Mathematical Problems of Computer Science 31, 122–129, 2008.

Interval Total Colorings of Certain Graphs

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Abstract

An interval total t–coloring of a graph G is a total coloring of G with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident with each vertex v together with v are colored by $(d_G(v) + 1)$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G. It is proved that complete graphs, complete bipartite graphs and n–dimensional cubes have interval total colorings and bounds are found for the possible number of colors in such colorings.

1 Introduction

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The concept of total coloring was introduced by V. Vizing [16] and independently by M. Behzad [4]. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of G. One of the most important long-standing open problems in this area is the conjecture of Vizing-Behzad stating that $\chi''(G) \leq \Delta(G) + 2$ for every graph G [4,16], where $\Delta(G)$ is the maximum degree of a vertex of G. This conjecture became known as Total Coloring Conjecture [9]. The Total Coloring Conjecture is proved for some classes of graphs, e.g., for complete graphs, for bipartite graphs, for complete multipartite graphs [18], for planar graphs G with $\Delta(G) \neq 6$ [6,9,14], for graphs with sufficiently small maximum degree. M. Rosenfeld [13] and N. Vijayaditya [15] independently proved that the total chromatic number of graphs G with $\Delta(G) = 3$ is at most 5. A. Kostochka in [10,11] proved that the total chromatic number of graphs with $\Delta(G) = 4$ (respectively $\Delta(G) = 5$) is at most 6 (respectively 7). Exact values for the total chromatic number are determined, e.g., for paths, cycles, complete and complete bipartite graphs [5], complete multipartite graphs of odd order [8], planar graphs G with $\Delta(G) \geq 11$ [7] and outerplanar graphs [19].

The key concept discussed in this article is the following. Given a graph G, we say that G is interval total colorable if there is $t \geq 1$ for which G has a total coloring with colors $1, 2, \ldots, t$ such that at least one vertex or edge of G is colored by i, $i = 1, 2, \ldots, t$, and the edges incident with each vertex v together with v are colored by $(d_G(v) + 1)$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G.

The concept of interval total colorings is a new one in graph coloring, synthesizing interval colorings [1-3] and total colorings. The introduced concept is valuable as it extends to total colorings of graphs one of the most important notions of classical mathematics - the one of continuity.

2 Definitions

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of a vertex of G - by $\Delta(G)$, the chromatic number of $G-$ by $\chi(G)$ and the chromatic index of $G-$ by $\chi'(G)$. For an edge coloring α of the graph G and for any $v \in V(G)$ we denote by $S(v,\alpha)$ the set of colors of colored edges incident with v. If β is a total coloring of a graph G then $\beta(v)$ and $\beta(e)$ denote the color of a vertex $v \in V(G)$ and the color of an edge $e \in E(G)$ in the coloring β. For a total coloring β of a graph G and for any $v \in V(G)$ define the set $S[v,\beta]$ as follows:

$$S[v,\beta] \equiv \{\beta(v)\} \cup \{\beta(e) \mid e \text{ is incident with } v\}$$

Let Z_+ denote the set of nonnegative integers, and $\lfloor a \rfloor$ means the greatest integer $\leq a$. For two integers $a \leq b$ the set $\{a, a+1, \ldots, b\}$ is denoted by $[a,b]$ and called an interval. For interval $[a,b]$ and a $p \in Z_+$, the notation $[a,b] \oplus p$ means: $[a+p, b+p]$.

An interval total $t-$coloring of a graph G is a total coloring of G with colors $1,2,\ldots,t$ such that at least one vertex or edge of G is colored by i, $i = 1,2,\ldots,t$, and the edges incident with each vertex v together with v are colored by $(d_G(v)+1)$ consecutive colors.

For $t \geq 1$ let $\mathcal{T}_t$ denote the set of graphs which have an interval total $t-$coloring, and assume: $\mathcal{T} \equiv \bigcup_{t\geq 1} \mathcal{T}_t$. For a graph $G \in \mathcal{T}$ the least and the greatest values of t, for which $G \in \mathcal{T}_t$, are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively.

In this paper it is proved that complete graphs, complete bipartite graphs and $n-$dimensional cubes have interval total colorings and bounds are found for the possible number of colors in such colorings.

The terms and concepts that we do not define can be found in [1,17,18].

3. Main results

Theorem 1. For any $n \in N$

(1) $K_n \in \mathcal{T}$,

(2) $w_\tau(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \frac{3}{2}n, & \text{if } n \text{ is even,} \end{cases}$

(3) $W_\tau(K_n) = 2n-1$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, $E(K_n) = \{(v_i,v_j) \mid 1 \leq i < j \leq n\}$. First of all let us show that for any $n \in N$ $K_n \in \mathcal{T}_{2n-1}$.

Define a total coloring α of the graph K_n in the following way:

1. for $i = 1,2,\ldots,n$ $\quad \alpha(v_i) = 2i-1$;
2. for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$, where $i \neq j$, $\quad \alpha((v_i,v_j)) = i+j-1$.

It is easy to see that α is an interval total $(2n-1)-$ coloring of the graph K_n. This proves that for any $n \in N$ $K_n \in \mathcal{T}$ and $W_\tau(K_n) \geq 2n-1$. On the other hand, it is not difficult to check that for any $n \in N$ $W_\tau(K_n) \leq 2n-1$. Hence (1) and (3) hold. Let us prove (2).

Case 1: n is odd. Since K_n is a regular graph with $\chi''(K_n) = n$ then $w_\tau(K_n) = \chi''(K_n) = n$.

Case 2: n is even. Now we show that $w_\tau(K_n) \leq \frac{3}{2}n$.

Define a total coloring β of the graph K_n in the following way:

$$\beta(v_1) = 1; \quad \beta(v_2) = n+1; \quad \beta(v_{n-1}) = \frac{n}{2}; \quad \beta(v_n) = \frac{3}{2}n;$$

for $i=3,\dots,\frac{n}{2}$ set $\beta(v_i)=i-1;\ \beta\left(v_{\frac{n}{2}+i-2}\right)=i+n-1;$
for $i=1,\dots,\left\lfloor\frac{n}{4}\right\rfloor,\ j=2,\dots,\frac{n}{2},\ i<j,\ i+j\leq\frac{n}{2}+1$ set $\beta((v_i,v_j))=i+j-1;$
for $i=2,\dots,\frac{n}{2}-1,\ j=\left\lfloor\frac{n}{4}\right\rfloor+2,\dots,\frac{n}{2},\ i<j,\ i+j\geq\frac{n}{2}+2$ set $\beta((v_i,v_j))=i+j+\frac{n}{2}-2;$
for $i=3,\dots,\frac{n}{2},\ j=\frac{n}{2}+1,\dots,n-2,\ j-i\leq\frac{n}{2}-2$ set $\beta((v_i,v_j))=\frac{n}{2}+1+j-i;$
for $i=1,\dots,\frac{n}{2},\ j=\frac{n}{2}+1,\dots,n,\ j-i\geq\frac{n}{2}$ set $\beta((v_i,v_j))=j-i+1;$
for $i=2,\dots,1+\left\lfloor\frac{n-2}{4}\right\rfloor,\ j=\frac{n}{2}+1,\dots,\frac{n}{2}+\left\lfloor\frac{n-2}{4}\right\rfloor,\ j-i=\frac{n}{2}-1$ set $\beta((v_i,v_j))=2i-1;$
for $i=\left\lfloor\frac{n-2}{4}\right\rfloor+2,\dots,\frac{n}{2},\ j=\frac{n}{2}+1+\left\lfloor\frac{n-2}{4}\right\rfloor,\dots,n-1,\ j-i=\frac{n}{2}-1$ set $\beta((v_i,v_j))=i+j-1;$
for $i=\frac{n}{2}+1,\dots,\frac{n}{2}+\left\lfloor\frac{n}{4}\right\rfloor-1,\ j=\frac{n}{2}+2,\dots,n-2,\ i<j,\ i+j\leq\frac{3}{2}n-1$ set $\beta((v_i,v_j))=i+j-n+1;$
for $i=\frac{n}{2}+1,\dots,n-1,\ j=\frac{n}{2}+\left\lfloor\frac{n}{4}\right\rfloor+1,\dots,n,\ i<j,\ i+j\geq\frac{3}{2}n$ set $\beta((v_i,v_j))=i+j-\frac{n}{2}.$

It is not difficult to check that:

$S[v_1,\beta]=S(v_1,\beta)\cup\{\beta(v_1)\}=[2,n]\cup\{1\}=[1,n];$
$S[v_2,\beta]=S(v_2,\beta)\cup\{\beta(v_2)\}=[2,n]\cup\{n+1\}=[2,n+1];$
$S\left[v_{\frac{n}{2}+i-2},\beta\right]=S\left(v_{\frac{n}{2}+i-2},\beta\right)\cup\left\{\beta\left(v_{\frac{n}{2}+i-2}\right)\right\}=[i,i+n-2]\cup\{i+n-1\}=[i,i+n-1],$
$S[v_i,\beta]=S(v_i,\beta)\cup\{\beta(v_i)\}=[i,i+n-2]\cup\{i-1\}=[i-1,i+n-2],\ i=3,\dots,\frac{n}{2};$
$S[v_{n-1},\beta]=S(v_{n-1},\beta)\cup\{\beta(v_{n-1})\}=\left[\frac{n}{2}+1,\frac{3}{2}n-1\right]\cup\left\{\frac{n}{2}\right\}=\left[\frac{n}{2},\frac{3}{2}n-1\right];$
$S[v_n,\beta]=S(v_n,\beta)\cup\{\beta(v_n)\}=\left[\frac{n}{2}+1,\frac{3}{2}n-1\right]\cup\left\{\frac{3}{2}n\right\}=\left[\frac{n}{2}+1,\frac{3}{2}n\right].$

This shows that β is an interval total $\frac{3}{2}n$–coloring of the graph K_n and, therefore, $w_\tau(K_n)\leq\frac{3}{2}n$. Let us prove that $w_\tau(K_n)\geq\frac{3}{2}n$. Suppose, to the contrary, that γ is an interval total $w_\tau(K_n)$–coloring of the graph K_n, where $n\leq w_\tau(K_n)\leq\frac{3}{2}n-1$. Since $w_\tau(K_n)\geq\chi''(K_n)$ then $w_\tau(K_n)\geq n+1$ and, therefore, $n+1\leq w_\tau(K_n)\leq\frac{3}{2}n-1$. Consider the vertices $v_1,v_2,\dots,v_n$. It is clear that for $i=1,2,\dots,n$ $1\leq\min S[v_i,\gamma]\leq w_\tau(K_n)-n+1$. Hence $\{w_\tau(K_n)-n+1,\dots,n\}\subseteq S[v_i,\gamma]$, $i=1,2,\dots,n$. Let us show that none of the vertices $v_1,v_2,\dots,v_n$ is colored by j, $j=w_\tau(K_n)-n+1,\dots,n$. Suppose that $\gamma(v_{i_0})=j_0$, $j_0\in\{w_\tau(K_n)-n+1,\dots,n\}$. It is clear that for $i=1,2,\dots,n$, $i\neq i_0$, $\gamma(v_i)\neq j_0$. This implies that any vertex v_i, except v_{i_0}, is incident with an edge of color j_0, which is a contradiction. The contradiction shows that for $i=1,2,\dots,n$ $\gamma(v_i)\notin\{w_\tau(K_n)-n+1,\dots,n\}$. Hence $\gamma(v_i)\in\{1,\dots,w_\tau(K_n)-n\}\cup\{n+1,\dots,w_\tau(K_n)\}$, $i=1,2,\dots,n$. On the other hand since $\chi(K_n)=n$ then $|\{1,\dots,w_\tau(K_n)-n\}|+|\{n+1,\dots,w_\tau(K_n)\}|\geq n$. From that we obtain $w_\tau(K_n)\geq\frac{3}{2}n$, which is a contradiction. The proof is complete.

Theorem 2. (1) For any $n\in N$ if $2n-1\leq t\leq 4n-3$ then $K_{2n-1}\in\mathcal{T}_t$.

(2) Let $n=p2^q$, where p is odd, and $q\in Z_+$. Then if $4n-p\leq t\leq 4n-1$ then $K_{2n}\in\mathcal{T}_t$.

Proof. First of all let us prove (1). For that we transform an interval total $(4n-3)$–coloring α of the graph K_{2n-1}, constructing in theorem 1, to interval total t–coloring β of the same graph.

For every $v\in V(K_{2n-1})$ set:

$$\beta(v)=\begin{cases}\alpha(v), & \text{if } 1\leq\alpha(v)\leq t,\\ \alpha(v)-2n+1, & \text{if } t+1\leq\alpha(v)\leq 4n-3.\end{cases}$$

For every $e\in E(K_{2n-1})$ set:

$$\beta(e)=\begin{cases}\alpha(e), & \text{if } 1\leq\alpha(e)\leq t,\\ \alpha(e)-2n+1, & \text{if } t+1\leq\alpha(e)\leq 4n-3.\end{cases}$$

It is not difficult to see that β is an interval total t−coloring of the graph K_{2n-1}. Now we prove (2). We use induction on q. Let $q = 0$. Let us show that if $3p \leq t \leq 4p - 1$ then $K_{2p} \in \mathcal{T}_t$. Consider a graph K_{2p} with $V(K_{2p}) = \{v_1, v_2, ..., v_{2p}\}$ and $E(K_{2p}) = \{(v_i, v_j) \mid v_i, v_j \in V(K_{2p}),\ i < j\}$.

Let G be a subgraph of the graph K_{2p}, induced by its vertices $v_1, v_2, ..., v_p$. Clearly G is isomorphic to the graph K_p. Since p is odd then from statement (1) of the theorem follows that there exists an interval total t'−coloring γ of G, where $p \leq t' \leq 2p - 1$.

Now we define a total coloring φ of K_{2p}.

For $i = 1, \ldots, p$ set: $\varphi(v_i) = \gamma(v_i)$, and for $j = p + 1, \ldots, 2p$ set: $\varphi(v_j) = \gamma(v_{j-p}) + 2p$.

For $i = 1, 2, \ldots, 2p$ and $j = 1, 2, \ldots, 2p$, where $i \neq j$, set:

$$\varphi((v_i, v_j)) = \begin{cases} \gamma((v_i, v_j)), & \text{if } 1 \leq i \leq p,\ 1 \leq j \leq p; \\ \gamma(v_i) + p, & \text{if } 1 \leq i \leq p,\ p+1 \leq j \leq 2p,\ i = j - p; \\ \gamma((v_i, v_{j-p})) + p, & \text{if } 1 \leq i \leq p,\ p+1 \leq j \leq 2p,\ i \neq j - p; \\ \gamma((v_{i-p}, v_{j-p})) + 2p, & \text{if } p+1 \leq i \leq 2p,\ p+1 \leq j \leq 2p. \end{cases}$$

It is not difficult to see that φ is an interval total $(t' + 2p)$−coloring of K_{2p}. This implies that for any t, $3p \leq t \leq 4p - 1$ $K_{2p} \in \mathcal{T}_t$. Now suppose that $q \geq 1$ and the theorem is true for all $0 \leq q' \leq q - 1$. We prove that theorem is true for case q. For that we show that if $p2^{q+2} - p \leq t \leq p2^{q+2} - 1$ then $K_{p2^{q+1}} \in \mathcal{T}_t$.

Let G' be a subgraph of the graph $K_{p2^{q+1}}$, induced by its vertices $v_1, v_2, ..., v_{p2^q}$. Clearly G' is isomorphic to the graph K_{p2^q} and, therefore, there exists an interval total t''−coloring λ of G', where $p2^{q+1} - p \leq t'' \leq p2^{q+1} - 1$ (by induction hypothesis).

Define a total coloring μ of $K_{p2^{q+1}}$.

For $i = 1, \ldots, p2^q$ set: $\mu(v_i) = \lambda(v_i)$, and for $j = p2^q + 1, \ldots, p2^{q+1}$ set: $\mu(v_j) = \lambda(v_{j-p2^q}) + p2^{q+1}$.

For $i = 1, 2, \ldots, p2^{q+1}$ and $j = 1, 2, \ldots, p2^{q+1}$, where $i \neq j$, set:

$$\mu((v_i, v_j)) = \begin{cases} \lambda((v_i, v_j)), & \text{if } 1 \leq i \leq p2^q,\ 1 \leq j \leq p2^q; \\ \lambda(v_i) + p2^q, & \text{if } 1 \leq i \leq p2^q,\ p2^q + 1 \leq j \leq p2^{q+1},\ i = j - p2^q; \\ \lambda((v_i, v_{j-p2^q})) + p2^q, & \text{if } 1 \leq i \leq p2^q,\ p2^q + 1 \leq j \leq p2^{q+1},\ i \neq j - p2^q; \\ \lambda((v_{i-p2^q}, v_{j-p2^q})) + p2^{q+1}, & \text{if } p2^q + 1 \leq i \leq p2^{q+1},\ p2^q + 1 \leq j \leq p2^{q+1}. \end{cases}$$

It is not difficult to see that μ is an interval total $(t'' + p2^{q+1})$−coloring of $K_{p2^{q+1}}$. This implies that for any t, $p2^{q+2} - p \leq t \leq p2^{q+2} - 1$ $K_{p2^{q+1}} \in \mathcal{T}_t$. The proof is complete.

Lemma 1. For any $m, n \in N$ $K_{m,n} \in \mathcal{T}_{m+n+1}$.

Proof. Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$, $E(K_{m,n}) = \{(u_i, v_j) \mid 1 \leq i \leq m, 1 \leq j$

Let us define a total coloring α of the graph $K_{m,n}$ in the following way:

1. for $i = 1, 2, \ldots, m$ $\alpha(u_i) = i$, and for $j = 1, 2, \ldots, n$ $\alpha(v_j) = m + 1 + j$;
2. for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ $\alpha((u_i, v_j)) = i + j$.

It is easy to see that α is an interval total $(m + n + 1)$ −coloring of the graph $K_{m,n}$. The proof is complete.

Theorem 3. For any $m, n \in N$

(1) $K_{m,n} \in \mathcal{T}$,

(2) $w_\tau(K_{m,n}) \leq m + n + 2 - \gcd(m, n)$ ($\gcd(m, n)$ is the greatest common divisor of m and n),

(3) $W_\tau(K_{m,n}) \geq m + n + 1$,

(4) if $m+n+2-\gcd(m,n) \le t \le m+n+1$ then $K_{m,n} \in \mathcal{T}_t$.

Proof. (1) and (3) follow from lemma 1. Next we show that if $t = m + n + 2 - \gcd(m,n) + k$, where $0 \le k \le \gcd(m,n) - 1$, then $K_{m,n} \in \mathcal{T}_t$. Let $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$, $E(K_{m,n}) = \{(u_i, v_j) \mid 1 \le i \le m, 1 \le j \le n\}$ and $\sigma = \gcd(m,n)$.

Let H be a subgraph of the graph $K_{m,n}$ induced by its vertices $u_1, u_2, \dots, u_\sigma$ and $v_1, v_2, \dots, v_\sigma$. Clearly, H is isomorphic to the graph $K_{\sigma,\sigma}$ and, consequently, $\chi'(H) = \Delta(H) = \sigma$. Define an edge coloring α of the graph H in the following way: $\alpha((u_i, v_j)) = i + j$, where $i, j = 1, 2, \dots, \sigma$.

For the graph H define an edge coloring $\beta : E(H) \longrightarrow \{2, 3, \dots, \sigma + k + 1\}$ as follows:

$$\beta((u_i, v_j)) = \begin{cases} \alpha((u_i, v_j)), & \text{if } 2 \le \alpha((u_i, v_j)) \le \sigma + k + 1, \\ \alpha((u_i, v_j)) - \sigma, & \text{if } \sigma + k + 2 \le \alpha((u_i, v_j)) \le 2\sigma. \end{cases}$$

It is not difficult to see that β is an interval edge coloring [2] of the graph H with colors $2, 3, \dots, \sigma + k + 1$.

Now we construct an interval total t-coloring of the graph $K_{m,n}$.

For $i \in N$ define a function $f_1(i)$ in the following way:

$$f_1(i) = \begin{cases} \sigma, & \text{if } \sigma \text{ divides } i, \\ i - \sigma \left\lfloor \frac{i}{\sigma} \right\rfloor, & \text{otherwise.} \end{cases}$$

For $j \in N$ define a function $f_2(j)$ in the following way:

$$f_2(j) = \begin{cases} \left\lfloor \frac{j}{\sigma} \right\rfloor - 1, & \text{if } \sigma \text{ divides } j, \\ \left\lfloor \frac{j}{\sigma} \right\rfloor, & \text{otherwise.} \end{cases}$$

For $i \in N$ and $j \in N$ define a function $f_3(i,j)$ in the following way:

$$f_3(i,j) = \begin{cases} \left\lfloor \frac{i}{\sigma} \right\rfloor + \left\lfloor \frac{j}{\sigma} \right\rfloor, & \text{if } \sigma \text{ does not divide } i \text{ or } j, \\ \left\lfloor \frac{i}{\sigma} \right\rfloor + \left\lfloor \frac{j}{\sigma} \right\rfloor - 1, & \text{if } (\sigma \text{ divides } i \text{ and does not divide } j) \text{ or} \\ & (\sigma \text{ divides } j \text{ and does not divide } i), \\ \left\lfloor \frac{i}{\sigma} \right\rfloor + \left\lfloor \frac{j}{\sigma} \right\rfloor - 2, & \text{if } \sigma \text{ divides } i \text{ and } j. \end{cases}$$

Define a total coloring γ of the graph $K_{m,n}$.

For $i = 1, 2, \dots, m$ set: $\gamma(u_i) = \min S\left(u_{f_1(i)}, \beta\right) + \sigma f_2(i) - 1$.

For $j = 1, 2, \dots, n$ set: $\gamma(v_j) = m + \min S\left(v_{f_1(j)}, \beta\right) + \sigma f_2(j)$.

For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ set: $\gamma((u_i, v_j)) = \beta\left(\left(u_{f_1(i)}, v_{f_1(j)}\right)\right) + \sigma f_3(i,j)$.

It is not difficult to check that for $i = 1, 2, \dots, m$

$$S[u_i, \gamma] = S(u_i, \gamma) \cup \{\gamma(u_i)\} = \left[\left(\bigcup_{k=1}^{\frac{n}{\sigma}} S\left(u_{f_1(i)}, \beta\right) \oplus \sigma(k-1)\right) \oplus \sigma f_2(i)\right] \cup \{\gamma(u_i)\} =$$

$$= \left[\min S\left(u_{f_1(i)}, \beta\right) + \sigma f_2(i), n + \min S\left(u_{f_1(i)}, \beta\right) + \sigma f_2(i) - 1\right] \cup$$

$$\cup \left\{\min S\left(u_{f_1(i)}, \beta\right) + \sigma f_2(i) - 1\right\} =$$

$$= \left[\min S\left(u_{f_1(i)},\beta\right)+\sigma f_2(i)-1, n+\min S\left(u_{f_1(i)},\beta\right)+\sigma f_2(i)-1\right]$$

and for $j=1,2,\ldots,n$

$$S\left[v_j,\gamma\right]=S\left(v_j,\gamma\right)\cup\{\gamma(v_j)\}=\left[\left(\bigcup_{k=1}^{m} S\left(v_{f_1(j)},\beta\right)\oplus\sigma(k-1)\right)\oplus\sigma f_2(j)\right]\cup\{\gamma(v_j)\}=$$

$$=\left[\min S\left(v_{f_1(j)},\beta\right)+\sigma f_2(j), m+\min S\left(v_{f_1(j)},\beta\right)+\sigma f_2(j)-1\right]\cup$$

$$\cup\left\{m+\min S\left(v_{f_1(j)},\beta\right)+\sigma f_2(j)\right\}=$$

$$=\left[\min S\left(v_{f_1(j)},\beta\right)+\sigma f_2(j), m+\min S\left(v_{f_1(j)},\beta\right)+\sigma f_2(j)\right].$$

Therefore, γ is an interval total $t-$coloring of the graph $K_{m,n}$. The proof is complete.

Corollary. For any $n \in N$ $w_\tau(K_{n,n})=n+2$.

Lemma 2. For any $n\geq 3$ if $n+1\leq t\leq 2n$ then $Q_n\in\mathcal{T}_t$.

Proof. First of all we show that Q_n has an interval total $(n+1)-$ coloring. Since $\chi''(Q_n)=n+1$, $n\geq 3$, then $Q_n\in\mathcal{T}$ and $w_\tau(Q_n)=n+1$, $n\geq 3$. Now suppose that $n+2\leq t\leq 2n$. Since Q_n is a bipartite graph then $V(Q_n)=V_1(Q_n)\cup V_2(Q_n)$, $V_1(Q_n)\cap V_2(Q_n)=\emptyset$, where $V_1(Q_n), V_2(Q_n)$ are the parts of the graph Q_n. From the result of [12] it follows that if $n\leq l\leq 2n-2$ then Q_n has an interval edge $l-$coloring.

Now we construct an interval total $t-$coloring of the graph Q_n. Let α be an interval edge $(t-2)-$coloring of the graph Q_n. For a graph Q_n define an edge coloring β in the following way: for every $e\in E(Q_n)$ $\beta(e)=\alpha(e)+1$. It is not difficult to see that β is an interval edge coloring [2] of the graph Q_n with colors $2,3,\ldots,t-1$.

Let us define a total coloring γ of the graph Q_n in the following way:

1. for every $v\in V(Q_n)$ set

$$\gamma(v)=\begin{cases}\min S(v,\beta)-1, & \text{if } v\in V_1(Q_n),\\ \max S(v,\beta)+1, & \text{if } v\in V_2(Q_n);\end{cases}$$

2. for every $e\in E(Q_n)$ set $\gamma(e)=\beta(e)$.

It is easy to see that γ is an interval total $t-$coloring of the graph Q_n, $n\geq 3$. The proof is complete.

Theorem 4. For any $n\in N$

(1) $Q_n\in\mathcal{T}$,

(2) $w_\tau(Q_n)=\begin{cases} n+2, & \text{if } n\leq 2,\\ n+1, & \text{if } n\geq 3,\end{cases}$

(3) $W_\tau(Q_n)\geq\frac{(n+1)(n+2)}{2}$,

(4) if $w_\tau(Q_n)\leq t\leq\frac{(n+1)(n+2)}{2}$ then $Q_n\in\mathcal{T}_t$.

Proof. Clearly, (1) and (2) are true for the case $n\leq 2$ and for $n\geq 3$ statements follow from lemma 2. Note that (3) follows from (4). Let us prove (4). Clearly, (4) is true in case $n\leq 2$. Assume that $n\geq 3$. Let us show that if $n+1\leq t\leq\frac{(n+1)(n+2)}{2}$ then $Q_n\in\mathcal{T}_t$.

We use induction on n. Now suppose that $n\geq 4$ and the statement is true for all $3\leq n'\leq n-1$. We prove that (4) is true for case n. Without loss of generality we may assume that $2n+1\leq t\leq\frac{(n+1)(n+2)}{2}$ (by lemma 2). Since $Q_n=K_2\times Q_{n-1}$, therefore there are two subgraphs $Q_{n-1}^{(1)}$ and $Q_{n-1}^{(2)}$ of Q_n, which satisfy conditions: $V(Q_{n-1}^{(1)})\cap V(Q_{n-1}^{(2)})=\emptyset$.

$Q_{n-1}^{(i)}$ is isomorphic to Q_{n-1}, $i = 1, 2$. It is clear that for $i = 1, 2$ $Q_{n-1}^{(i)} \in T$. Clearly, $Q_{n-1}^{(1)}$ is isomorphic to $Q_{n-1}^{(2)}$, therefore there exists a bijection $f : V(Q_{n-1}^{(1)}) \longrightarrow V(Q_{n-1}^{(2)})$ such that $(x, y) \in E(Q_{n-1}^{(1)})$ iff $(f(x), f(y)) \in E(Q_{n-1}^{(2)})$. Let α be an interval total $(t-n-1)$ −coloring of the graph $Q_{n-1}^{(1)}$ (by induction hypothesis).

For a graph $Q_{n-1}^{(2)}$ define a total coloring β in the following way:

(1) for every $u \in V(Q_{n-1}^{(2)})$ $\quad \beta(u) = \alpha(f^{-1}(u)) + n + 1$;

(2) for every $(u, v) \in E(Q_{n-1}^{(2)})$ $\quad \beta((u, v)) = \alpha((f^{-1}(u), f^{-1}(v))) + n + 1$.

Now we define a total coloring γ of the graph Q_n.

For every $x \in V(Q_n)$ set:

$$\gamma(x) = \begin{cases} \alpha(x), & \text{if } x \in V(Q_{n-1}^{(1)}), \\ \beta(x), & \text{if } x \in V(Q_{n-1}^{(2)}). \end{cases}$$

For every $(x, y) \in E(Q_n)$ set:

$$\gamma((x, y)) = \begin{cases} \alpha((x, y)), & \text{if } x, y \in V(Q_{n-1}^{(1)}); \\ \min S[x, \alpha] + n, & \text{if } x \in V(Q_{n-1}^{(1)}), y \in V(Q_{n-1}^{(2)}),\ y = f(x); \\ \beta((x, y)), & \text{if } x, y \in V(Q_{n-1}^{(2)}). \end{cases}$$

It is not difficult to see that γ is an interval total t−coloring of Q_n. This implies that if $n + 1 \le t \le \frac{(n+1)(n+2)}{2}$ then $Q_n \in T_t$. The proof is complete.

Acknowledgement

The author would like to thank Rafayel R. Kamalian for his attention to this work.

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Որոշ գրաֆների միջակայքային լիակատար ներկումներ

Պ. Պետրոսյան

Ամփոփում

G գրաֆի լիակատար ներկումը $1,2,...t$ գույներով կանվանենք միջակայքային լիակատար t-ներկում, եթե ամեն մի i գույնով. $i = 1,2,...t$ ներկված է առնվազն մեկ գագաթ կամ կող և յուրաքանչյուր գագաթին կից կողերը և գագաթը ներկված են $(d_G(v)+1)$ հաջորդական գույներով, որտեղ $d_G(v)$- ով նշանակված է գագաթի աստիճանը G գրաֆում: Ապացուցված է, որ լրիվ գրաֆները, լրիվ երկկողմանի գրաֆները և չափանի խորանարդը ունեն միջակայքային լիակատար ներկումներ և գտնված են գնահատականներ այդ ներկումների մեջ մասնակցող գույների հնարավոր թվի համար: