of certain $2(L-1)$ elements of the reliabieity matrix $\mathbf{E}(\Phi^*)$ the test provides best values for all other elements of the matrix.

Our aim is to find LAO test from the set of compound tests $\Phi = (\Phi^1, \Phi^2)$ when strictly positive elements $E_{m,m|m,L}$ and $E_{m,m|L,m}$, $m = \overline{1, L-1}$ of the reliability matrix are given.

Consider for a given positive elements $E_{m,m|m,L}$ and $E_{m,m|L,m}$, $m = \overline{1, L-1}$, the family of regions for $i = 1, 2$:

$$\mathcal{R}_m^{(i)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) \leq E_{m,m|L,m}\}, \quad m = \overline{1, L-1},$$

$$\mathcal{R}_L^{(i)} \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_m) > E_{m,m|L,m}, \quad m = \overline{1, L-1}\}.$$

There are two error probabilities for each couple (r_1, r_2), $r_i = \overline{1, L}$, $i = 1, 2$, the probability $\alpha_{(l_1,l_2) \neq (r_1,r_2)|(m_1,m_2)=(r_1,r_2)}^{(N)}$ to accept (l_1, l_2) different from (r_1, r_2), when (r_1, r_2)

is realised, and the probability $\alpha^{(N)}_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ that (r_1,r_2) is accepted, when it is not correct. The probability $\alpha^{(N)}_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)}$ coincides with the probability $\alpha^{(N)}_{(r_1,r_2)|(r_1,r_2)}$. Our aim is to determine the dependence of $E^{(N)}_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ on given $E^{(N)}_{(r_1,r_2)|(r_1,r_2)}$.

We need to use the probabilities of different hypotheses. Let us assume that the hypotheses $H_l : l=\overline{1,L}$ have, say, probabilities $\Pr(r)$, $r=\overline{1,L}$. The only supposition we shall use is that $\Pr(r)>0$, $r=\overline{1,L}$. We will see, that the result formulated in the following theorem does not depend on values of $\Pr(r)$, $r=\overline{1,L}$, if they all are strictly positive. Now we can make the following reasoning for each $r_i=\overline{1,L}$, $i=1,2$:

$$\alpha^{(N)}_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}=\frac{\Pr^N((l_1,l_2)=(r_1,r_2),(m_1,m_2)\neq(r_1,r_2))}{\Pr((m_1,m_2)\neq(r_1,r_2))}=$$

$$=\frac{1}{\sum\limits_{m:(m_1,m_2)\neq(r_1,r_2)}\Pr(m_1,m_2)}\sum_{m:(m_1,m_2)\neq(r_1,r_2)}\alpha_{(m_1,m_2)|(r_1,r_2)}\Pr^{(N)}(m_1,m_2).$$

Finally we obtain that:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}=\min_{(m_1,m_2):(m_1,m_2)\neq(r_1,r_2)}E_{(r_1,r_2)|(m_1,m_2)}. \tag{6}$$

For every LAO test Φ^* from (3), (4) and (5) it follows that:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}=\min_{m_1\neq r_1,m_2\neq r_2}(E^1_{r_1|m_1},E^2_{r_2|m_2}), \tag{7}$$

where $E^1_{r_1|m_1},E^2_{r_2|m_2}$ are determined for, correspondingly, the first and the second objects. For every LAO test Φ^* from (3) and (4) we see that

$$E_{(r_1,r_2)|(r_1,r_2)}=\min_{m_1\neq r_1,m_2\neq r_2}(E^1_{r_1|m_1},E^2_{r_2|m_2})=\min(E^1_{r_1|r_1},E^2_{r_2|r_2}). \tag{8}$$

and each of $E^1_{r_1|r_1},E^2_{r_2|r_2}$ satisfy the following conditions(see in [?]).

$$0<E^1_{r_1|r_1}<\min\left[\min_{l=\overline{1,r_1-1}}E^*_{l|m}(E^1_{l|l}),\ \min_{l=\overline{r_1+1,L}}D(Q_l\circ P_l\|Q_l\circ P_{r_1})\right], \tag{9}$$

$$0<E^2_{r_2|r_2}<\min\left[\min_{l=\overline{1,r_2-1}}E^*_{l|m}(E^2_{l|l}),\ \min_{l=\overline{r_2+1,L}}D(Q_l\circ P_l\|Q_l\circ P_{r_2})\right], \tag{10}$$

From (5) we see that the elements $E^*_{l|m}(E^1_{l|l})$, $r_1=\overline{1,r_1-1}$ and $E^*_{l|m}(E^2_{l|l})$, $r_2=\overline{1,r_2-1}$ are determined only by $E^1_{l|l}$ and $E^2_{l|l}$. But we are considering only elements $E^1_{r_1|r_1}$ and $E^2_{r_2|r_2}$. By applying theorem 1 of [6] and (9), (10) we have

$$0<E^1_{r_1|r_1}<\min\left[\min_{l=\overline{1,r_1-1}}D(Q_l\circ P_l\|Q_l\circ P_{r_1}),\ \min_{l=\overline{r_1+1,L}}D(Q_l\circ P_l\|Q_l\circ P_{r_1})\right], \tag{11}$$

$$0<E^2_{r_2|r_2}<\min\left[\min_{l=\overline{1,r_2-1}}D(Q_l\circ P_l\|Q_l\circ P_{r_2}),\ \min_{l=\overline{r_2+1,L}}D(Q_l\circ P_l\|Q_l\circ P_{r_2})\right]. \tag{12}$$

Let $r=\max(r_1,r_2)$ and $k=\min(r_1,r_2)$. From (8) we have that, when $E_{(r_1,r_2)|(r_1,r_2)}=E^1_{r_1|r_1}$, then $E^1_{r_1|r_1}\leq E^2_{r_2|r_2}$ and when $E_{(r_1,r_2)|(r_1,r_2)}=E^2_{r_2|r_2}$, then $E^1_{r_1|r_1}\geq E^2_{r_2|r_2}$. Therefore we conclude that given strictly positive elements $E_{(r_1,r_2)|(r_1,r_2)}$ must verity both inequalities (11), (12) and the juxtaposition of them bring us to

$$0 < E_{(r_1,r_2)|(r_1,r_2)} < \min\left[\min_{l=\overline{1,r-1}} D(Q_l \circ P_l \| Q_l \circ P_r), \min_{l=\overline{r+1,L}} D(Q_l \circ P_l \| Q_l \circ P_k)\right].$$

By (9) and (10) we can find reliability $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ in function of $E_{(r_1,r_2)|(r_1,r_2)}$ in the following way:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}(E_{(r_1,r_2)|(r_1,r_2)}) =$$
$$\min_{m_1\neq r_1, m_2\neq r_2}(E_{r_1|m_1}(E_{(r_1,r_2)|(r_1,r_2)}), E_{r_2|m_2}(E_{(r_1,r_2)|(r_1,r_2)}), \quad (13)$$

where $(E_{r_1|r_1}(E_{(r_1,r_2)|(r_1,r_2)})$ and $E_{r_2|r_2}(E_{(r_1,r_2)|(r_1,r_2)})$ to be found by (7) and (5). This results can be summarized in:

Theorem 1: If the distrbutions H_m, $m = \overline{1,L}$, are distinct and the given positive number $E_{(r_1,r_2)|(r_1,r_2)}$ satisfy (7)–(12), then the reliability $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ is defined in (13).

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Երկու Մարկովյան շղթաների բաշխումների հուսալի նույնականացման մասին

Լ. Նավայի

Ամփոփում

Հոդվածում ստացված է երկու Մարկովյան շղթայով բնութագրվող օբյեկտների բաշխումների ասիմպտոտորեն օպտիմալ նույնականացման խնդրի լուծումը: Մեկ օբյեկտի դեպքում խնդիրը լուծված է [6] հոդվածում: