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The Clique Covering Number for the Strong Product of Generalized Cycles

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Abstract

In this paper the clique covering number for the strong product of generalized cycles is investigated. A method is given to construct a minimal clique cover in case some conditions hold.

1. Preliminaries

A set of vertices of a graph is a clique if every two distinct vertices in it are adjacent and if it's maximal with respect to this property. A collection C of cliques is a clique-cover of graph G if $\bigcup_{Q\in C} Q = V(G)$, where $V(G)$ is the set of vertices of G. The clique-covering number of G, $\sigma(G)$, is the number of cliques in a minimum clique-cover of G. A graph is called k-regular if the degree of each vertex is k.

For real number $c \in R$ we shall use the following notations:
$[c]$ - greatest integer less than or equal to c,
$]c[$ - least integer greater than or equal to c.

Generalized cycles are defined as follows:
Let's denote by C_n^k the $2k$-regular graph with n vertices which can be ordered on a circle so that each vertex is adjacent to the k vertices coming after and before it on the circle $(n > 2; 1 \le k \le [\frac{n-1}{2}])$.

The strong product of G_1 and G_2 is a graph G with vertices $V(G)$ and edges $E(G)$, where $V(G) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in E(G)$ if and only if:

1. $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or
2. $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$, or
3. $(u_1, v_1) \in E(G_1)$ and $(u_2, v_2) \in E(G_2)$.

A non-negative real-valued function f on $V(G)$ is called admissible if for each clique C, $\sum_{v\in C} f(v) \le 1$.

The Rosenfeld number $\rho(G)$ of a graph G is defined as [1, 2]:
$\rho(G) = \max_f \sum_{v\in V(G)} f(v)$, running over all f admissible functions.

One can deduce that

$$\rho(C_{2n+1}) = n + \frac{1}{2},$$

$$\sigma(C_{2n+1}) =]\rho(C_{2n+1})[= n+1,$$
$$\rho(C_n^k) = \frac{n}{k+1},$$
$$\sigma(C_n^k) =]\rho(C_n^k)[,$$

where C_n is a cycle of length n. The following inequalities are known for each of graphs G and H [1, 2]

$$\sigma(G \times H) \leq \sigma(G) \times \sigma(H),$$
$$\sigma(G \times H) \geq \rho(G) \times \sigma(H).$$

Hales [2] obtained the following result for the clique-covering number of strong product of two odd cycles ($2 \leq k \leq n$):

$$\sigma(C_{2n+1} \times C_{2k+1}) =]\rho(C_{2n+1}) \times \sigma(C_{2k+1})[.$$

2. The Strong Product of Generalized Cycles

The above result due to Hales is generalized here. We'll use the following notations
$r_{mp} = m mod(p+1)$,
$r_{nk} = n mod(k+1)$.

Theorem 1: Let C_m^p and C_n^k be generalized cycles. If the following conditions hold:
1)$p+1 \geq 2r_{mp}, r_{mp} \neq 0$,
2)$(\sigma(C_m^p)-1)(k+1-r_{nk}) \leq [\sigma(C_n^k)/2](k+1)$, then

$$\sigma(C_m^p \times C_n^k) =]\sigma(C_m^p) \times \rho(C_n^k)[= \sigma(C_m^p) \times \sigma(C_n^k) - [\sigma(C_m^p)\frac{k+1-r_{nk}}{k+1}]. \quad (1)$$

Proof: Since the right hand is a lower bound for σ it's enough to construct a clique cover to attain that bound. Let's denote the vertices of C_m^p and C_n^k by numbers $0, 1, ..., m-1$ and $0, 1, ..., n-1$ correspondingly. Then for the vertex $(x, y) \in V(C_m^p \times C_n^k)$ let

$$Q(x, y) = \{(x+i, y+j) : i = 0, ..., p; j = 0, ..., k\}$$

be a clique in the product graph $C_m^p \times C_n^k$ ($x+i$ and $y+j$ are taken by modulo m and n respectively). We will also use the notations below
$t = [\sigma(C_n^k)/2]$,
$\sigma_{mp} = \sigma(C_m^p)$,
$\sigma_{nk} = \sigma(C_n^k)$.
Consider the following families of cliques

$$Q_0^0 = \{Q(0, (k+1)i) : i = 0, ..., t-1\},$$
$$Q_0^1 = \{Q(r_{mp}, (k+1)i) : i = t, ..., \sigma_{nk}-1\},$$
$$Q_1^0 = \{Q(p+1, k+1-r_{nk}+(k+1)i) : i = 0, ..., t-1\},$$
$$Q_1^1 = \{Q(p+1+r_{mp}, k+1-r_{nk}+(k+1)i) : i = t, ..., \sigma_{nk}-1\},$$

....

$$Q_{\sigma_{mp}-2}^0 = \{Q((p+1)(\sigma_{mp}-2), (k+1-r_{nk})(\sigma_{mp}-2)+(k+1)i) : i = 0, ..., t-1\},$$

$Q^1_{\sigma_{mp}-2} = \{Q((p+1)(\sigma_{mp}-2)+r_{mp}, (k+1-r_{nk})(\sigma_{mp}-2)+(k+1)i) : i = t, ..., \sigma_{nk}-1\}$,

$Q_{\sigma_{mp}-1} = \{Q((p+1)(\sigma_{mp}-1), (k+1-r_{nk})(\sigma_{mp}-1)+(k+1)i) : i = 0, ..., \sigma_{nk}-[\sigma_{mp}\frac{k+1-r_{nk}}{k+1}]-1\}$.

We will show that the union of the families above is a clique cover for the product graph. Obviously its cardinal number is equal to the right hand of the equality (1). Let $(x,y) \in V(C^p_m \times C^k_n)$, then the following 3 cases are possible

1)$r_{mp} \leq x \leq m - r_{mp} - 1 - (\sigma_{mp}-1)(p+1) - 1$.
Then $x = s(p+1)+c$, $0 \leq s \leq \sigma_{mp}-2$, $0 \leq c \leq p$. If $s = 0$, then $r_{mp} \leq c \leq p$ and clearly $(x,y) \in Q^0_0 \cup Q^1_0$, otherwise $(x,y) \in Q^1_{s-1} \cup Q^0_s \cup Q^1_s$.

2)$m - r_{mp} \leq x \leq m$.
Then $x = s(p+1)+c$, $s = \sigma_{mp}-1$, $0 \leq c \leq r_{mp}-1$. If
$0 \leq y \leq (k+1-r_{nk})(\sigma_{mp}-1)-1$, then $(x,y) \in Q^1_{\sigma_{mp}-2}$, otherwise we have

$$\begin{aligned}(k+1-r_{nk})(\sigma_{mp}-1)+(k+1)(\sigma_{nk}-[\sigma_{mp}\frac{k+1-r_{nk}}{k+1}]) \geq \\ \geq (k+1)\sigma_{nk}+(k+1-r_{nk})(\sigma_{mp}-1)-\sigma_{mp}(k+1-r_{nk}) \geq \\ \geq (k+1)\sigma_{nk}-(k+1-r_{nk}) \geq n,\end{aligned} \quad (2)$$

therefore $(x,y) \in Q_{\sigma_{mp}-1}$.

3)$0 \leq x \leq r_{mp}-1$.
In this case if $0 \leq y \leq t(k+1)-1$, then $(x,y) \in Q^0_0$, otherwise we have the conditions of the theorem and inequality (2). According to the 1st condition of the theorem, $Q_{\sigma_{mp}-1}$ covers a part of vertices with first coordinate up to $r_{mp}-1$(and more if $p+1$ is strictly greater than $2r_{mp}$). According to the 2nd condition of the theorem and inequality (2), there are no uncovered vertices with second coordinate $t(k+1) \leq y \leq n-1 (0 \leq x \leq r_{mp}-1)$, hence $(x,y) \in Q_{\sigma_{mp}-1}$.

Therefore the union of the mentioned families is a clique cover of the product graph with required cardinality.

Corollary 1: Let C^p_m and C^k_n be generalized cycles. If $p+1 = 2r_{mp}$ and $k+1 = 2r_{nk}$, then
$\sigma(C^p_m \times C^k_n) = max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[)$.

Proof: The right hand of the suggested equality is a lower bound for $\sigma(C^p_m \times C^k_n)$. If the 2nd condition of theorem holds then the proof is immediate, otherwise we have
$(\sigma(C^p_m)-1)(k+1-r_{nk}) > [\sigma(C^k_n)/2](k+1)$ and since $k+1 = 2r_{nk}$ we get
$\sigma(C^p_m)/2 > [\sigma(C^k_n)/2] + 1/2 \geq \sigma(C^k_n)/2$,
$[\sigma(C^p_m)/2] \geq \frac{\sigma(C^k_n)-1}{2}$.
The latter is the second condition of theorem and with $k+1 = 2r_{nk}$ equality it implies that
$\sigma(C^p_m \times C^k_n) =]\sigma(C^k_n) \times \rho(C^p_m)[\leq max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[)$, hence
$\sigma(C^p_m \times C^k_n) = max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[)$.

References

[1] M. Rosenfeld," On a problem of C. E. Shannon in graph theory", *Proc. Amer. Math. Soc. 18*, 315-319, 1967.

[2] R. S. Hales, "Numerical invariants and the strong product of graphs", *Combin. Theory (B) 15*, 146-155, 1973.

Ծածկույթի թիվը ընդհանրացված ցիկլերի ուժեղ արտադրյալի համար

Ս. Բաղայան

Ամփոփում

Սույն աշխատանքում ուսումնասիրված է ընդհանրացված ցիկլերի ուժեղ արտադրյալի ծածկույթի թիվը: Որոշ պայմանների առկայության դեպքում տրված է եղանակ մինիմալ ծածկույթը կառուցելու համար: