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The Stable Set Number for the Strong Product of Generalized Cycles

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Abstract

The strong product of an odd cycle and a generalized cycle and the strong product of two generalized cycles are investigated. For both cases a method is given to construct a stable set of vertices in product graph to achieve the known upper bound $\alpha(G\times H) \leq \rho(G)\times\alpha(H)$ in case some conditions hold. For the stable set number of strong product of generalized cycles a lower bound is found.

1. Introduction

The investigation of the stable set number of product graphs has come from the problem in information theory due to Shannon [1, 2]. Ore in [3] raised the following problem:

Given a finite graph G, what are the necessary and sufficient conditions on G in order that $\alpha(G\times H) = \alpha(G)\times\alpha(H)$, for every finite graph H, where $\alpha(G)$ is the stable set number of G.

For the equality above a sufficient condition is found by Shannon [1]. Then Rosenfeld [4] proved its being not necessary and gave a necessary and sufficient condition, thereby introducing an invariant called ρ, the Rosenfeld number. In [5], Hales obtained the non-multiplicative behavior of the stable set number on the strong product of odd cycles. This work is closely related to it.

2. Preliminaries

A set of vertices of a graph is stable if no two vertices in it are adjacent. A stable set containing k vertices is called k-stable set. Let's denote by $\alpha(G)$-the number of vertices in a maximum stable set of G. A graph is called k- regular if the degree of each vertex is k.

For real number $c \in R$ we shall use the following notations:

$[c]$ - greatest integer less than or equal to c,

$]c[$ - least integer greater than or equal to c.

Generalized cycles are defined as follows:

Let's denote by C_n^k the $2k$ - regular graph with n vertices which can be ordered on a circle so that each vertex is adjacent to the k vertices coming after and before it on the circle $(n > 2; 1 \leq k \leq [\frac{n-1}{2}])$.

The strong product of G_1 and G_2 is a graph G with vertices $V(G)$ and edges $E(G)$, where $V(G) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in E(G)$ if and only if:

1. $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or
2. $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$, or
3. $(u_1, v_1) \in E(G_1)$ and $(u_2, v_2) \in E(G_2)$.

A non-negative real-valued function f on $V(G)$ is called admissible if for each clique C, $\sum_{v \in C} f(v) \leq 1$.

The Rosenfeld number $\rho(G)$ of a graph G is defined as[4, 5]:

$\rho(G) = \max_f \sum_{v \in V(G)} f(v)$, running over all f admissible functions.

One can deduce that,

$$\rho(C_{2n+1}) = n + \frac{1}{2},$$

$$\alpha(C_{2n+1}) = \lfloor \rho(C_{2n+1}) \rfloor = n,$$

$$\rho(C_n^k) = \frac{n}{k+1},$$

$$\alpha(C_n^k) = \lfloor \rho(C_n^k) \rfloor,$$

where C_n is a cycle of length n. The following inequalities are known for each of graphs G and H [4, 5],

$$\alpha(G \times H) \geq \alpha(G) \times \alpha(H),$$

$$\alpha(G \times H) \leq \rho(G) \times \alpha(H).$$

Hales [5] obtained the following result for the stable set number of strong product of two odd cycles $(1 \leq k \leq n)$:

$$\alpha(C_{2n+1} \times C_{2k+1}) = \alpha(C_{2n+1}) \times \alpha(C_{2k+1}) + \lfloor \alpha(C_{2k+1})/2 \rfloor = \lfloor \rho(C_{2n+1}) \times \alpha(C_{2k+1}) \rfloor.$$

For the related results on the stable set number of products of cycles refer to [6, 7, 8].

3. The Strong Product of an Odd Cycle and a Generalized Cycle

The above result due to Hales is generalized on the presence of some conditions in the following way

$$\alpha(C_{2n+1} \times C_h^k) = \alpha(C_{2n+1}) \times \alpha(C_h^k) + \lfloor \alpha(C_h^k)/2 \rfloor = \lfloor \rho(C_{2n+1}) \times \alpha(C_h^k) \rfloor. (1)$$

Let's enumerate the vertices of C_{2n+1} with numbers $0, \ldots, 2n$. In order to obtain (1) we should construct $2n+1$ stable sets $S_0, \ldots, S_{2n}$ in C_h^k such that:

1. $S_i \cap S_{(i+1) mod (2n+1)} = \emptyset, i = 0, \ldots, 2n;$
2. $S_i \cup S_{(i+1) mod (2n+1)}$ sets are stable in $C_h^k, i = 0, \ldots, 2n;$
3. $\sum_{i=0}^{2n} |S_i| = \alpha(C_{2n+1}) \times \alpha(C_h^k) + \lfloor \alpha(C_h^k)/2 \rfloor = n \times \alpha(C_h^k) + \lfloor \alpha(C_h^k)/2 \rfloor.$

In that case the stable set below in the product graph will have the required cardinality

$$S = \bigcup_{i=0}^{2n} \{(i, v)/v \in S_i\}.$$

It's obvious that the conditions 1-3 above are also necessary conditions for equality (1). Those conditions imply that $|S_i| \geq [\alpha(C_h^k)/2], i = 0, \ldots, 2n$. Indeed, let's assume that there is a set with a cardinality less than the mentioned number. Applying the 2nd condition we can imply that the remaining sets cardinal numbers sum cannot exceed $n \times \alpha(C_h^k)$, hence it's not possible to get the equality suggested by 3rd condition.

Lemma 1: Consider the graph C_h^k and $hmod(k+1) = 0$. In that case equality (1) doesn't hold.

Proof: Suppose the equality holds. Hence it's possible to mention $(2n+1)$ stable sets in C_h^k graph with 1-3 properties. As stated above $|S_i| \geq [\alpha(C_h^k)/2], i = 0, \ldots, 2n$. Clearly all the stable sets cardinal numbers are either $[\alpha(C_h^k)/2]$ or $]\alpha(C_h^k)/2[$. Since no cardinal numbers of two consecutive stable sets can be $]\alpha(C_h^k)/2[$ at the same time (provided that $\alpha(C_h^k)/2$ is not integer), there exist two consecutive stable sets with cardinality $[\alpha(C_h^k)/2]$. Without loss of generality we may assume $|S_0| = |S_{2n}| = [\alpha(C_h^k)/2]$. $hmod(k+1) = 0$ implies that for each A_1 stable set in C_h^k there will be at most one stable set A_2 such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2$ is a maximum stable set in C_h^k. Since $|S_{2n}| = [\alpha(C_h^k)/2]$, using the 3rd condition we'll obtain that $S_i \cup S_{(i+1)mod(2n+1)} (i = 0, 2, \ldots, 2n-2)$ are maximum stable sets. Therefore $S_1 = S_3 = \ldots = S_{2n-1}$ and $S_0 = S_2 = S_4 = \ldots = S_{2n-2}$. Thus, $S_0 \cup S_{2n} \cup S_{2n-1}$ is a stable set with cardinality $\alpha(C_h^k) + [\alpha(C_h^k)/2]$. This yields a contradiction.

Particularly for even cycles $hmod(k+1) = hmod2 = 0$, that is, the equality doesn't hold.

Theorem 1: Consider the graph C_h^k. If $mn \geq [\alpha(C_h^k)/2](k+1)$, where $m = hmod(k+1)$, then

$$\alpha(C_{2n+1} \times C_h^k) = \alpha(C_{2n+1}) \times \alpha(C_h^k) + [\alpha(C_h^k)/2] = [\rho(C_{2n+1}) \times \alpha(C_h^k)].$$

Proof: We can assume that n is the minimum number for which the inequality holds. Since if the $S_0, S_1, \ldots, S_{2n}$ sets satisfy 1-3 conditions then for $(n+k)$ we shall have $2(n+k)+1$ sets $S_0, S_1, \ldots, S_{2n}, \underbrace{S_{2n-1}, S_{2n}, \ldots, S_{2n-1}, S_{2n}}_{2k}$ which will also satisfy 1-3 conditions if without loss of generality to assume that $(S_{2n-1} \cup S_{2n})$ is a maximum stable set.

Let's notice that the inequality

$$\alpha(C_{2n+1} \times C_h^k) \leq \alpha(C_{2n+1}) \times \alpha(C_h^k) + [\alpha(C_h^k)/2],$$

always holds. To show the equality it's enough to construct a stable set in the product graph with vertices count equal to the right side of the equation.

Let's denote vertices of C_h^k by $0, 1, \ldots, h-1$ numbers and consider the stable set $\{0, (k+1), \ldots, (t-1)(k+1)\}$ with cardinality $t = \alpha(C_h^k)$(Fig.1).

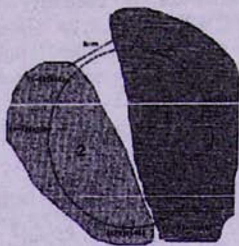

Figure 1: The vertices of the maximum stable set in C_h^k are marked in the picture. The set is devided into two parts.

Now we can construct stable sets in C_h^k satisfying 1-3. Thus we will obtain the required stable set in product graph. Consider the following sets,

$S_0 = \{0, \ldots, ([t/2]-1)(k+1)\}$,
$S_1 = \{[t/2](k+1), \ldots, (t-1)(k+1)\}$,
$S_2 = \{-m, \ldots, ([t/2]-1)(k+1)-m\}$,
$S_3 = \{[t/2](k+1)-m, \ldots, (t-1)(k+1)-m\}$,
...
$S_{2n-2} = \{-(n-1)m, \ldots, ([t/2]-1)(k+1)-(n-1)m\}$,
$S_{2n-1} = \{[t/2](k+1)-(n-1)m, \ldots, (t-1)(k+1)-(n-1)m\}$,
$S_{2n} = \{-nm, \ldots, ([t/2]-1)(k+1)-nm\}$.

We take the numbers above by $modh$. One can verify that these sets satisfy the 1st and 3rd points. Clearly, the 2nd point is also satisfied provided that conditions of the theorem hold.

With the help of these sets we shall construct a stable set with cardinality $[t/2]+nt$ in product graph,

$$S = \bigcup_{i=0}^{2n} \{(i,v)/v \in S_i\}.$$

Hence,

$$\alpha(C_{2n+1} \times C_h^k) \geq \alpha(C_{2n+1}) \times \alpha(C_h^k) + [\alpha(C_h^k)/2].$$

4. The Strong Product of Generalized Cycles

Now let's determine the relation between $\alpha(C_m^p \times C_n^k)$ and $\alpha(C_m^p)$, $\alpha(C_n^k)$ numbers. We'll use the following notations

$$\alpha_{mp} = \alpha(C_m^p), r_{mp} = m mod(p+1),$$
$$\alpha_{nk} = \alpha(C_n^k), r_{nk} = n mod(k+1).$$

It is known

$$\alpha(C_m^p \times C_n^k) \leq [\rho(C_m^p) \times \alpha(C_n^k)] = \alpha_{mp}\alpha_{nk} + [r_{mp}\alpha_{nk}/(p+1)],$$

$$\alpha(C_m^p \times C_n^k) \leq [\alpha(C_m^p) \times \rho(C_n^k)] = \alpha_{mp}\alpha_{nk} + [r_{nk}\alpha_{mp}/(k+1)],$$

but from the other side we have,

$$\alpha(C_m^p \times C_n^k) \geq \alpha(C_m^p) \times \alpha(C_n^k) = \alpha_{mp}\alpha_{nk}.$$

Thus,

$$\alpha_{mp}\alpha_{nk} \leq \alpha(C_m^p \times C_n^k) \leq \alpha_{mp}\alpha_{nk} + \min([r_{mp}\alpha_{nk}/(p+1)], [r_{nk}\alpha_{mp}/(k+1)]).$$

One can notice that Lemma 1 is a simple consequence of the inequality above, since if $r_{mp} = 0$ or $r_{nk} = 0$ then,

$$\alpha(C_m^p \times C_n^k) = \alpha(C_m^p) \times \alpha(C_n^k).$$

The theorem below suggests a stronger lower bound:

Theorem 2: If $\alpha_{mp} r_{nk} \geq [\alpha_{nk}/(p+1)](k+1) r_{mp}$, then

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{nk}/(p+1)] r_{mp}.$$

Proof: To prove the theorem it's enough to construct a stable set in the product graph. We shall construct $t = \alpha_{mp}$, α_{nk}-stable sets $S_0, \ldots, S_{t-1}$ in C_n^k graph, then shall decompose each of them into $p+1$ parts. Afterwards by constructing more r_{mp} stable sets in C_n^k we shall have m stable sets, $P_0, P_1, \ldots, P_{m-1}$. Finally, we shall show that the required stable set in the product graph is the following:

$$S = \bigcup_{i=0}^{m-1} B_i, B_i = \{(i, v)/v \in P_i\}.$$

Let's try to decompose α_{nk} number into $p+1$ almost equal parts. It will be used to decompose S_i sets. Let

$v = \alpha_{nk} mod(p+1)$, then

$\alpha_{nk} = v]\alpha_{nk}/(p+1)[+(p+1-v)[\alpha_{nk}/(p+1)]$.

Let's define also a_i numbers according to the equality above,

$a_i =]\alpha_{nk}/(p+1)[, i = 0, \ldots, v-1;$

$a_i = [\alpha_{nk}/(p+1)], i = v, \ldots, p.$

Clearly in that case, $\alpha_{nk} = \sum_{i=0}^{p} a_i$. Suppose l is the minimum non-negative integer satisfying the inequality,

$$(l+1) r_{nk} \geq [\alpha_{nk}/(p+1)](k+1) r_{mp},$$

according to the supposition of the theorem $l < \alpha_{mp}$. Consider the following α_{nk}-stable sets in C_n^k graph,

$S_0 = \{0, (k+1), 2(k+1), \ldots, (\alpha_{nk}-1)(k+1)\}$,

$S_1 = \{-r_{nk}, (k+1) - r_{nk}, 2(k+1) - r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - r_{nk}\}$,

$S_2 - \{-2r_{nk}, (k+1) \quad 2r_{nk}, 2(k+1) - 2r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - 2r_{nk}\}$,

...

$S_l = \{-lr_{nk}, (k+1) - lr_{nk}, 2(k+1) - lr_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - lr_{nk}\}$,

...

$S_{t-1} = \{-lr_{nk}, (k+1) - lr_{nk}, 2(k+1) - lr_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - lr_{nk}\}$,

$$R = \{-(l+1)r_{nk}, (k+1) - (l+1)r_{nk}, 2(k+1) - (l+1)r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - (l+1)r_{nk}\}.$$

Operations here are considered to be done by $modn$. Consider the elements of sets $S_0, \dots, S_{t-1}$ in the specified order and split each of them into $p+1$ parts (so that i-th set cardinality is a_i). We shall get $P_0, P_1, \dots, P_{(p+1)t-1}$ sets. Now let's consider the elements of R in the specified order and separate the first r_{mp} sets with cardinality $[\alpha_{nk}/(p+1)]$ from them. We shall get $P_0, P_1, \dots, P_{m-1}$ stable sets in C_n^k graph.

To finalize the proof of the theorem it remains to show that the constructed set S is a stable set in the product graph. It suffices to show that each sequential $p+1$ sets in the cyclic sequence of $P_0, P_1, \dots, P_{m-1}$ sets are pair-wise disjoint and the union of the $p+1$ sets is a stable set in C_n^k. Consider any such sequence of sets $P_{imod(n)}, P_{(i+1)mod(n)}, \dots, P_{(i+p)mod(n)}$. If P_0 and P_{m-1} aren't present in the sequence at the same time, then the statement is true according to the construction, otherwise the statement is implied from the definition of number l above.

Corollary 1: For every C_m^p and C_n^k generalized cycles holds,

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + \min([\alpha_{nk}/(p+1)]r_{mp}, [\alpha_{mp}/(k+1)]r_{nk}),$$

particularly,

$$\alpha(C_n^k \times C_n^k) \geq \alpha_{nk}^2 + [\alpha_{nk}/(k+1)]r_{nk}.$$

Proof: Clearly, it suffices to prove only the first inequality. If the condition of Theorem Theorem 2 is satisfied $\alpha_{mp}r_{nk} \geq [\alpha_{nk}/(p+1)](k+1)r_{mp}$, then the proof of corollary is immediate, otherwise we have

$$\alpha_{mp}r_{nk} < [\alpha_{nk}/(p+1)](k+1)r_{mp},$$

hence

$$\alpha_{nk}r_{mp} \geq [\alpha_{mp}/(k+1)](p+1)r_{nk},$$

applying Theorem 2 we get

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{mp}/(k+1)]r_{nk} \geq \alpha_{mp}\alpha_{nk} + \min([\alpha_{nk}/(p+1)]r_{mp}, [\alpha_{mp}/(k+1)]r_{nk})$$

and the corollary is proved.

Corollary 2: If the following conditions hold
1. $\alpha_{mp}r_{nk} \geq [\alpha_{nk}/(p+1)](k+1)r_{mp}$,
2. $\alpha_{nk}mod(p+1) = 0$,

then

$$\alpha(C_m^p \times C_n^k) = [\rho(C_m^p) \times \alpha(C_n^k)] = \alpha_{mp}\alpha_{nk} + [r_{mp}\alpha_{nk}/(p+1)].$$

Proof: Since $\alpha(C_m^p \times C_n^k) \leq \rho(C_m^p) \times \alpha(C_n^k)$ and $\alpha(C_m^p \times C_n^k)$ is a natural number then $\alpha(C_m^p \times C_n^k) \leq [\rho(C_m^p) \times \alpha(C_n^k)]$.

According to the 2nd condition, $\alpha_{nk}/(p+1)$ is an integer number and according to Theorem 2

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{nk}/(p+1)]r_{mp} = \alpha_{mp}\alpha_{nk} + r_{mp}\alpha_{nk}/(p+1) = [\rho(C_m^p) \times \alpha(C_n^k)].$$

Corollary 3: If the following conditions hold
1. $\alpha_{mp}r_{nk} \geq]\alpha_{nk}/(p+1)[(k+1)r_{mp}$,
2. $p+1 = r_{mp}q$, and q is a divider for α_{nk},

then

$$\alpha(C_m^p \times C_n^k) = [\rho(C_m^p) \times \alpha(C_n^k)] = \alpha_{mp}\alpha_{nk} + [r_{mp}\alpha_{nk}/(p+1)].$$

Proof: To prove the corollary it suffices to construct $P_0, \ldots, P_{(p+1)l-1}, \ldots, P_{m-1}$ sets in Theorem 2 so that the last r_{mp} sets cardinal numbers sum to $[r_{mp}\alpha_{nk}/(p+1)]$. As in Theorem 2 all S_h sets will be decomposed in the same way into $p+1$ parts so that i-th part has cardinality $a_i (i = 0, \ldots, p)$. Let's make the following notations and define first r_{mp} a_i numbers (Fig. 2)

$v = \alpha_{nk} mod(p+1)$, as q divides α_{nk} and $p+1$, it also divides v, so we can define:
$a_i =]\alpha_{nk}/(p+1)[, i = 0, \ldots, v/q - 1;$
$a_i = [\alpha_{nk}/(p+1)], i = v/q, \ldots, r_{mp} - 1.$

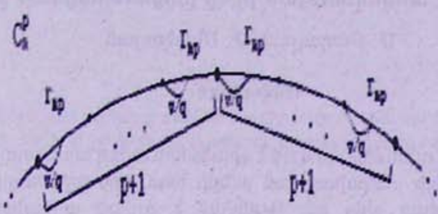

Figure 2: A part from graph C_m^p is shown. For r_{mp} consecutive elements first v/q are marked. Those are elements for which $a_i =]\alpha_{nk}/(p+1)$. [For the other elements $a_i = [\alpha_{nk}/(p+1)]$.

Taking into account that r_{mp} is a divider for $p+1$, the remaining $(p+1) - r_{mp}$ a_i numbers can be defined recursively $a_i = a_{i-r_{mp}}, i = r_{mp}, \ldots, p$.

Having the definitions above for a_i numbers, each of t α_{nk}-stable sets can be decomposed into $p+1$ parts. From these parts v will have cardinality $]\alpha_{nk}/(p+1)[$ and $p+1-v$ of them $[\alpha_{nk}/(p+1)]$ (since q is a divider for v and $p+1-v$ numbers). Finally, let's take the last r_{mp} P_j sets in Theorem 2 with cardinality a_i(instead of $[\alpha_{nk}/(p+1)]$)$i = 0, \ldots, r_{mp} - 1$. In that case we have:

$$\sum_{i=0}^{r_{mp}-1} a_i = \frac{\sum_{i=0}^{p} a_i}{q} = \frac{\alpha_{nk}}{q} = [\frac{p+1}{q} \times \frac{\alpha_{nk}}{p+1}] = [r_{mp}\alpha_{nk}/(p+1)].$$

Consider also number l in Theorem 2 to be the minimum non-negative integer number satisfying $(l+1)r_{nk} \geq \sum_{i=0}^{r_{mp}-1} a_i(k+1)$. This finalizes the proof of the corollary.

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ՈՒժեղ արտադրյալի անկախության թիվը ընդհանրացված ցիկլերի համար

Ս. Բադալյան, Ս. Մարկոսյան

Ամփոփում

Սույն աշխատանքում ուսումնասիրվում է գրաֆների ուժեղ արտադրյալի անկախության թվի կապը արտադրիչների անկախության թվերի հետ, երբ արտադրիչներից մեկը կամ երկուսն էլ ընդհանրացված ցիկլ են: Գտնված է ճշված անկախության թիվը որոշ պայմանների դեպքում և տրված է մեթոդ համապատասխան անկախ բազմությունը կառուցելու համար: Ընդհանրացված ցիկլերի ուժեղ արտադրյալի անկախության թվի համար գտնված է ստորին գնահատական: