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Necessary and Sufficient Condition for Existence of Locally-balanced 2-partition of a Tree under the Extended Definition of a Neighbourhood of a Vertex

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Abstract

A necessary and sufficient condition is obtained for the problem of partitioning of the set of vertices of a tree G into two disjoint sets V_1 and V_2 such that it satisfies the condition $||\lambda(v) \cap V_1| - |\lambda(v) \cap V_2|| \leq 1$ for any vertex v of G, where $\lambda(v)$ is the set of all vertices of G the distance of which from v does not exceed 1.

We consider finite, undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. If $v \in V(G)$ then $ex_G(v)$ denotes the eccentricity of a vertex v in a graph G. For a graph G let $\Delta(G)$ be the greatest degree of a vertex of G. Let $\rho_G(x, y)$ denote the distance between the vertices $x \in V(G)$ and $y \in V(G)$ in a graph G. For $\nu \in V(G)$ let us denote $\lambda(\nu) \equiv \{\nu\} \cup \{\omega \in V(G)/(\omega, \nu) \in E(G)\}$. A function $f : V(G) \to \{0, 1\}$ is called 2-partition of a graph G. 2-partition f of a graph G is called locally-balanced iff for $\forall \nu \in V(G)$

$$||\{\omega \in \lambda(\nu)/f(\omega) = 1\}| - |\{\omega \in \lambda(\nu)/f(\omega) = 0\}|| \leq 1.$$

Non-defined concepts can be found in [1, 2, 3, 4, 5].

Let $x \in V(G)$ be an arbitrary vertex of a tree G.

We define the subset $N_i(x)$ of the set $V(G)$, where $0 \leq i \leq ex_G(x)$, as follows:

$$N_i(x) \equiv \{z \in V(G)/\rho_G(x, z) = i\}.$$

Obviously, for any $u \in N_i(x)$, where $1 \leq i \leq ex_G(x)$, there exists a single vertex $u^{(-1)} \in N_{i-1}(x)$ satisfying the condition $(u, u^{(-1)}) \in E(G)$.

Let us assume we have some partition of the set $V(G)\backslash\{x\}$ into sets $A(x)$, $B(x)$, $C(x)$, which satisfies following conditions:

$$V(G)\backslash\{x\} = A(x) \cup B(x) \cup C(x),$$
$$A(x) \cap B(x) = \emptyset, \quad B(x) \cap C(x) = \emptyset, \quad A(x) \cap C(x) = \emptyset.$$

For $\forall u \in V(G)\backslash N_{ex_G(x)}(x)$ define:

$$a(u) \equiv |\{v \in N_{\rho_G(x,u)+1}(x)/(u,v) \in E(G), v \in A(x)\}|,$$
$$b(u) \equiv |\{v \in N_{\rho_G(x,u)+1}(x)/(u,v) \in E(G), v \in B(x)\}|,$$
$$c(u) \equiv |\{v \in N_{\rho_G(x,u)+1}(x)/(u,v) \in E(G), v \in C(x)\}|.$$

Note 1. *From the definitions of functions a, b and c it follows that if for $\forall i$, $1 \le i \le ex_G(x)$ for all $u \in N_i(x)$ it is already determined whether $u \in A(x)$, $u \in B(x)$ or $u \in C(x)$, then for an arbitrary $u \in N_{i-1}(x)$ the values $a(u)$, $b(u)$ and $c(u)$ are unambiguously calculated.*

Note 2. $a(x) + b(x) + c(x) = d_G(x)$; *for $\forall u \in V(G)\backslash(N_{ex_G(x)}(x) \cup \{x\})$ the equality $a(u) + b(u) + c(u) + 1 = d_G(u)$ holds.*

Let us inductively define sets $A(x)$, $B(x)$ and $C(x)$ as follows:

$$N_{ex_G(x)}(x) \subseteq B(x),\ N_{ex_G(x)}(x) \cap A(x) = \emptyset,\ N_{ex_G(x)}(x) \cap C(x) = \emptyset.$$

Assume that for i, $2 \le i \le ex_G(x)$, the partitioning of $N_i(x)$ is already defined:

$$N_i(x) = (N_i(x) \cap A(x)) \cup (N_i(x) \cap B(x)) \cup (N_i(x) \cap C(x)).$$

It follows from the note 1 that the values of functions a, b and c can be calculated for each $u \in N_{i-1}(x)$.

Define the partitioning of $N_{i-1}(x)$ as follows: for $\forall u \in N_{i-1}(x)$

$$u \in \begin{cases} A(x), & \text{if } 1 \le b(u) - (d_G(u) - b(u)) \le 2, \\ B(x), & \text{if } -1 \le a(u) - (d_G(u) - a(u)) \le 0, \\ C(x), & \text{if inequalities } 1 \le b(u) - (d_G(u) - b(u)) \le 2, \\ & -1 \le a(u) - (d_G(u) - a(u)) \le 0 \text{ are false.} \end{cases}$$

Let us show that under the given definition the following conditions are true.

$$(N_{i-1}(x) \cap A(x)) \cap (N_{i-1}(x) \cap C(x)) = \emptyset,$$
$$(N_{i-1}(x) \cap B(x)) \cap (N_{i-1}(x) \cap C(x)) = \emptyset,$$
$$(N_{i-1}(x) \cap A(x)) \cap (N_{i-1}(x) \cap B(x)) = \emptyset.$$

Obviously, it will be enough to show the correctness of the last condition only. Assume the opposite: $\exists u_0 \in (N_{i-1}(x) \cap A(x) \cap B(x))$. This means that following inequalities take place $1 \le b(u_0) - (d_G(u_0) - b(u_0)) \le 2$, $-1 \le a(u_0) - (d_G(u_0) - a(u_0)) \le 0$.

This implies $1 + d_G(u_0) \le 2 \cdot b(u_0) \le 2 + d_G(u_0)$, $1 + d_G(u_0) \le 2 + 2 \cdot a(u_0) \le 2 + d_G(u_0)$.

Since $2 \cdot b(u_0)$ and $2 + 2 \cdot a(u_0)$ are even integers, and $1 + d_G(u_0)$ and $2 + d_G(u_0)$ are consecutive integers then the following two cases only are possible.

Case 1. $2 \cdot b(u_0) = 2 + 2 \cdot a(u_0) = 1 + d_G(u_0)$

Obviously, $a(u_0) = \frac{d_G(u_0)-1}{2}$, $b(u_0) = \frac{d_G(u_0)+1}{2}$. It follows from the note 2 that $d_G(u_0) \ge 1 + a(u_0) + b(u_0) = 1 + \frac{d_G(u_0)-1}{2} + \frac{d_G(u_0)+1}{2} = 1 + d_G(u_0)$, which is impossible.

Case 2. $2 \cdot b(u_0) = 2 + 2 \cdot a(u_0) = 2 + d_G(u_0)$

Obviously, $a(u_0) = \frac{d_G(u_0)}{2}$, $b(u_0) = 1 + \frac{d_G(u_0)}{2}$. It follows from the note 2 that $d_G(u_0) \ge 1 + a(u_0) + b(u_0) = 1 + \frac{d_G(u_0)}{2} + 1 + \frac{d_G(u_0)}{2} = 2 + d_G(u_0)$, which is impossible.

The obtained contradiction shows that $(N_{i-1}(x) \cap A(x)) \cap (N_{i-1}(x) \cap B(x)) = \emptyset$

It is easy to see that the sets $A(x)$, $B(x)$ and $C(x)$ are unambiguously defined and, moreover, $V(G)\backslash\{x\} = A(x) \cup B(x) \cup C(x)$, $A(x) \cap B(x) = \emptyset$, $B(x) \cap C(x) = \emptyset$, $A(x) \cap C(x) = \emptyset$.

Note that we have also defined the following functions $a : (V(G)\backslash N_{ex_G(x)}(x)) \to Z_+$, $b : (V(G)\backslash N_{ex_G(x)}(x)) \to Z_+$, $c : (V(G)\backslash N_{ex_G(x)}(x)) \to Z_+$.

Further we shall assume, that consideration of any tree G automatically implies the choice of a vertex $x \in V(G)$, the realization of the partitioning of the set $V(G)\backslash\{x\}$ into sets $A(x), B(x), C(x)$ mentioned above and the definition of functions a, b, c on the set $V(G)\backslash N_{ex_G(x)}(x)$.

Lemma 1. *If G is a tree and f – its locally-balanced 2-partition, then for $\forall u \in V(G)\backslash\{x\}$ following properties hold* $u \in A(x) \Rightarrow f(u^{(-1)}) = f(u)$, $u \in B(x) \Rightarrow f(u^{(-1)}) = 1 - f(u)$.

Proof is done by reverse induction on $\rho_G(x, u)$.

First of all let us prove the lemma for vertices of the set $N_{ex_G(x)}(x)$.

Obviously, $A(x) \cap N_{ex_G(x)}(x) = \emptyset$, so there is nothing to prove.

Let $u \in B(x) \cap N_{ex_G(x)}(x)$. Obviously, $d_G(u) = 1$. Since f is a locally-balanced 2-partition of G, then $f(u) = 1 - f(u^{(-1)})$, which is the statement of the lemma.

Assume that the lemma holds for all vertices of the set $N_i(x)$, where $2 \le i \le ex_G(x)$. Let us prove the lemma for vertices of the set $N_{i-1}(x)$.

Let $u \in N_{i-1}(x)$ be an arbitrary vertex.

Case 1. $u \in A(x)$.

From the definition of A it follows that $1 \le b(u) - (d_G(u) - b(u)) \le 2$, so $0 \le b(u) - (d_G(u) - b(u) + 1)$. From this inequality, the inductive assumption and the fact that f is a locally-balanced 2-partition of G we conclude that for $\forall \omega \in \lambda(u)\backslash B(x)$ $f(\omega) = f(u)$ and, particularly, $f(u^{(-1)}) = f(u)$.

Case 2. $u \in B(x)$.

From the definition of B it follows that $-1 \le a(u) - (d_G(u) - a(u)) \le 0$, so $0 \le a(u) + 1 - (d_G(u) - a(u))$. From this inequality, the inductive assumption and the fact that f is a locally-balanced 2-partition of G we conclude that for $\forall \omega \in \lambda(u)\backslash(A(x) \cup \{u\})$ $f(\omega) = 1 - f(u)$ and, particularly, $f(u^{(-1)}) = 1 - f(u)$. Lemma is proved.

Theorem 1. *For a given tree G there exists a locally-balanced 2-partition iff for $\forall u \in V(G)\backslash N_{ex_G(x)}(x)$ following inequalities simultaneously hold:*

$$b(u) - (d_G(u) - b(u)) \le 2, \quad a(u) - (d_G(u) - a(u)) \le 0.$$

Proof. Necessity. Suppose that f is a locally-balanced 2-partition of the tree G.

Let us prove that for any vertex $u \in V(G)\backslash N_{ex_G(x)}(x)$ following inequalities simultaneously hold: $b(u) - (d_G(u) - b(u)) \le 2$, $a(u) - (d_G(u) - a(u)) \le 0$.

Assume the opposite. This means that there exists a vertex $u_0 \in V(G)\backslash N_{ex_G(x)}(x)$ for which at least one of the mentioned inequalities is false.

Let us assume that the inequality $b(u_0) - (d_G(u_0) - b(u_0)) \le 2$ is false. Then the inequality $b(u_0) - (d_G(u_0) - b(u_0) + 1) > 1$ is true. But this inequality, taking into account the statement of the lemma 1, contradicts the fact that f is a locally-balanced 2-partition of the tree G.

Let us assume that the inequality $a(u_0) - (d_G(u_0) - a(u_0)) \le 0$ is false. Then the inequality $a(u_0) + 1 - (d_G(u_0) - a(u_0)) > 1$ is true. But this inequality, taking into account the statement of the lemma 1, contradicts the fact that f is a locally-balanced 2-partition of the tree G.

Sufficiency. Suppose that for $\forall u \in V(G)\setminus N_{exc_G(x)}(x)$ following inequalities simultaneously hold: $b(u) - (d_G(u) - b(u)) \leq 2$, $a(u) - (d_G(u) - a(u)) \leq 0$.

Let us inductively define a function $f : V(G) \to \{0, 1\}$.

Let us set $f(x) \equiv 1$.

Let us assume that for all vertices of the set $N_i(x)$, where $0 \leq i \leq exc_G(x) - 1$, the function f is already defined. Let us define the function f for vertices of the set $N_{i+1}(x)$.

For each vertex $u \in N_i(x)$ let us define the function f for vertices of the set $N_{i+1}(x) \cap \lambda(u)$.

Obviously, without loss of generality it can be supposed that all vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap C(x)$, if it is not empty, are numbered: $h_1(u), h_2(u), \ldots, h_{c(u)}(u)$.

First of all let us define the function f on vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap A(x)$ by the following way: for $\forall z \in N_{i+1}(x) \cap \lambda(u) \cap A(x)$ set $f(z) \equiv f(u)$.

Now let us define the function f on vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap B(x)$ by the following way: for $\forall z \in N_{i+1}(x) \cap \lambda(u) \cap B(x)$ set $f(z) \equiv 1 - f(u)$.

Note 3. *On all vertices of the set $\lambda(u)\setminus(C(x) \cap N_{i+1}(x))$ the function f is already defined.*

Let us denote $\epsilon(u) = |\{\omega \in \lambda(u)\setminus(C(x) \cap N_{i+1}(x))/f(\omega) = f(u)\}|$ and $\sigma(u) = |\{\omega \in \lambda(u)\setminus(C(x) \cap N_{i+1}(x))/f(\omega) = 1 - f(u)\}|$.

It follows from the note 3 that values of $\epsilon(u)$ and $\sigma(u)$ are already defined.

Now let us define the function f on vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap C(x)$ by the following way: for $\forall z \in N_{i+1}(x) \cap \lambda(u) \cap C(x)$ set:

$$f(z) \equiv \begin{cases} f(u), & \text{if } z = h_j(u), \text{ where } 1 \leq j \leq \sigma(u) - \epsilon(u). \\ 1 - f(u), & \text{if } z = h_j(u), \text{ where } 1 \leq j \leq \epsilon(u) - \sigma(u), \\ f(u), & \text{if } z = h_j(u), \text{ where } |\epsilon(u) - \sigma(u)| < j \leq c(u) \text{ and} \\ & j - |\epsilon(u) - \sigma(u)| \text{ is an odd number,} \\ 1 - f(u), & \text{if } z = h_j(u), \text{ where } |\epsilon(u) - \sigma(u)| < j \leq c(u) \text{ and} \\ & j - |\epsilon(u) - \sigma(u)| \text{ is an even number.} \end{cases}$$

So we have defined the function f on all vertices of the set $N_{i+1}(x)$.

Therefore, the function f is defined on whole $V(G)$.

Let us check that the function f defined above is a locally-balanced 2-partition of the tree G, indeed.

Let $u \in V(G)$ be an arbitrary vertex.

Case 1. $u = x$.

Since $b(x) - (d_G(x) - b(x)) \leq 2$ and $a(x) - (d_G(x) - a(x)) \leq 0$ then, taking into account the note 2, we obtain: $b(x) - (a(x) + 1) \leq 1 + c(x)$ and $(a(x) + 1) - b(x) \leq 1 + c(x)$.

Case 1a). $c(x) \leq b(x) - (a(x) + 1)$.

Obviously $|\{\omega \in \lambda(x)/f(\omega) = f(x)\}| = a(x) + 1 + c(x)$, $|\{\omega \in \lambda(x)/f(\omega) = 1 - f(x)\}| = b(x)$.

Let us show that $|b(x) - (a(x) + 1 + c(x))| \leq 1$.

It is clear that in this case that $|b(x) - (a(x) + 1 + c(x))| = b(x) - (a(x) + 1 + c(x)) \leq 1$.

Case 1b). $c(x) \leq (a(x) + 1) - b(x)$.

Obviously $|\{\omega \in \lambda(x)/f(\omega) = f(x)\}| = a(x) + 1$, $|\{\omega \in \lambda(x)/f(\omega) = 1 - f(x)\}| = b(x) + c(x)$.

Let us show that $|(a(x) + 1) - (b(x) + c(x))| \leq 1$.

It is clear that in this case that $|(a(x)+1)-(b(x)+c(x))| = (a(x)+1)-(b(x)+c(x)) \leq 1$.

Case 1c). $c(x) > |b(x) - (a(x) + 1)|$.

It is clear that in this case the equality

$$|\{\omega \in (\lambda(x)\backslash\{h_{|b(x)-(a(x)+1)|+1}(x),\dots,h_{c(x)}(x)\})/f(\omega)=f(x)\}| =$$
$$|\{\omega \in (\lambda(x)\backslash\{h_{|b(x)-(a(x)+1)|+1}(x),\dots,h_{c(x)}(x)\})/f(\omega)=1-f(x)\}|.$$

implies the inequality $||\{\omega \in \lambda(x)/f(\omega)=f(x)\}| - |\{\omega \in \lambda(x)/f(\omega)=1-f(x)\}|| \leq 1$.

Case 2. $u \neq x$.

Case 2a). $u \in A(x)$.

In this case $1 \leq b(u)-(d_G(u)-b(u)) \leq 2$.

From the note 2 it follows that following inequalities are true $0 \leq b(u)-a(u)-c(u)-2 \leq 1$, $c(u) \leq b(u)-(a(u)+2)$.

Obviously $|\{\omega \in \lambda(u)/f(\omega)=f(u)\}| = a(u)+2+c(u)$, $|\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}| = b(u)$.

Let us show that $|b(u)-(a(u)+c(u)+2)| \leq 1$.

It is clear that in this case that $|b(u)-(a(u)+c(u)+2)| = b(u)-(a(u)+c(u)+2) \leq 1$.

Case 2b). $u \in B(x)$.

In this case $-1 \leq a(u)-(d_G(u)-a(u)) \leq 0$.

From the note 2 it follows that following inequalities are true $0 \leq a(u)-b(u)-c(u) \leq 1$, $c(u) \leq a(u)-b(u)$.

Obviously $|\{\omega \in \lambda(u)/f(\omega)=f(u)\}| = a(u)+1$, $|\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}| = b(u)+1+c(u)$.

Let us show that $|a(u)+1-(b(u)+c(u)+1)| \leq 1$.

It is clear that in this case that $|a(u)+1-(b(u)+c(u)+1)| = |a(u)-b(u)-c(u)| = a(u)-b(u)-c(u) \leq 1$.

Case 2c). $u \in C(x)$.

In this case following inequalities are false $1 \leq b(u)-(d_G(u)-b(u)) \leq 2$, $-1 \leq a(u)-(d_G(u)-a(u)) \leq 0$.

Consequently, using the condition of the theorem, we obtain $b(u)-(d_G(u)-b(u)) < 1$, $a(u)-(d_G(u)-a(u)) < -1$.

Now, taking into account the note 2, we conclude that $b(u)-a(u)-2 < c(u)$, $a(u)-b(u) < c(u)$, which imply $|b(u)-(a(u)+2)| \leq c(u)+1$ and $|a(u)-b(u)| \leq c(u)+1$.

Case 2c)1. $f(u^{(-1)}) = f(u)$.

Case 2c)1a). $|b(u)-(a(u)+2)| \geq c(u)$.

It is clear that in this case the inequality $|b(u)-(a(u)+2)| \leq c(u)+1$ implies the inequality $||\{\omega \in \lambda(u)/f(\omega)=f(u)\}| - |\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}|| \leq 1$.

Case 2c)1b). $|b(u)-(a(u)+2)| < c(u)$.

It is clear that in this case the equality

$$|\{\omega \in (\lambda(u)\backslash\{h_{|b(u)-(a(u)+2)|+1}(u),\dots,h_{c(u)}(u)\})/f(\omega)=f(u)\}| =$$
$$|\{\omega \in (\lambda(u)\backslash\{h_{|b(u)-(a(u)+2)|+1}(u),\dots,h_{c(u)}(u)\})/f(\omega)=1-f(u)\}|.$$

implies the inequality $||\{\omega \in \lambda(u)/f(\omega)=f(u)\}| - |\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}|| \leq 1$.

Case 2c)2. $f(u^{(-1)}) = 1-f(u)$.

Case 2c)2a). $|a(u)-b(u)| \geq c(u)$.

It is clear that in this case the inequality $|a(u)-b(u)| \leq c(u)+1$ implies the inequality $||\{\omega \in \lambda(u)/f(\omega)=f(u)\}| - |\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}|| \leq 1$.

Case 2c)2b). $|a(u)-b(u)| < c(u)$.

It is clear that in this case the equality

$$|\{\omega \in (\lambda(u)\setminus\{h_{|a(u)-b(u)|+1}(u),\ldots,h_{c(u)}(u)\})/f(\omega)=f(u)\}| =$$
$$|\{\omega \in (\lambda(u)\setminus\{h_{|a(u)-b(u)|+1}(u),\ldots,h_{c(u)}(u)\})/f(\omega)=1-f(u)\}|.$$

implies the inequality $||\{\omega \in \lambda(u)/f(\omega)=f(u)\}| - |\{\omega \in \lambda(u)/f(\omega)=1-f(u)\}|| \leq 1$. Theorem is proved.

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Ծառում լոկալ-հավասարակշռված 2-տրոհման գոյության համար անհրաժեշտ և բավարար պայման գագաթի շրջակայքի ընդլայնված սահմանման դեպքում

Ս. Բալիկյան, Ռ. Քամալյան

Ամփոփում

Ստացված է անհրաժեշտ և բավարար պայման ծառի գագաթների բազմության V_1 և V_2 չհատվող ենթաբազմությունների այնպիսի տրոհման գոյությունը պարզելու համար, որ ծառի յուրաքանչյուր v գագաթի համար տեղի ունենա $||\lambda(v) \cap V_1| - |\lambda(v) \cap V_2|| \leq 1$ անհավասարությունը, որտեղ (v)-ով նշանակված է այն գագաթների բազմությունը, որոնց հեռավորությունը v-ից չի գերազանցում 1-ին: