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On Interval-Separable Subsets of Vertices of a Complete Graph

Hakob Z. Arakelyan[1] and Rafayel R. Kamalian[2]
[1]Department of Informatics and Applied Mathematics, YSU,
[2]Institute for Informatics and Automation Problems of NAS of RA,
e-mail: arak_hakob@yahoo.com, rrkamalian@yahoo.com

Abstract

A subset R of the set of vertices of a graph G is called interval-separable iff there exists a proper edge coloring of G in which colors of edges incident with any vertex x of G form an interval of integers iff $x \in R$. All interval-separable subsets of the set of vertices of the complete graph are found.

1. Preliminaries

We consider undirected graphs without loops and multiple edges [1]. Let V(G) and E(G) denote the sets of vertices and edges of a graph G, respectively. If $x \in V(G)$ then let $d_G(x)$ denote the degree of a vertex x in a graph G. For a graph G let $\Delta(G)$ be the greatest degree of a vertex of G, $\chi'(G)$ be the chromatic index of G [2]. Let $\beta(G)$ denote the cardinality of the greatest matching of a graph G.

The set of positive integers is denoted by N. If D is a finite non-empty subset of N then let $l(D)$ and $L(D)$ denote the least and the greatest element of D, respectively. A non-empty finite subset D of N is referred as interval if $l(D) \le t \le L(D)$, $t \in N$ implies that $t \in D$. An interval D is called h – interval if $|D| = h$. An interval D is called (q,h) – interval and is denoted by $Int(q,h)$ if $l(D) = q, |D| = h$. A function $\varphi : E(G) \to Int(1,t)$ is referred as a proper edge t – coloring of a graph G if

1) for each $i \in Int(1,t)$, there is $e \in E(G)$ such that $\varphi(e) = i$,
2) for any adjacent edges $e' \in E(G), e'' \in E(G)$ $\varphi(e') \ne \varphi(e'')$.

If φ is a proper edge t – coloring of a graph G, where $\chi'(G) \le t \le |E(G)|$ and $x \in V(G)$ then let $S(x,\varphi) = \{i \in N / \exists e_0 \in E(G)$ such that x is adjacent to e_0 and $\varphi(e_0) = i\}$. A proper edge t – coloring φ of a graph G is called interval edge t – coloring of G [3], if for $\forall x \in V(G)$ $S(x,\varphi)$ is a $d_G(x)$ – interval. Let $\mathfrak{N}_t$ denote the set of graphs for which there is an interval t – coloring and assume

$$\mathfrak{N} \equiv \bigcup_{t \ge 1} \mathfrak{N}_t$$

For $G \in \mathfrak{N}$ let $w(G)$ and $W(G)$ be the least and the greatest possible value of t, respectively, for which $G \in \mathfrak{N}_t$.

Proposition 1. [4] If G is a regular graph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.

Corollary 1. $K_p \in \mathfrak{N}$ if and only if p is even.

Theorem 1. [4] If G is a regular graph with $\chi'(G) = \Delta(G)$, then for $\forall t (\Delta(G) \le t \le W(G))$ $G \in \mathfrak{N}_t$.

Theorem 2. [5.6] For $\forall n \in N$ $W(K_{2n}) \ge 3n-2$.

A proper edge t – coloring φ of a graph G is called interval on $R \subseteq V(G)$ edge t – coloring of a graph G [4], if for $\forall x \in R$ $S(x,\varphi)$ is a $d_G(x)$ – interval. For G and $R \subseteq V(G)$ let $w_R(G)$ and $W_R(G)$

be the least and the greatest possible value of t, respectively, for which there is an interval on R edge t-coloring of a graph G.

2. Definitions and Terms

Definition 1. A proper edge t-coloring φ of a graph G has a $(R,\overline{R})$-feature, where $\chi'(G)\le t\le|E(G)|$ and $R\subseteq V(G)$, if for each $x\in V(G)$ $S(x,\varphi)$ is an interval if and only if $x\in R$.

Definition 2. A subset $R\subseteq V(G)$ is called interval-separable subset of the set of vertices of G (or, shorter, "interval-separable subset of G"), if there is t_0, $\chi'(G)\le t_0\le|E(G)|$, for which there is a proper edge t_0-coloring of a graph G, which has the $(R,\overline{R})$-feature.

Definition 3. For a graph G and an interval-separable subset $R\subseteq V(G)$ of its vertices let $w_{R,\overline{R}}(G)$ and $W_{R,\overline{R}}(G)$ be the least and the greatest possible value of t, respectively, for which there is a proper edge t-coloring of a graph G with $(R,\overline{R})$-feature.

Note 1. For each graph G and the interval-separable subset $R\subseteq V(G)$ of G the following inequality holds: $\chi'(G)\le w_R(G)\le w_{R,\overline{R}}(G)\le W_{R,\overline{R}}(G)\le W_R(G)\le|E(G)|$

For each graph $G\in\mathfrak{N}$ and the interval-separable subset $R\subseteq V(G)$ of G the following inequality holds: $\chi'(G)\le w(G)\le w_R(G)\le w_{R,\overline{R}}(G)\le W_{R,\overline{R}}(G)\le W_R(G)\le W(G)\le|E(G)|$

Note 2. If $G\in\mathfrak{N}$ we can't state that its arbitrary subset of vertices is interval-separable. It can be shown by the following

Example : Let G is a tree with $|V(G)|\ge 3$ and R is the subset of all vertices $x\in V(G)$ which satisfies the condition $d_G(x)\ge 2$. It is clear that even $G\in\mathfrak{N}$ [7] , but R is not an interval-separable subset of G.

The goal of this work is finding all interval-separable subsets of the complete graphs K_p, where $p\ge 2$. All non-defined terms can be found in [1,4]. Everywhere in this work we assume that $V(K_p)=\{x_1,x_2,\dots,x_p\}$. It is clear that $|V(K_p)|=p$, $|E(K_p)|=\frac{p(p-1)}{2}$, $\Delta(K_p)=p-1$, $\beta(K_p)=\left\lfloor\frac{p}{2}\right\rfloor$ and $\chi'(K_p)=\begin{cases}p-1, \text{if } p \text{ is even}\\ p \quad, \text{if } p \text{ is odd}\end{cases}$.

3. Some intermediate results

Proposition 2. If $R=V(K_{2n})$ where $n\ge 1$ then for each t, which satisfies the inequality $2n-1\le t\le 3n-2$, there is an interval on R t-coloring of a graph K_{2n}.

Proof. Follows from the corollary 1 , theorems 1 and 2.

Corollary 2. For each $R\subseteq V(K_{2n})$ where $n\ge 1$ $w_R(K_{2n})=2n-1$.

Corollary 3. For each $R\subseteq V(K_{2n})$ where $n\ge 1$ $W_R(K_{2n})\ge 3n-2$.

Proposition 3. If $R=V(K_{2n+1})$ where $n\ge 1$ then for each t, which satisfies the inequality $2n+1\le t\le n(2n+1)$, there is no interval on R t-coloring of a graph K_{2n+1}. The proof is trivial.

Proposition 4. For each $n\ge 2$ and for each $R\subset V(K_n)$ there is an interval on R $(2n-3)$-coloring of a graph K_n.

Proof. As the proof is trivial for the case $n=2$, let us suppose that $n\geq 3$. As $R\subset V(K_n)$ then without loss of generality, we may assume that $R\subseteq\{x_2,x_3,...,x_n\}$. Let define the function $\varphi:E(K_n)\to Int(1,2n-3)$. For $\forall e\in E(K_n)$ let

$$\varphi(e)=\begin{cases}2j-3\ , if\ e=(x_1,x_j)\ for\ which\ j\in Int(2,n-1)\\ i+j-3, if\ e=(x_i,x_j)\ for\ which\ i\in Int(2,n-2), j\in Int(3,n-2), i<j\end{cases}$$

It is easy to see that φ is a proper edge $(2n-3)$-coloring of the graph K_n, and for $\forall i\in Int(2,n-1)$ $S(x_i,\varphi)=Int(i-1,n-1)$. Thus φ is an interval on R $(2n-3)$-coloring of a graph K_n. Proposition 4 is proved.

Corollary 4. For each $n\geq 2$ and each $R\subset V(K_n)$ $w_R(K_n)\leq 2n-3$.

Corollary 5. For each $n\geq 2$ and each $R\subset V(K_n)$ $W_R(K_n)\geq 2n-3$.

Lemma 1. For each $n\geq 1$ and each proper edge $(2n+1)$-coloring α of a graph K_{2n+1} the following statement holds: for each $i_0\in Int(1,2n+1)$ there is only $j(i_0)\in Int(1,2n+1)$ such that $i_0\notin S(x_{j(i_0)},\alpha)$.

Proof. Existence follows from the fact that for each $n\geq 1$ there is no perfect matching for K_{2n+1}. Let's prove uniqueness. Let us assume the opposite: there are $j'(i_0)\in Int(1,2n+1)$, $j''(i_0)\in Int(1,2n+1)$, $j'(i_0)\neq j''(i_0)$ which satisfy the following conditions: $i_0\notin S(x_{j'(i_0)},\alpha)$, $i_0\notin S(x_{j''(i_0)},\alpha)$. As $|E(K_{2n+1})|=n(2n+1)$ and $\beta(K_{2n+1})=n$ then for $\forall i\in Int(1,2n+1)$ $|\{e\in E(K_{2n+1})/\alpha(e)=i\}|=n$. But it's clear that $|\{e\in E(K_{2n+1})/\alpha(e)=i_0\}|\leq\left\lfloor\frac{2n-1}{2}\right\rfloor=n-1$. This contradiction shows incorrectness of the assumption. Lemma 1 is proved.

Lemma 2. For each $n\geq 1$ and each proper edge $(2n+1)$-coloring α of a graph K_{2n+1} there are i' and i'', which satisfy the following conditions:

1) $1\leq i'\leq 2n+1, 1\leq i''\leq 2n+1, i'\neq i''$,
2) $S(x_{i'},\alpha)=Int(2,2n)$, $S(x_{i''},\alpha)=Int(1,2n)$,
3) for $\forall i\in Int(1,2n+1)\setminus\{i',i''\}$ $S(x_i,\alpha)$ is not interval.

Proof. From lemma 1 it follows that there are the only $j(1)\in Int(1,2n+1)$ such that $1\notin S(x_{j(1)},\alpha)$ and the only $j(2n+1)\in Int(1,2n+1)$ such that $(2n+1)\notin S(x_{j(2n+1)},\alpha)$. As α is a proper edge $(2n+1)$-coloring of a graph K_{2n+1} and $d_{K_{2n+1}}(x_{j(1)})=d_{K_{2n+1}}(x_{j(2n+1)})=2n$, then $S(x_{j(1)},\alpha)=Int(2,2n)$ and $S(x_{j(2n+1)},\alpha)=Int(1,2n)$. For completing of reasoning it's enough to assume that $i'\equiv j(1)$, $i''\equiv j(2n+1)$. Lemma 2 is proved.

Proposition 5. For each $n\geq 1$ and $R\subset V(K_{2n+1})$ which satisfies the inequality $|R|\leq 2$ $w_R(K_{2n+1})=2n+1$. The proof follows from the lemma 2.

Lemma 3. For each $n\geq 4$ there is a proper edge $|E(K_n)|$-coloring α of a graph K_n, where for $\forall i\in Int(1,n)$ $S(x_i,\alpha)$ is not interval.

Proof. Case 1. $n=4$. It is clear that in this case edge coloring α of a graph K_n, which is defined by the following equalities: $\alpha((x_1,x_2))=1$, $\alpha((x_1,x_3))=2$, $\alpha((x_1,x_4))=4$, $\alpha((x_2,x_3))=5$, $\alpha((x_2,x_4))=3$, $\alpha((x_3,x_4))=6$ implies the statement.

Case 2. $n \geq 5$. Let us assume

$E_3 \equiv \{(x_1, x_{n-1}), (x_2, x_n)\}$, $E_4 \equiv E(K_n) \setminus (E_1 \cup E_2 \cup E_3)$. It's clear that $|E_4| = \frac{n(n-1)}{2} - 2n + 1$.

Let us number all edges in E_4 by the random order: $e[1], e[2], \ldots, e\left[\frac{n(n-1)}{2} - 2n + 1\right]$.

Then let us number all edges in $E(K_n)$ by the following order :

for $i \in Int(1, n-1)$ *assume* $e_i \equiv (x_i, x_{i+1})$

for $i \in Int(n, n-2)$ *assume* $e_i \equiv (x_{i-n+1}, x_{i-n+3})$

$e_{2n-2} \equiv (x_1, x_{n-1})$

$e_{2n-1} \equiv (x_2, x_n)$

for $i \in Int(2n, \frac{n(n-1)}{2} - 2n + 1)$ *assume* $e_i \equiv e[i - 2n + 1]$

Now let define the function $\alpha : E(K_n) \to Int(1, |E(K_n)|)$. For $\forall i \in Int(1, |E(K_n)|)$ assume $\alpha(e_i) \equiv i$. It is easy to see that in the examined case α is a proper edge $|E(K_n)|-$coloring of a graph K_n, which implies the statement. Lemma 3 is proved.

Proposition 6. For a graph K_n, where $2 \leq n \leq 3$, a subset $R \subseteq V(K_n)$ is interval-separable if and only if $|R| = 2$. The proof is trivial.

Lemma 4. In the case $n \geq 4$ and $R \subset V(K_n)$, which satisfies the condition $0 \leq |R| \leq 1$, there is a proper edge $|E(K_n)|-$coloring of a graph K_n with $(R, \overline{R})-$feature.

Proof. Case 1. $R = \varnothing$. The proof follows from the lemma 3.

Case 2. $|R| = 1$. Without loss of generality, we may assume that $R = \{x_n\}$.

Case 2.1. $n = 4$. It is clear that the proper edge 6-coloring φ_1 of the graph K_4, which is defined by the following equalities: $\varphi_1((x_1, x_2)) = 4$, $\varphi_1((x_1, x_3)) = 5$, $\varphi_1((x_1, x_4)) = 1$, $\varphi_1((x_2, x_3)) = 6$, $\varphi_1((x_2, x_4)) = 2$, $\varphi_1((x_3, x_4)) = 3$ has the $(R, \overline{R})-$feature.

Case 2.2. $n \geq 5$. Let G' is a subgraph of a graph K_n, which is induced by the subset $\{x_1, x_2, \ldots, x_{n-1}\}$ of the vertices of K_n. It's clear that $G' \cong K_{n-1}$. From the lemma 3 it follows that there is a proper edge $|E(K_{n-1})|-$coloring φ_0 of the graph G', where for $\forall x \in V(G')$ $S(x, \varphi_0)$ is not interval. Let us define the function $\varphi_2 : E(K_n) \to Int(1, |E(K_n)|)$. For $\forall e \in E(K_n)$ assume

$$\varphi_2(e) \equiv \begin{cases} i & \text{, if } e = (x_i, x_n), \text{ where } i \in Int(1, n-1) \\ n - 1 + \varphi_0(e), & \text{if } e \text{ is not adjacent to } x_n \end{cases}$$

It is clear that φ_2 is a proper edge $|E(K_n)|-$coloring of a graph K_n with $(R, \overline{R})-$feature. Lemma 4 is proved.

Corollary 6. In the case $n \geq 4$ and $R \subset V(K_n)$, $0 \leq |R| \leq 1$ follows $W_{R,\overline{R}}(K_n) = |E(K_n)|$.

Lemma 5. In the case $n \geq 4$ and $R \subset V(K_n)$, $2 \leq |R| \leq n-2$, there is a proper edge $(2n - 3 + |E(K_{n-|R|})|)-$coloring of a graph K_n with $(R, \overline{R})-$feature.

Proof. Without loss of generality, we may assume that $R=\{x_2,\dots,x_{|R|},x_n\}$. Let us assume that $E_1=\{(x_1,x_i)/i\in Int(2,|R|-1)\cup\{n\}\}$, $E_2=\{(x_i,x_j)/i\in Int(2,|R|-1),\ j\in Int(3,n-2),\ i<j\}$, $E_3=\{(x_n,x_j)/j\in Int(|R|+1,n-|R|-1)\}$, $E_4=E(K_n)\setminus(E_1\cup E_2\cup E_3)$.

It's clear that

$|E_1|=|R|$, $|E_2|=\frac{(|R|-1)(|R|-2)}{2}+(|R|-1)(n-|R|)$, $|E_3|=n-|R|-1$, $|E_4|=\frac{(n-|R|)(n-|R|-1)}{2}$. It's easy to see that E_4 coincides with a set of all edges of the subgraph G' of a graph K_n, which is induced by the subset $\{x_1,x_{|R|+1},\dots,x_{n-1}\}$ of the vertices of K_n. It is clear that

$G'\cong K_{n-|R|}$, $|V(G')|=n-|R|$, $|E(G')|=|E_4|$, $\Delta(G')=n-|R|-1$.

Case 1. $n-|R|=2$. Let us define the function $\varphi_1:E(K_n)\to Int(1,2n-2)$. For $\forall e\in E(K_n)$ assume

$$\varphi_1(e)=\begin{cases}2i-3 & \text{, if } e\in E_1 \text{ and } e=(x_1,x_i)\\ i+j-3, & \text{if } e\in E_2 \text{ and } e=(x_i,x_j)\\ 2n-4 & \text{, if } e=(x_n,x_{n-1})\\ 2n-2 & \text{, if } e=(x_1,x_{n-1})\end{cases}$$

It is easy to see that φ_1 is a proper edge $(2n-2)-$coloring of a graph K_n, where for $\forall i\in Int(2,n-3)\cup\{n\}$ $S(x_i,\varphi_1)=Int(i-1,n-1)$ but the sets $S(x_1,\varphi_1)$ and $S(x_{n-1},\varphi_1)$ are not intervals.

Case 2. $n-|R|=3$. In the examined case it's clear that $n\geq 5$. Let us define the function $\varphi_2:E(K_n)\to Int(1,2n)$. For $\forall e\in E(K_n)$ assume

$$\varphi_2(e)=\begin{cases}2i-3 & \text{, if } e\in E_1 \text{ and } e=(x_1,x_i)\\ i+j-3, & \text{if } e\in E_2 \text{ and } e=(x_i,x_j)\\ 2n-5 & \text{, if } e=(x_{n-2},x_n)\\ 2n-4 & \text{, if } e=(x_{n-1},x_n)\\ 2n-2 & \text{, if } e=(x_{n-2},x_{n-1})\\ 2n-1 & \text{, if } e=(x_1,x_{n-2})\\ 2n & \text{, if } e=(x_1,x_{n-1})\end{cases}$$

It is easy to see that φ_2 is a proper edge $2n-$coloring of a graph K_n, where for $\forall i\in Int(2,n-4)\cup\{n\}$ $S(x_i,\varphi_2)=Int(i-1,n-1)$ but the sets $S(x_1,\varphi_2)$, $S(x_{n-2},\varphi_2)$ and $S(x_{n-1},\varphi_2)$ are not intervals.

Case 3. $n-|R|\geq 4$. In the examined case it's clear that $\Delta(G')\geq 3$ and $n\geq 6$. From the lemma 3 it follows that there is a proper $|E_4|-$coloring φ_0 of a graph G', where for $\forall x\in V(G')$ the set $S(x,\varphi_0)$ is not interval. Let us define the function $\varphi_3:E(K_n)\to Int\left(1,2n-3+\frac{(n-|R|)(n-|R|-1)}{2}\right)$. For $\forall e\in E(K_n)$ assume

$$\varphi_3(e)=\begin{cases}2i-3 & \text{, if } e\in E_1 \text{ and } e=(x_1,x_i)\\ i+j-3 & \text{, if } e\in E_2 \text{ and } e=(x_i,x_j)\\ n+j-3 & \text{, if } e\in E_3 \text{ and } e=(x_n,x_j)\\ 2n-3+\varphi_0(e), & \text{if } e\subset E_4\end{cases}$$

It's easy to see that φ_3 is a proper edge $\left(2n-3+\frac{(n-|R|)(n-|R|-1)}{2}\right)$ – coloring of a graph K_n, where for $\forall i\in Int(2,|R|-1)\cup\{n\}$ $S(x_i,\varphi_3)=Int(i-1,n-1)$ but for $\forall i\in Int(|R|+1,n-|R|-1)$ $S(x_i,\varphi_3)$ is not interval and the set $S(x_1,\varphi_3)$ is not interval. Lemma 5 is proved.

Corollary 7. When $n\geq 4$ and $R\subset V(K_n)$, $|R|=2$ then $W_{R,\bar{R}}(K_n)=|E(K_n)|$.

Corollary 8. When $n\geq 4$ and $R\subset V(K_n)$, $|R|=n-2$ then $W_{R,R}(K_n)\geq 2n-2$.

Corollary 9. When $n\geq 4$ and $R\subset V(K_n)$, $|R|=n-3$ then $W_{R,\bar{R}}(K_n)\geq 2n$.

Lemma 6. When $n\geq 4$ and $R\subset V(K_n)$ satisfies the condition $|R|=n-1$ then there is a proper edge $(2n-3)$ – coloring of a graph K_n with $(R,\overline{R})$ – feature.

Proof. Without loss of generality, we may assume that $R=\{x_2,x_3,\dots,x_n\}$. It's easy to see that the coloring φ which was constructed in the proof of proposition 4 is a proper edge $(2n-3)$ – coloring of a graph K_n, which has $(R,\overline{R})$ – feature. Lemma 6 is proved.

4. Main results

Theorem 3. In the case $n\geq 2$ each subset $R\subset V(K_{2n+1})$ is an interval-separable subset of K_{2n+1}. The proof follows from the lemmas 4-6.

Theorem 4. In the case $n\geq 2$ each subset $R\subseteq V(K_{2n})$ is an interval-separable subset of K_{2n}.

Proof. Case 1. $R=V(K_{2n})$. The proof follows from the proposition 2.

Case 2. $R\subset V(K_{2n})$. The proof follows from the lemmas 4-6. Theorem 4 is proved.

Corollary 10. There are graphs $G\in\mathfrak{N}$ for which each subset $R\subseteq V(G)$ of its vertices is an interval-separable.

Proposition 7. When $n\geq 2$ and $R\subset V(K_n)$, $|R|=n-1$ then $W_R(K_n)=2n-3$.

Proof. In the case $2\leq n\leq 3$ the proof is trivial. Let $n\geq 4$. From the corollary 5 it follows that for proving of statement it's enough to show that for $n\geq 4$ and $R\subset V(K_n)$, which satisfies the condition $|R|=n-1$, the inequality $W_R(K_n)\leq 2n-3$ is right. Let us assume the opposite: there is $n_0\geq 4$ and a subset $R_0\subset V(K_{n_0})$ satisfies the condition $|R_0|=n_0-1$, such that there is an interval on R_0 edge t_0 – coloring φ_0 of the graph K_{n_0}, where $t_0\geq 2n_0-2$. Let $e_1\in E(K_{n_0})$ and $e_2\in E(K_{n_0})$ such that $\varphi_0(e_1)=1$ and $\varphi_0(e_2)=2n_0-2$.

Case 1. e_1 and e_2 are not adjacent. Without loss of generality, we may assume that $e_1=(x_1,x_2)$, $e_2=(x_3,x_4)$. As $|R_0|=n_0-1$ so at least one of the following statements is correct :

a) the sets $S(x_1,\varphi_0)$ and $S(x_4,\varphi_0)$ are intervals,

b) the sets $S(x_2,\varphi_0)$ and $S(x_3,\varphi_0)$ are intervals.

Without loss of generality, we may assume that the statement a) is correct. As $S(x_1,\varphi_0)$ is an $(n_0-1)-$interval and $\varphi_0(e_1)=1$, so $\varphi_0((x_1,x_4))\le n_0-1$. From the other side, as $S(x_4,\varphi_0)$ is an $(n_0-1)-$interval and $\varphi_0(e_2)=2n_0-2$, so $\varphi_0((x_1,x_4))\ge 2n_0-2-(n_0-2)=n_0>n_0-1\ge\varphi_0((x_1,x_4))$, which is impossible.

Case 2. e_1 and e_2 are adjacent. Without loss of generality, we may assume that $e_1=(x_1,x_2)$, $e_2=(x_2,x_3)$. In the examined case it's clear that $S(x_2,\varphi_0)$ is not interval. It implies that $S(x_1,\varphi_0)$ and $S(x_3,\varphi_0)$ are intervals. As $\varphi_0(e_1)=1$ and $S(x_1,\varphi_0)$ is $(n_0-1)-$interval, so $\varphi_0((x_1,x_3))\le n_0-1$. From the other side, as $\varphi_0(e_2)=2n_0-2$ and $S(x_3,\varphi_0)$ is $(n_0-1)-$interval, so $\varphi_0((x_1,x_3))\ge 2n_0-2-(n_0-2)=n_0>n_0-1\ge\varphi_0((x_1,x_3))$, which is impossible.

The contradiction shows incorrectness of the assumption. Proposition 7 is proved.

From lemma 6 and proposition 7 it follows

Corollary 11. When $n\ge 3$ and $R\subset V(K_n)$, $|R|=n-1$ then $W_{R,\bar{R}}(K_n)=2n-3$.

Corollary 12. When $n\ge 3$ and $R\subset V(K_n)$, $|R|=n-1$ then $w_{R,\bar{R}}(K_n)\le 2n-3$.

From lemma 2 it follows

Corollary 13. When $n\ge 1$ and $R\subset V(K_{2n+1})$, $|R|=2$ then $w_{R,\bar{R}}(K_{2n+1})=2n+1$.

Proposition 8. When $n\ge 2$ and $R\subset V(K_{2n+1})$, $0\le |R|\le 1$ then $w_{R,\bar{R}}(K_{2n+1})=2n+2$.

Proof. From the equality $\chi'(K_{2n+1})=2n+1$ and lemma 2 it follows that for proving of the statement it's enough to show that for $n\ge 2$ and $R\subset V(K_{2n+1})$, which satisfies the condition $0\le |R|\le 1$, there is a proper edge $(2n+2)-$coloring of the graph K_{2n+1} with $(R,\overline{R})-$feature.

Case 1. $|R|=1$. Let $R=\{\bar{x}\}$. From the lemma 2 it follows that there is a proper $(2n+1)-$coloring α_1 of the graph K_{2n+1} such that there is a vertex $\bar{\bar{x}}\in V(K_{2n+1})$ for which the following statements are correct :

1) $S(\bar{x},\alpha_1)=Int(1,2n)$, $S(\bar{\bar{x}},\alpha_1)=Int(2,2n)$,
2) for $\forall x\in V(K_{2n+1})\setminus\{\bar{x},\bar{\bar{x}}\}$ $S(x,\alpha_1)$ is not interval.

It is clear that $\exists x'\in V(K_{2n+1})$ such that $\alpha_1((\bar{\bar{x}},x'))=2n+1$. As $S(\bar{x},\alpha_1)=Int(1,2n)$ so $x'\ne\bar{x}$. Let us define the function $\varphi_1:E(K_{2n+1})\to Int(1,2n+2)$. For $\forall e\in E(K_{2n+1})$ assume

$$\varphi_1(e)=\begin{cases}\alpha_1(e)\ , if\ e\ne(\bar{\bar{x}},x')\\ 2n+2, if\ e=(\bar{\bar{x}},x')\end{cases}.$$

As α_1 is a proper edge $(2n+1)-$coloring of the graph K_{2n+1} and the equalities $\chi'(K_{2n+1})=2n+1$, $\beta(K_{2n+1})=n$, $|E(K_{2n+1})|=n(2n+1)$ hold, so $\left|\{e\in E(K_{2n+1})\setminus\{(\bar{\bar{x}},x')\}/\varphi_1(e)=2n+1\}\right|=n-1\ge 1$. This and the definition of φ_1 imply that φ_1 is the proper edge $(2n+2)-$coloring of the graph K_{2n+1} which has $(R,\overline{R})-$feature.

Case 2. $R=\varnothing$. Let α_2 be a proper edge $(2n+1)-$coloring of the graph K_{2n+1}. From the lemma 2 it follows that there are i' and i'' which satisfy the following conditions :

1) $1\le i'\le 2n+1, 1\le i''\le 2n+1, i'\ne i''$,
2) $S(x_{i'},\alpha_2)=Int(2,2n), S(x_{i''},\alpha_2)=Int(1,2n)$,

3) for $\forall i \in Int(1,2n+1)\setminus\{i',i''\}$ $S(x_i,\alpha_2)$ is not interval.

It is clear that $\alpha_2((x_{i'},x_{i''}))\neq 2n+1$, $\alpha_2((x_{i'},x_{i''}))\neq 1$.

Case 2.1. $\alpha_2((x_{i'},x_{i''}))\neq 2$. Let us define the function $\varphi_2 : E(K_{2n+1})\to Int(1,2n+2)$. For $\forall e\in E(K_{2n+1})$ assume $\varphi_2(e)\equiv\begin{cases}\alpha_2(e)\ , if\ e\neq(x_{i'},x_{i''})\\ 2n+2,\ if\ e=(x_{i'},x_{i''})\end{cases}$.

As α_2 is a proper edge $(2n+1)-$coloring of the graph K_{2n+1} and the equalities $\chi'(K_{2n+1})=2n+1$, $\beta(K_{2n+1})=n$, $|E(K_{2n+1})|=n(2n+1)$ hold, so $|\{e\in E(K_{2n+1})\setminus\{(x_{i'},x_{i''})\}/\varphi_2(e)=\alpha_2((x_{i'},x_{i''}))\}|=n-1\geq 1$. This and the definition of φ_2 imply that φ_2 is the proper edge $(2n+2)-$coloring of the graph K_{2n+1} which has $(R,\overline{R})-$feature.

Case 2.2. $\alpha_2((x_{i'},x_{i''}))=2$. Let us define the function $\varphi_3 : E(K_{2n+1})\to Int(1,2n+2)$. For $\forall e\in E(K_{2n+1})$ assume $\varphi_3(e)\equiv\begin{cases}1+\alpha_2(e)\ , if\ e\neq(x_{i'},x_{i''})\\ 1\qquad\quad , if\ e=(x_{i'},x_{i''})\end{cases}$.

As α_2 is a proper edge $(2n+1)-$coloring of the graph K_{2n+1} and the equalities $\chi'(K_{2n+1})=2n+1$, $\beta(K_{2n+1})=n$, $|E(K_{2n+1})|=n(2n+1)$ hold, so $|\{e\in E(K_{2n+1})\setminus\{(x_{i'},x_{i''})\}/\varphi_3(e)=3\}|=n-1\geq 1$. This and the definition of φ_3 imply that φ_3 is the proper edge $(2n+2)-$coloring of the graph K_{2n+1} which has $(R,\overline{R})-$feature. Proposition 8 is proved.

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Լրիվ գրաֆի գագաթների բազմության միջակայքայնորեն առանձնացվող ենթաբազմությունների մասին

Հ. Առաքելյան Ռ. Քամալյան

Ամփոփում

G գրաֆի գագաթների բազմության R ենթաբազմությունը կոչվում է միջակայքայնորեն առանձնացվող այն և միայն այն ժամանակ, երբ գոյություն ունի G գրաֆի այնպիսի ճիշտ կողային ներկում, որ կամայական x գագաթին կից կողերի գույները կազմում են բնական թվերի բազմության մեջ միջակայք այն և միայն այն դեպքում, երբ $x\in R$: Գտնված են լրիվ գրաֆի գագաթների բազմության բոլոր միջակայքայնորեն առանձնացվող ենթաբազմությունները: Նկարագրվել են առաջարկված կանոնի կիրառության արդյունավետությունը ցուցադրող համապատասխան պատկերներ և թվային արդյունքներ: