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Pointwise Reconstruction of Interferometric Phase

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Abstract

In this work the problem of interferometric phase reconstruction is considered. The pointwise approximation approach is proposed, which provides stable results even for bad quality of interferogramms. Experimental results show that the developed algorithm demonstrates better performance in comparison with some of the state-of-the-art techniques.

1. Introduction

Two dimensional phase measurements have many important applications in different areas. For example, in Synthetic Aperture Radar Interferometry (InSAR) the interferometric phase can be used to make extremely fine measurements of surface topography, deformation, or velocity [1, 2, 3]. In adaptive optics the phase measurements provide estimates of atmospheric turbulence effects on an optical imaging system [4, 5, 6]. These atmospheric distortions are then removed through the use of a deformable focusing mirror. In magnetic resonance imaging (MRI) phase measurements from 2-D, or 3-D MR images can be used for such purposes as estimating blood flow rates [7], or separating water and fat signals [8, 9].

In each of these cases, the *absolute* phase extracted from an actual signal is wrapped into the interval $(-\pi, \pi]$ and called *principal* or *wrapped* phase. If absolute phase value is outside the interval $(-\pi, \pi]$, the observed value is wrapped into this interval by addition or subtraction of some multiples of 2π. The relationship between the wrapped phase ψ and the unwrapped (absolute) phase ϕ is stated as

$$\psi = \phi + 2\pi k, \qquad \psi \in (-\pi, \pi]. \tag{1}$$

In the applications mentioned above the wrapped phase ψ is useless until 2π phase discontinuities are removed, which is realized by using *phase unwrapping* algorithms. Phase unwrapping is an ill-posed problem, if no additional information is added. In fact, given any wrapped phase data, there is an infinite number of possible corresponding unwrapped phase data. Simply stated, the phase unwrapping problem is to obtain an estimate φ for the absolute phase ϕ from the wrapped values ψ.

Measured values of wrapped phase are usually corrupted by noise which makes phase unwrapping problem more difficult. The phase unwrapping from noisy data starts from the following observation model:

$$g_\psi = W(\phi + \Delta\phi), \tag{2}$$

where $\Delta\phi$ denotes a random error additive to ϕ, and g_ψ is the observed noisy wrapped phase. W is a wrapping operator transforming the noisy absolute phase $\phi+\Delta\phi$ to the basic interval $(-\pi,\pi]$. The phase unwrapping problem for noisy data is to restore the absolute phase $\phi(x,y)$ from the noisy wrapped observations $g_\psi(x,y)$, $x,y \in X$. (In this work X is considered as an integer 2D grid, $X = \{x, y : x = 1,2,...,N, y = 1,2,...,M\}$).

Many various approaches to 2-D phase unwrapping have been proposed over the past several decades, but only a limited number of them are currently in common use. The methods developed for phase unwrapping problem can be roughly separated in two large families: Path-following (local) methods and Minimum-norm (global) methods. The first family of algorithms relies on performing integration of the discrete gradients (wrapped differences) along paths. The algorithms of the second family rely on global approximation of the absolute phase. A comprehensive review of these two families of algorithms is given in [11]. The developed technique belongs to the second family of the algorithms with the only difference that approximation is performed on each point (local).

2. Observation Model

A variety of models exist for phase observation depending on measurement principals. In this paper we use the following one (Figure 1):

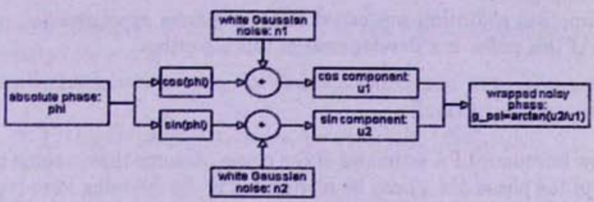

Figure 1: Observation model.

Let

$$\phi = \{\phi(x,y) \in \Re, x = 1,...,N, y = 1,...,M\}, \tag{3}$$

be the original absolute phase. The observation model is stated as

$$u_1 = \cos\phi + n_1, \quad u_2 = \sin\phi + n_2, \tag{4}$$

where u_1 and u_2 are the so-called *in-phase* (cosine) and *quadrature* (sine) components of the absolute phase ϕ, and n_1 and n_2 are independent white Gaussian noises. Then the wrapped phase g_ψ is calculated as follows:

$$g_\psi = \arctan\frac{u_2}{u_1}. \tag{5}$$

We mention that, particularly in optical interferometry and InSAR, the presence of additive white Gaussian noise in the in-phase and quadrature components is in fact the commonly adopted model [12, 13, 14, 15].

If we consider *cos* and *sin* of Equation 1, the difference between wrapped and unwrapped phases disappears ($\cos\psi = \cos\phi$ and $\sin\psi = \sin\phi$) and we can use a fit of these transformed observations for the absolute phase reconstruction. We consider observation in transformed domain (using $cos\psi$ and $sin\psi$) because in phase domain wrapped phase is discontinuous even for a continuous absolute phase because of non-linear characteristics of wrapping operator W. We also assume that the absolute phase $\phi(x,y)$ is a continuous function of the arguments x and y and allows a good polynomial approximation in a neighborhood of the estimation point (x,y).

The developed algorithm is based on the pointwise approximation of spatially varying absolute phase from its wrapped observations. For pointwise approximation we use *local polynomial approximation* (LPA) [16]. LPA is applied for direct pointwise phase approximation using polynomial fit in the sliding window. The window size is a key parameter of the algorithm.

We assume that observed data is given in phase form (5). Then we calculate

$$g_1 = \cos(g_\psi), \quad g_2 = \sin(g_\psi), \tag{6}$$

which are correspond to transformed noisy observations. According to Equation 2 we can rewrite them in the following form:

$$g_1 = \cos(\phi + \Delta\phi), \quad g_2 = \sin(\phi + \Delta\phi), \tag{7}$$

where $\Delta\phi$ is an error additive to ϕ caused by observation errors in g_ψ.

The LPA is applied in order to approximate absolute phase ϕ as an argument of harmonic functions in Equation 7

The proposed algorithm was called PAP (*pointwise approximation of phase*). The contribution of this paper is a development of this algorithm.

3. Proposed Approach

Let us now introduce LPA estimates of the phase. Assume that in some neighborhood of the point (x,y) the phase $\phi(x,y)$ can be represented in the following form (vector representation of the truncated Taylor series) [16]:

$$\tilde{\phi}(x_s, y_s|p) = q^T(x_s, y_s)p, \tag{8}$$

where $q = (q_1, q_2, q_3)$ is a vector of first order polynomials $q_1 = 1, q_2 = x, q_3 = y$, and $p = (p_1, p_2, p_3)^T$ is a vector of unknown parameters. The local fit loss function is defined as follows:

$$\begin{aligned} J_h(x,y,p) &= \frac{1}{2}\sum_s \omega_{h,s}\left[g_1(x+x_s, y+y_s) - \cos\tilde{\phi}(x_s, y_s|p)\right]^2 \\ &\quad + \omega_{h,s}\left[g_2(x+x_s, y+y_s) - \sin\tilde{\phi}(x_s, y_s|p)\right]^2 \\ &= \sum_s \omega_{h,s}\left[1 - \cos\left(g_\psi(x+x_s, y+y_s) - \tilde{\phi}(x_s, y_s|p)\right)\right], \\ &\quad \omega_{h,s} = \omega_h(x_s, y_s) \geq 0. \end{aligned} \tag{9}$$

The vector of unknown parameters p is defined as a solution of following optimization problem:

$$\hat{p} = \arg\min_p J_h(x,y,p). \tag{10}$$

The LPA estimates of the phase ϕ and the first derivatives $\phi_x^{(1)}, \phi_y^{(1)}$ are as follows [16]:

$$\hat{\phi}(x,y) = \hat{p_1}(x,y), \quad \hat{\phi}_x(x,y) = \hat{p_2}(x,y), \quad \hat{\phi}_y(x,y) = \hat{p_3}(x,y). \tag{11}$$

The window $\omega_{h,s}$ defines a set of neighborhood observations and their weights in estimation for point (x,y). The scale parameter h in $\omega_{h,s}$ defines the size of the window and is usually used in the following form: $\omega_h(x,y)=\omega\left(\frac{x}{h},\frac{y}{h}\right), h>0$. For example, for the square uniform window $\omega_h=1$ for $|x|\le h, |y|\le h$ and $\omega_h=0$, otherwise.

(10) shows that we obtain simultaneously the estimates of the phase ϕ and its first derivatives ϕ_x, ϕ_y. These estimates depend on the coordinates (x,y) and the window size h.

We wish to mention the nonparametric nature of the introduced estimates, because the polynomial approximation (8) is used only for a single "central" point $x_s=y_s=0$. For the phase it gives $\hat{\phi}(x,y)=\tilde{\phi}(0,0|p)=\hat{p}_1(x,y)$ and for the derivatives $\hat{\phi}_x(x,y)=\left(\frac{\partial\tilde{\phi}(x,y|p)}{\partial x}\right)_{x=0,y=0}=\hat{p}_2(x,y)$, $\hat{\phi}_y(x,y)=\left(\frac{\partial\tilde{\phi}(x,y|p)}{\partial y}\right)_{x=0,y=0}=\hat{p}_3(x,y)$.

Minimization of $J_h(x,y,p)$ with respect to vector p can not be expressed in an explicit form and requires numerical recursive calculations using the vector-gradient: $\partial_p L_h(x,y,p)=(\partial_{p_i}J_h(x,y,p))_{M\times 1}$, and the second derivative (*Hessian*) matrix: $\partial_p\partial_{p^T}L_h(x,y,p)=(\partial_{p_i}\partial_{p_j}J_h(x,y,p))_{M\times M}$, where M denotes the dimension of the vector p (in our case $M=3$). There are different procedures for calculation of the estimates. We consider the *Newton* method, which can be expressed in the following form:

$$p^{(k+1)}=p^{(k)}-\alpha_k\frac{\partial_p J_h(x,y,p^{(k)})}{H},\quad k=0,1,\ldots, \tag{12}$$

where $p^{(k)}$ are sequential iterations of p, $0<\alpha_k\le 1$ is the step size parameter, and the gradient $\partial_p J_h$ is calculated for $p=p^{(k)}$.

The straightforward manipulations give the Hessian matrix and the vector-gradient in the form:

$$\partial_p J_h=\sum_s\omega_{h,s}\sin\left(g_\psi(x+x_s,y+y_s)-\tilde{\phi}(x_s,y_s|p)\right)q(x_s,y_s), \tag{13}$$

$$\begin{aligned} H &= \partial_p\partial_{p^T}J_h\\ &= \sum_s\omega_{h,s}\cos\left(g_\psi(x+x_s,y+y_s)-\tilde{\phi}(x_s,y_s|p)\right)q(x_s,y_s)q^T(x_s,y_s). \end{aligned} \tag{14}$$

Assuming that the error approximation of $\cos\left(g_\psi(x+x_s,y+y_s)\right)$ by $\tilde{\phi}(x_s,y_s|p)$ is small, we can rewrite (14) in the following form:

$$H=\sum_s\omega_{h,s}q(x_s,y_s)q^T(x_s,y_s). \tag{15}$$

The Hessian matrix (14) can be used to analyze the convexity of the function $J_h(x,y,p)$. For the noiseless case we have $g_\psi(x+x_s,y+y_s)=\phi(x+x_s,y+y_s)$. Substituting these expressions in (14), we find that

$$\partial_p\partial_{p^T}J_h = \sum_s\omega_{h,s}\cos\left(\phi(x+x_s,y+y_s)-\tilde{\phi}(x_s,y_s|p)\right)q(x_s,y_s)q^T(x_s,y_s). \tag{16}$$

Let us assume that the polynomials $q(x_s,y_s)$ are linearly independent in area where $w_{h,s}>0$. It follows that the matrix $\sum_s\omega_{h,s}q(x_s,y_s)q^T(x_s,y_s)$ is positive definite. Then we can conclude for Equation 16 that if

$$|\phi(x+x_s,y+y_s)-\tilde{\phi}(x_s,y_s|p)|<\frac{\pi}{2} \tag{17}$$

$\cos\left(\phi(x+x_s, y+y_s) - \tilde{\phi}(x_s, y_s|p)\right) > 0$ and the matrix $\partial_p\partial_{p^T}J_h$ is also positive definite. It proves that the function $J_h(x, y, p)$ is locally convex and the convergence of the Newton method can be guaranteed at least locally provided a proper selection of the step size parameter α.

The recursive procedure (12) gives the estimate for any (x, y), provided that in the neighborhood of this point there is a sufficient number of observations (x_s, y_s). With independent initialization for each point this is only a denoising algorithm which does not assume the phase unwrapping. Because of that this pointwise estimate is used as an element of a more complex procedure with a special sequence of the estimation points (x, y) arranged with underlying intention to reconstruct continuous phase function $\phi(x, y)$. For instance, it can be a line-by-line sequence. Let us introduce a sequence of the neighboring points $\left\{x^{(n)}, y^{(n)}\right\}_{n=1,\ldots,NM}$ of a rectangular phase data, starting from the point $(1, 1)$ and going along the first line, further along the points of the second line, and in a similar way up to the last line. In this way we order all points of the phase data as a sequence.

The flow chart of the algorithm is presented in the Figure 2. A straightforward idea of the algorithm is to use for initialization of recursive estimator (12) for the given point (x, y) the estimates already obtained for its neighboring points.

Let $p^{(n)}\left(x^{(n)}, y^{(n)}|p\right)$ be the estimate for the point $\left(x^{(n)}, y^{(n)}\right)$, provided that the recursive pointwise algorithm (12) is initiated by the vector p. The proposed phase unwrapping algorithm is defined in the following sequential form:

$$p^{(n)} = p^{(n)}\left(x^{(n)}, y^{(n)}|p^{(n-1)}\right), \tag{18}$$

$$\begin{aligned} \hat{\phi}\left(x^{(n)}, y^{(n)}\right) &= p_1^{(n)}, \\ \hat{\phi}_x^{(1)}\left(x^{(n)}, y^{(n)}\right) &= p_2^{(n)}, \\ \hat{\phi}_y^{(1)}\left(x^{(n)}, y^{(n)}\right) &= p_3^{(n)}, \end{aligned} \tag{19}$$

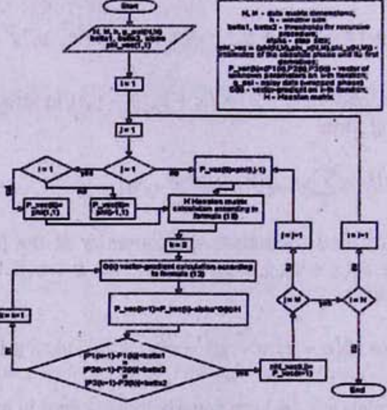

Figure 2: The Flow chart of the algorithm

The recursive pointwise estimator (12) is included in this recursive procedure. It is initiated by the vector $p^{(1)}$, which is the estimate for the first point $\left(x^{(1)}, y^{(1)}\right)$. This estimate can be defined from the boundary condition or can be taken from the original observations.

Presented algorithm solves two important goals: noise suppression and absolute phase reconstruction. Experiments show that the accuracy of the algorithm is high provided that the absolute phase differences in the neighboring points are not larger than $0.5 \div 1$ radians. The accuracy is as high as small this difference, even for a high level of the random noise.

The window size h is a crucial parameter for the accuracy of estimation. When the window size is small, the LPA gives a good smooth fit of signals, but then fewer number of observations are used and the estimates are more variable and sensitive with respect to the noise. The best choice of h involves a trade-off between the bias and variance, which depends on the degree of the LPA, the noise variance, and the derivatives of ϕ of the orders beyond the degree used in the LPA.

Theoretical analysis and experiments show that the efficiency of the local approximation estimates can be essentially improved provided a correct selection of the window size h. It can be varying or invariant but must be properly selected.

4. Experimental Results

For the algorithm's performance test we use $Z\pi M$ algorithm, which is considered as one of the best algorithms developed for noisy data [17]. For the accuracy measurements of the developed algorithm we use the root-mean-squared-error: $RMSE = \sqrt{\frac{1}{N_x N_y} \sum \left(\phi(x_s, y_s) - \hat{\phi}(x_s, y_s)\right)^2}$. With the LPA we use the uniform square windows ω_h defined on the integer grid $U_h = \{x, y : x = -h, -h+1, \ldots, 0, \ldots, h-1, h, y = -h, -h+1, \ldots, 0, \ldots, h-1, h\}$.

Figure 3 illustrates the original absolute phase ϕ (a) and noisy wrapped phase g_ψ (b) obtained from ϕ according to Equation 5, with the standard deviation of the white Gaussian noise $\sigma = 0.6$. This figure also presents results of the reconstruction for the different window sizes (c)-(f). Comparing the original and reconstracted phases, one may conclude that the noise suppression and phase reconstruction are performed quite accurately.

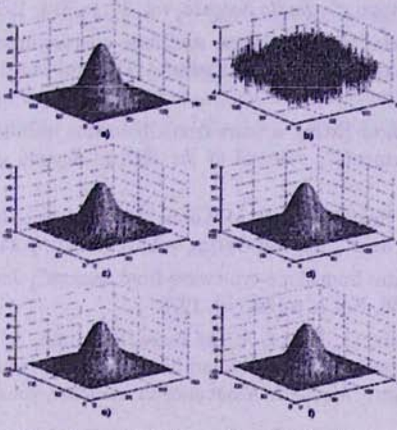

Figure 3: a) Original absolute phase ϕ, b) observed wrapped phase g_ψ with additive white Gaussian noise, c) reconstructed phase $\hat{\phi}$ for window size $h = 1$, d) reconstructed phase $\hat{\phi}$ for window size $h = 2$,

e) reconstructed phase $\hat{\phi}$ for window size $h = 3$, f) reconstructed phase $\hat{\phi}$ for window size $h = 4$.

Table below contains the numerical evaluation of the two algorithms for the original absolute phase ϕ presented in Figure 3 (a). For estimation we use different window sizes $h = \{1, 2, 3, 4\}$ and different values of standard deviation σ of the white Gaussian noise in (4). Based on the obtained results we can conclude that the proposed algorithm gives

Algorithm/σ	0.1	0.2	0.3	0.4	0.5	0.6
PAP, h=1	0.04	0.07	0.11	0.15	0.20	0.25
PAP, h=2	0.05	0.06	0.08	0.10	0.13	0.16
PAP, h=3	0.09	0.10	0.10	0.11	0.12	0.15
PAP, h=4	0.15	0.15	0.16	0.16	0.17	0.18
$Z\pi M$	0.05	0.08	0.11	0.15	0.19	0.22

significant improvement of the accuracy of reconstruction, which in fact depends on correct selection of the window size h. As follows from results presented in table, the best results for low level of noise were obtained with window size $h = 2$, while for the high noise level the best window size is $h = 3$.

One of the possible directions for further research is development of adaptive procedure for window size selection.

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Ինտերֆերենցիոն փուլի կետ առ կետ վերականգնում

Յ. Բարսեղյան

Ամփոփում

Աշխատանքում քննարկվում է ինտերվերոմետրիկական փուլի վերականգնման խնդիրը: Առաջարկվում է կետ առ կետ մոտարկման մեթոդ, որը ապահովում է կայուն արդյունքներ նույնիսկ ինտերֆերենցիոն պատկերների վատ որակի դեպքում: Փորձերի արդյունքները ցույց են տալիս, որ առաջարկված ալգորիթմը՝ համեմատած այլ առաջատար մեթոդների հետ, տալիս էփուլի վերականգնման ճշտության զգալի լավացում: