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The Minimum Linear Arrangement Problem on Bipartite Γ-oriented Graphs

Levon R. Rafayelyan

Yerevan State University
levonr@synopsys.com

Abstract

In general case the minimum linear arrangement (MINLA) problem is NP-complete. It is NP-complete also for bipartite graphs. In this paper it is proved that a minimum linear arrangement for bipartite Γ-oriented graphs can be found in polynomial time. The formula for the cost of optimal arrangement is given as well.

1. Introduction

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such a way that a certain objective function is optimized. The minimum linear arrangement (MINLA) is a well known problem in this class.

MINLA is defined as follows. Consider a set of n pins and a required number of wire connections between each pair of the pins. The problem is to put the n pins into n holes such that the total wire length is a minimum. The holes are all on a line with adjacent holes at unit distance apart. We can abstract the pins and wire connections as a graph G with n nodes. In general case the problem is NP-complete for both directed and undirected graphs [1]. In [2] the problem is proved to be NP-complete also for undirected bipartite graphs. For directed bipartite graphs it is unknown whether there exists a polynomial solution or not.

Because of its many applications it is important to find subclasses of graphs for which MINLA can be found in polynomial time. Here the problem is solved for bipartite Γ-oriented graphs.

2. Problem Formulation

Given an oriented graph G(V, E). A linear arrangement of G is a bijection $\varphi : V \to \{1, 2, \cdots, |V|\}$. The arrangement φ is called to be feasible if for every $(v, u) \in E$, $\varphi(v) < \varphi(u)$. Clearly, a necessary condition for a feasible arrangement to exist is that G(V, E) contains no directed circuits. The cost of feasible arrangement φ is defined as follows:

$$L(G, \varphi) = \sum_{(v,u)\in E} (\varphi(u) - \varphi(v)).$$

The problem is to find a feasible arrangement with minimum cost. If $(u,v) \in E$, we will say that v is an image of u. For every $u \in V$ we define the sets $\Gamma u = \{v/(u,v) \in E\}$, $\Gamma^{-1}u = \{v/(v,u) \in E\}$. G is called to be Γ-oriented if for every $u, v \in V$ either $\Gamma u \subseteq \Gamma v$ or $\Gamma v \subseteq \Gamma u$. It is well known that graph G is Γ-oriented if and only if the complement of G is an interval graph.

Given a bipartite Γ-oriented graph G(V, U; X) where $V = \{v_1, v_2, \cdots, v_n\}, U = \{u_1, u_2, ..., u_m\}, X \subseteq \{(v_i, u_j)/1 \le i \le n, 1 \le j \le m\}$. Let's consider the MINLA problem for graph G(V, U; X).

3. Solution of the problem

Without loss of generality we can assume that

$$\Gamma v_1 \subseteq \Gamma v_2 \subseteq \cdots \subseteq \Gamma v_n, |\Gamma^{-1}u_1| \ge |\Gamma^{-1}u_2| \ge \cdots \ge |\Gamma^{-1}u_m|.$$

We can suppose that $\Gamma v_n = U$, because otherwise every vertex from the set $U \backslash \Gamma v_n$ will be isolated, but isolated vertices aren't essential for the MINLA problem, and we can remove them from G.

Let's define $S(a) = 1 + 2 + ... + a$, $a \in N$ and $L(v, \varphi) = \sum_{(v,u) \in E} (\varphi(u) - \varphi(v))(v \in V)$. In this case we will have

$$L(G, \varphi) = \sum_{i=1}^{n} L(v_i, \varphi).$$

Let's define two feasible arrangements φ_1 and φ_2 for graph G(V, U; X). Define arrangement φ_1 as follows:

$$\varphi_1(v_i) = i (1 \le i \le n),$$
$$\varphi_1(u_j) = n + j (1 \le j \le m).$$

One can easily check that

$$L(G, \varphi_1) = \sum_{i=1}^{n} [S(|\Gamma v_i|) + (n-i)|\Gamma v_i|].$$

Define arrangement φ_2 as it is shown in Fig. 1

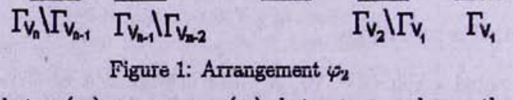

Figure 1: Arrangement φ_2

It can be seen that $\varphi_2(v_n) < \cdots < \varphi_2(v_1)$, between v_i and v_{i-1} the vertices of set $\Gamma v_i \backslash \Gamma v_{i-1}, (2 \le i \le n)$, are placed and after v_1 the vertices of set Γv_1 are placed. It is not difficult to see that

$$L(G, \varphi_2) = \sum_{i=1}^{n} [S(|\Gamma v_i|) + |\Gamma v_{i-1}| + |\Gamma v_{i-2}| + \cdots + |\Gamma v_1|] = \sum_{i=1}^{n} [S(|\Gamma v_i|) + (n-i)|\Gamma v_i|].$$

So

$$L(G,\varphi_1) = L(G,\varphi_2).$$

Theorem 1: For every feasible arrangement φ the following lower bound is true:

$$L(G,\varphi) \geq \sum_{i=1}^{n}[S(|\Gamma v_i|) + (n-i)|\Gamma v_i|].$$

Proof: Suppose that φ is a feasible arrangement and $\varphi(v_{i_1}) < \varphi(v_{i_2}) < \cdots < \varphi(v_{i_n})$ where $v_{i_k} \in V$ $(1 \leq k \leq n)$. Let's try to obtain a lower bound for $L(v_{i_k},\varphi)$ where $v_{i_k} \in V$ $(1 \leq k \leq n)$. For every $v_i, v_j \in V (i \neq j)$ define set $\Gamma(\varphi, v_i, v_j)$ to be the images of v_i placed on right hand side of vertex v_j if $\varphi(v_i) < \varphi(v_j)$, otherwise the images of v_j placed on right hand side of vertex v_i.

It is not difficult to see that

$$L(v_{i_k},\varphi) \geq S(|\Gamma v_{i_k}|) + |\Gamma(\varphi, v_{i_k}, v_{i_{k+1}})| + \cdots + |\Gamma(\varphi, v_{i_k}, v_{i_n})|.$$

Therefore

$$L(G,\varphi) \geq \sum_{k=1}^{n}[S(|\Gamma v_{i_k}|) + |\Gamma(\varphi, v_{i_k}, v_{i_{k+1}})| + \cdots + |\Gamma(\varphi, v_{i_k}, v_{i_n})|].$$

Let's consider the following sum:

$$\sum_{k=1}^{n}[|\Gamma(\varphi, v_{i_k}, v_{i_{k+1}})| + \cdots + |\Gamma(\varphi, v_{i_k}, v_{i_n})|].$$

It can be interpreted as a sum of corresponding $|\Gamma(\varphi, v_i, v_j)|$ values for all the possible unordered pairs $(v_i, v_j), (v_i, v_j \in V, i \neq j)$. Thus

$$\sum_{k=1}^{n}[|\Gamma(\varphi, v_{i_k}, v_{i_{k+1}})| + \cdots + |\Gamma(\varphi, v_{i_k}, v_{i_n})|] = \sum_{i=1}^{n}[|\Gamma(\varphi, v_i, v_{i+1})| + \cdots + |\Gamma(\varphi, v_i, v_n)|].$$

We have assumed that $|\Gamma v_1| \leq |\Gamma v_2| \leq \cdots \leq |\Gamma v_i| \leq |\Gamma v_{i+1}| \leq \cdots \leq |\Gamma v_n|$. Let's consider the vertex v_i. If for some k $(1 \leq k \leq n-i)$ $\varphi(v_i) < \varphi(v_{i+k})$, then all images of the vertex v_i are placed on the right hand side of the vertex v_{i+k}, which means that $|\Gamma(\varphi, v_i, v_{i+k})| = |\Gamma v_i|$. If $\varphi(v_i) > \varphi(v_{i+k})$ then v_{i+k} has at least $|\Gamma v_i|$ images in right hand side of the vertex v_i, which means that $|\Gamma(\varphi, v_i, v_{i+k})| \geq |\Gamma v_i|$. Thus

$$|\Gamma(\varphi, v_i, v_{i+k})| \geq |\Gamma v_i| (1 \leq i \leq n, 1 \leq k \leq n-i),$$

$$\sum_{i=1}^{n}[|\Gamma(\varphi, v_i, v_{i+1})| + \cdots + |\Gamma(\varphi, v_i, v_n)|] \geq \sum_{i=1}^{n}(n-i)|\Gamma v_i|,$$

$$L(G,\varphi) \geq \sum_{i=1}^{n}[S(|\Gamma v_i|) + (n-i)|\Gamma v_i|].$$

Theorem 2: A minimum linear arrangement for bipartite Γ-oriented graph G(V, U; X) can be found in polynomial time, and the cost of every optimal solution φ can be represented by the following formula:

$$L(G,\varphi) = \sum_{i=1}^{n}[S(|\Gamma v_i|) + (n-i)|\Gamma v_i|].$$

Proof: It is immediate from theorem 1 that arrangements φ_1 and φ_2 are optimal solutions for MINLA problem. Clearly, φ_1 and φ_2 can be constructed in polynomial time.

References

[1] M. R. Garey, D. S. Johnson, *Computers and Intractability*. A guide to the theory of NP-completeness. Freeman and Company, 1979.

[2] S. Even, Y. Shiloah, NP-completeness of several arrangement problems, Report No. 43, Dept. of Computer Science, Technion, Haifa, Israel, 1975.

Մինիմալ գծային համարակալում խնդրի լուծումը երկկողմանի Γ-կողմնորոշված գրաֆների համար

Լ. Ռաֆայելյան

Ամփոփում

Գրաֆի մինիմալ գծային համարակալում խնդիրը ընդհանուր դեպքում NP- լրիվ է: Այն NP- լրիվ է նաև երկկողմանի գրաֆների համար: Այստեղ ապացուցված է, որ երկկողմանի Γ -կողմնորոշված գրաֆների համար մինիմալ գծային համարակալում կարելի է գտնել բազմանդամային ժամանակում: Տրված է նաև օպտիմալ լուծման արժեքների հաշվարկող բանաձևը: