Mathematical Problems of Computer Science 32, 70–73, 2009.

On Interval Total Colorings of Trees

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Abstract

An interval total t-coloring of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by $i, i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. In this paper we prove that if T ($T \neq K_1$) is a tree and $\Delta(T) + 2 \leq t \leq M(T)$ then T has an interval total t-coloring, where $\Delta(T)$ is the maximum degree of vertices in T and $M(T)$ is a parameter which can be effectively found for any T.

1. Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of vertices in G - by $\Delta(G)$. A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. If α is a total coloring of a graph G then $\alpha(v)$ and $\alpha(e)$ denote the color of a vertex $v \in V(G)$ and the color of an edge $e \in E(G)$ in the coloring α. For a total coloring α of a graph G and for any $v \in V(G)$ define the set $S[v, \alpha]$ as follows:

$$S[v, \alpha] \equiv \{\alpha(v)\} \cup \{\alpha(e) \mid e \text{ is incident to } v\}$$

An interval total t-coloring [1, 2] of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors.

Terms and concepts that we do not define can be found in [3, 4].

2. The main result

Let T be a tree and $V(T) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$. Let $P(v_i, v_j)$ be the simple path joining v_i with v_j, $VP(v_i, v_j)$ and $EP(v_i, v_j)$ denote the sets of vertices and edges of this path, respectively.

For a simple path $P(v_i, v_j)$ define $L(v_i, v_j)$ as follows:

$$L(v_i, v_j) \equiv |EP(v_i, v_j)| + |\{(u, w)|(u, w) \in E(T), u \in VP(v_i, v_j), w \notin VP(v_i, v_j)\}|.$$

Define:

$$L(T) \equiv \max_{1<i<n, 1<j<n} L(v_i, v_j).$$

If $L(v_{i_0}, v_{j_0}) = L(T)$ then define $M(T)$ as follows:

$$M(T) \equiv L(T) + |VP(v_{i_0}, v_{j_0})|.$$

Theorem 1 *If T $(T \neq K_1)$ is a tree and $\Delta(T)+2 \leq t \leq M(T)$ then T has an interval total $t-$coloring.*

Proof. We use induction on $|E(T)|$. Clearly, the theorem is true for the case $|E(T)| = 1$. Suppose that $|E(T)| = k > 1$ and the theorem is true for all trees T', where $|E(T')| < k$.

Case 1: $L(T) < |E(T)|$.

Clearly, there is an edge $e = (u, v) \in E(T), d_T(u) = 1$ such that $M(T') = M(T)$, where $T' = T - u$. Since $|E(T)| > 1$ then $d_T(v) \geq 2$. Clearly, $d_{T'}(v) = d_T(v) - 1, \Delta(T') \leq \Delta(T)$ and $|E(T')| = |E(T)| - 1 < k, \Delta(T') + 2 \leq t \leq M(T')$. Let α be an interval total $t-$coloring of the tree T' (by induction hypothesis). Consider the vertex v. Let

$$S[v, \alpha] = \{s(1), s(2), \ldots, s(d_{T'}(v) + 1)\},$$

where $1 \leq s(1) < s(2) < \ldots < s(d_{T'}(v) + 1) \leq t$.

Subcase 1.1: $s(1) = 1$.

Clearly, $s(d_{T'}(v) + 1) = d_{T'}(v) + 1 = d_T(v)$. In this case we color the edge e with color $d_T(v) + 1$ and the vertex u with color $d_T(v) + 2$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 1.2: $s(1) = 2$.

Subcase 1.2.1: $\alpha(v) = 2$.

Clearly, $s(d_{T'}(v) + 1) = d_T(v) + 1$. In this case we color the edge e with color $d_T(v) + 2$ and the vertex u with color $d_T(v) + 1$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 1.2.2: $\alpha(v) \neq 2$ and $\Delta(T') = \Delta(T)$.

We color the edge e with color 1 and the vertex u with color 2. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 1.2.3: $\alpha(v) \neq 2$ and $\Delta(T') < \Delta(T)$.

If $t \geq \Delta(T') + 3$ then we color the edge e with color 1 and the vertex u with color 2. Clearly, obtained coloring is an interval total $t-$coloring of the tree T.

Assume that $t = \Delta(T') + 2$.

Define a total coloring β of the tree T' in the following way:

1. $\forall w \in V(T')$ $\beta(w) = \alpha(w) + 1$;
2. $\forall e' \in E(T')$ $\beta(e') = \alpha(e') + 1$.

Now we color the edge e with color 2 and the vertex u with color 1. It is not difficult to see that obtained coloring is an interval total $(\Delta(T) + 2)-$coloring of the tree T.

Subcase 1.3: $s(1) \geq 3$.

We color the edge e with color $s(1) - 1$ and the vertex u with color $s(1) - 2$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Case 2: $L(T) = |E(T)|$.

Subcase 2.1: $t \leq M(T) - 2$.

Let $e = (u, v) \in E(T)$ and $d_T(u) = 1$. Since $|E(T)| > 1$ then $d_T(v) \geq 2$. Consider the tree $T' = T - u$. Clearly, $d_{T'}(v) = d_T(v) - 1, \Delta(T') \leq \Delta(T), M(T) - 2 \leq M(T') \leq M(T)$. This implies that $\Delta(T') + 2 \leq \Delta(T) + 2 \leq t \leq M(T) - 2 \leq M(T')$ and $|E(T')| = |E(T)| - 1 < k$. Let γ be an interval total $t-$coloring of the tree T' (by induction hypothesis). Consider the vertex v. Let

$$S[v, \gamma] = \{s(1), s(2), \ldots, s(d_{T'}(v) + 1)\},$$

where $1 \leq s(1) < s(2) < \ldots < s(d_{T'}(v) + 1) \leq t$.

Subcase 2.1.1: $s(1) = 1$.

Clearly, $s(d_{T'}(v) + 1) = d_{T'}(v) + 1 = d_T(v)$. In this case we color the edge e with color $d_T(v) + 1$ and the vertex u with color $d_T(v) + 2$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 2.1.2: $s(1) = 2$.

Subcase 2.1.2.1: $\gamma(v) = 2$.

Clearly, $s(d_{T'}(v) + 1) = d_T(v) + 1$. In this case we color the edge e with color $d_T(v) + 2$ and the vertex u with color $d_T(v) + 1$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 2.1.2.2: $\gamma(v) \neq 2$ and $\Delta(T') = \Delta(T)$.

We color the edge e with color 1 and the vertex u with color 2. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 2.1.2.3: $\gamma(v) \neq 2$ and $\Delta(T') < \Delta(T)$.

If $t \geq \Delta(T') + 3$ then we color the edge e with color 1 and the vertex u with color 2. Clearly, obtained coloring is an interval total $t-$coloring of the tree T.

Assume that $t = \Delta(T') + 2$.

Define a total coloring ϕ of the tree T' in the following way:

1. $\forall w \in V(T')\ \phi(w) = \gamma(w) + 1$;
2. $\forall e' \in E(T')\ \phi(e') = \gamma(e') + 1$.

Now we color the edge e with color 2 and the vertex u with color 1. It is not difficult to see that obtained coloring is an interval total $(\Delta(T) + 2)-$coloring of the tree T.

Subcase 2.1.3: $s(1) \geq 3$.

We color the edge e with color $s(1) - 1$ and the vertex u with color $s(1) - 2$. It is easy to see that obtained coloring is an interval total $t-$coloring of the tree T.

Subcase 2.2: $t = M(T) - 1, M(T)$.

First we show that T has an interval total $M(T)-$coloring.

Let $L(u_1, u_2) = L(T)$ and

$$P(u_1, u_2) = (w_0, e_1, w_1, \ldots, w_{i-1}, e_i, w_i, \ldots, w_{k-1}, e_k, w_k),$$

where $w_0 = u_1, w_k = u_2, k \geq 1$. Clearly, $d_T(u_1) = d_T(u_2) = 1$. Now we construct an interval total $M(T)-$coloring of the tree T. First we color the vertex u_1 with color 1 and the vertex u_2 with color $M(T)$, further we color the vertex w_i with color $3 + \sum_{j=1}^{i-1} d_T(w_j), i = 1, 2, \ldots, k-1$. Next we color the edge (w_i, w_{i+1}) with color $2 + \sum_{j=1}^{i} d_T(w_j), i = 0, 1, \ldots, k-1$, further uncolored $d_T(w_i) - 2$ edges incident to $w_i, i = 1, 2, \ldots, k-1$, and uncolored $d_T(w_i) - 2$ vertices incident to these edges we color with $4 + \sum_{j=1}^{i-1} d_T(w_j), \ldots, 1 + \sum_{j=1}^{i} d_T(w_j)$ and $5 +$

$\sum_{j=1}^{i-1} d_T(w_j), \ldots, 2 + \sum_{j=1}^{i} d_T(w_j)$ colors, respectively. It is easy to see that obtained coloring is an interval total $M(T)$−coloring of the tree T.

Next we construct an interval total $(M(T)-1)$−coloring of the tree T.

If $d_T(w_{k-1}) = 2$ then an interval total $(M(T)-1)$−coloring of the tree T can be obtained from aforementioned coloring by recoloring the vertex w_{k-1} with color $M(T)-1$, the edge (w_{k-1}, w_k) with color $M(T)-2$ and the vertex w_k with color $M(T)-3$.

If $d_T(w_{k-1}) \geq 3$ then an interval total $(M(T)-1)$−coloring of the tree T can be obtained from aforementioned coloring by recoloring the vertex w_k with color $M(T)-2$.

It is not difficult to see that obtained coloring is an interval total $(M(T)-1)$−coloring of the tree T. ■

Acknowledgement

We would like to express our gratitude to Vahan V. Mkrtchyan for useful discussions over the subject.

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Ծառերի միջակայքային լիակատար ներկումների մասին

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Ամփոփում

G գրաֆի լիակատար ներկումը $1, 2, \ldots, t$ գույներով կանվանենք միջակայքային լիակատար $1, 2, \ldots, t$ −ներկում, եթե ամեն մի i գույնով, $i = 1, 2, \ldots, t$, ներկված է առնվազն մեկ գագաթ կամ կող և յուրաքանչյուր v գագաթին կից կողերը և այդ գագաթը ներկված են $d_G(v)+1$ հաջորդական գույներով, որտեղ $d_G(v)$ -ով նշանակված է v գագաթի աստիճանը G գրաֆում: Այս աշխատանքում ապացուցված է, որ եթե T $(T \neq K_1)$ -ն ծառ է և $\Delta(T)+2 \leq t \leq M(T)$, ապա T -ն ունի միջակայքային լիակատար t −ներկում, որտեղ $\Delta(T)$-ն T -ի մաքսիմալ գագաթի աստիճան է, իսկ $M(T)$−ն արդյունավետ հաշվարկելի պարամետր է T−ի համար: