Mathematical Problems of Computer Science 31, 79–89, 2008.

Fast Generalized Haar Transforms

Hakob G. Sarukhanyan

Institue for Informatics and Automation Problems of NAS of RA
e-mail: hakop@ipia.sci.am

Abstract

The fast generalized Haar transform algorithms of orders 4^n, 3^n, and 5^n are presented.

1 Introduction

The evolution of imaging and audio/video applications over the past two decades has pushed the data storage and transmission technologies to their limits and beyond. One of the main and most important steps in data compression, as well as in various pattern recognition and communication tasks, is the application of discrete orthogonal (spectral) transforms to the input signal-image classes. This step allows transforming the original signals into the much less redundant spectral domain and performing the actual compression/recognition on spectral coefficients rather than the original signals [1]-[8]. Developed in the 60-s and 70-s fast trigonometric transforms such as FFT and DCT (discrete cosine transform) facilitated the use of such techniques for a variety of efficient data representation problems. Particularly, the DCT-based algorithms have become industry standard (JPEG/MPEG) in digital image/video compression systems [9].

In this paper we consider the generalized Haar transform, develop corresponding fast algorithms and evaluate its complexities.

2 The Generalized Haar Functions

The generalized Haar functions for any integer p, n are defined as follows [3]:

$$\begin{aligned} &H_{0,0}^{0,0}(k) = 1, \quad 0 \le k < 1, \\ &H_{i,t}^{q,r}(k) = (\sqrt{p})^{i-1} \exp\{j\tfrac{2\pi}{p}(t-1)r\}, \quad \tfrac{q+(t-1)/p}{p^{i-1}} \le k < \tfrac{q+t/p}{p^{i-1}}, \\ &H_{i,t}^{q,r}(k) = 0, \quad \text{at all other points,} \end{aligned} \tag{1}$$

where $j = \sqrt{-1}$, $i = \overline{1,n}$, $r = \overline{1,p-1}$, $q = \overline{0,p^{i-1}-1}$, $t = \overline{1,p}$.

For $p = 2$ from (1) we obtain the definition of classical Haar transform matrix of order 2^n.

$$\begin{array}{ll} H^0_{0,0}(k) = 1, & 0 \le k < 1, \\ H^q_{i,1}(k) = (\sqrt{2})^{i-1}, & 2q/2^i \le k < (2q+1)/2^i, \\ H^q_{i,2}(k) = -(\sqrt{2})^{i-1}, & (2q+1)/2^i \le k < (2q+2)/2^i, \\ H^q_{i,t}(k) = 0, & \text{at all other points}, \end{array} \quad (2)$$

where $i = \overline{1,n}$, $q = \overline{1, 2^{i-1}-1}$. For $p = 3, n = 2$ we have ($a = \exp(j\frac{2\pi}{p})$)

Row 1:
$$\begin{array}{ll} H^{0,1}_{1,1}(k) = 1, & 0 \le k < 1/3, \\ H^{0,1}_{1,2}(k) = a, & 1/3 \le k < 2/3, \\ H^{0,1}_{1,3}(k) = a^2, & 2/3 \le k < 1; \end{array}$$

Row 2:
$$\begin{array}{ll} H^{0,2}_{1,1}(k) = 1, & 0 \le k < 1/3, \\ H^{0,2}_{1,2}(k) = a^2, & 1/3 \le k < 2/3, \\ H^{0,2}_{1,3}(k) = a, & 2/3 \le k < 1; \end{array}$$

Row 3:
$$\begin{array}{ll} H^{0,1}_{2,1}(k) = \sqrt{3}, & 0 \le k < 1/9, \\ H^{0,1}_{2,2}(k) = \sqrt{3}a, & 1/9 \le k < 2/9, \\ H^{0,1}_{2,3}(k) = \sqrt{3}a^2, & 2/9 \le k < 1/3; \end{array}$$

Row 4:
$$\begin{array}{ll} H^{1,1}_{2,1}(k) = \sqrt{3}, & 1/3 \le k < 4/9, \\ H^{1,1}_{2,2}(k) = \sqrt{3}a, & 4/9 \le k < 5/9, \\ H^{1,1}_{2,3}(k) = \sqrt{3}a^2, & 5/9 \le k < 2/3; \end{array}$$

Row 5:
$$\begin{array}{ll} H^{2,1}_{2,1}(k) = \sqrt{3}, & 2/3 \le k < 7/9, \\ H^{2,1}_{2,2}(k) = \sqrt{3}a, & 7/9 \le k < 8/9, \\ H^{2,1}_{2,3}(k) = \sqrt{3}a^2, & 8/9 \le k < 1; \end{array}$$

Row 6:
$$\begin{array}{ll} H^{0,2}_{2,1}(k) = \sqrt{3}, & 0 \le k < 1/9, \\ H^{0,2}_{2,2}(k) = \sqrt{3}a^2, & 1/9 \le k < 2/9, \\ H^{0,2}_{2,3}(k) = \sqrt{3}a, & 2/9 \le k < 1/3; \end{array}$$

Row 7:
$$\begin{array}{ll} H^{1,2}_{2,1}(k) = \sqrt{3}, & 1/3 \le k < 4/9, \\ H^{1,2}_{2,2}(k) = \sqrt{3}a^2, & 4/9 \le k < 5/9, \\ H^{1,2}_{2,3}(k) = \sqrt{3}a, & 5/9 \le k < 2/3; \end{array}$$

Row 8:
$$\begin{array}{ll} H^{2,2}_{2,1}(k) = \sqrt{3}, & 2/3 \le k < 7/9, \\ H^{2,2}_{2,2}(k) = \sqrt{3}a^2, & 7/9 \le k < 8/9, \\ H^{2,2}_{2,3}(k) = \sqrt{3}a, & 8/9 \le k < 1. \end{array}$$

Therefore, the complete orthogonal Haar transform matrix for $p = 3, n = 2$ has the following form (here $s = \sqrt{3}$)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a & a & a & a^2 & a^2 & a^2 \\ 1 & 1 & 1 & a^2 & a^2 & a^2 & a & a & a \\ s & sa & sa^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & sa & sa^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s & sa & sa^2 \\ s & sa^2 & sa & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & sa^2 & sa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s & sa^2 & sa \end{pmatrix} \quad (3)$$

3 2^n–Point Haar Transform

Introduce the following notations: $i_2 = (1, 1)$, $j_2 = (1, -1)$. From relations (2) we obtain

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}, \quad H_4 = \begin{pmatrix} H_2 \otimes i_2 \\ \sqrt{2}I_2 \otimes j_2 \end{pmatrix}, \quad H_8 = \begin{pmatrix} H_4 \otimes i_2 \\ 2I_4 \otimes j_2 \end{pmatrix}.$$

Continuing this process we obtain recursive representation of Haar matrices of any order 2^n

$$H_{2^n} = \begin{pmatrix} H_{2^{n-1}} \otimes i_2 \\ (\sqrt{2})^{n-1} I_{2^{n-1}} \otimes j_2 \end{pmatrix}, \quad H_1 = 1, \quad n = 1, 2, \dots. \tag{4}$$

where $\otimes$ is the Kronecker product.

Now we compute the complexity of Haar transform of order 2^n. Note that for n=1 we have $C^+(H_2) = 2$, $C^\times(H_2) = 0$. Calculate the complexity of H_4 transform. Let $X^T = (x_0, x_1, x_2, x_3)$ be a real-valued vector of length 4.

The one dimensional forward Haar transform of order 4 can be performed as follows:

$$H_4 X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} == \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{5}$$

where $y_0 = (x_0 + x_1) + (x_2 + x_3)$, $y_1 = (x_0 + x_1) - (x_2 + x_3)$, $y_2 = \sqrt{2}(x_0 - x_1)$, $y_3 = \sqrt{2}(x_2 - x_3)$.

Then the complexity of H_4 transform is: $C^+(H_4) = 6$, and $C^\times = 2$.

Now let $X^T = (x_0, x_1, \dots, x_{N-1})$ be a real-valued vector of length $N = 2^n$ $(n > 2)$. Introduce the following notations: P_i is (0,1) column-vector of length $N/2$ whose only i-th $(i = 1, \dots, N/2)$ element equals 1, and $(X^i)^T = (x_{2i-2}, x_{2i-1})$. The one dimensional forward Haar transform of order N can be performed as follows:

$$H_N X = \begin{pmatrix} (H_{N/2} \otimes i_2) X \\ (\sqrt{2}^{n-1} I_{N/2} \otimes j_2) X \end{pmatrix}$$

Using the above given notations we have

$$\begin{aligned} (H_{N/2} \otimes i_2) X &= (H_{N/2} \otimes i_2)(P_1 \otimes X^1 + P_2 \otimes X^2 + \dots + P_{N/2} \otimes X^{N/2}) \\ &= H_{N/2} P_1 \otimes i_2 X^1 + H_{N/2} P_2 \otimes i_2 X^2 + \dots + H_{N/2} P_{N/2} \otimes i_2 X^{N/2} \\ &= H_{N/2} P_1 (x_0 + x_1) + H_{N/2} P_2 (x_2 + x_3) + \dots + H_{N/2} P_{N/2} (x_{N-2} + x_{N-1}) \\ &= H_{N/2} \begin{pmatrix} x_0 + x_1 \\ x_2 + x_3 \\ \vdots \\ x_{N-2} + x_{N-1} \end{pmatrix} \end{aligned}$$

Then we can write

$$C^+(H_{N/2} \otimes i_2) = C^+(H_{N/2}) + \frac{N}{2}, \quad C^\times(H_{N/2} \otimes i_2) = C^\times(H_{N/2}).$$

Now compute the complexity of $(\sqrt{2}^{n-1} I_{N/2} \otimes j_2) X$ transform

$$\begin{aligned} ((\sqrt{2})^{n-1} I_{N/2} \otimes j_2) X &= ((\sqrt{2})^{n-1} I_{N/2} \otimes j_2)(P_1 \otimes X^1 + \dots + P_{N/2} \otimes X^{N/2}) \\ &= (\sqrt{2})^{n-1} (P_1 \otimes a_1 X^1 + \dots + P_{N/2} \otimes a_1 X^{N/2}), \end{aligned}$$

from which we obtain

$$C^+((\sqrt{2})^{n-1}I_{N/2}\otimes j_2)=N/2,\quad C^\times((\sqrt{2})^{n-1}I_{N/2}\otimes j_2)=N/2.$$

Finally, the complexity of H_{2^n} transform can be calculated as follows:

$$C^+(H_{2^n})=2^{n+1}-2,\quad C^\times(H_{2^n})=2^n-2,\quad n=1,2,3,\dots \tag{6}$$

For example, we have $C^+(H_4)=6$, $C^\times(H_4)=2$, $C^+(H_8)=14$, $C^\times(H_8)=6$, $C^+(H_{16})=30$, $C^\times(H_{16})=14$.

4 3^n–Point Generalized Haar Transform

Introduce the following notations: $i_3=(1,1,1)$, $b_3=(1,a,a^2)$, where $a=\exp\{j\frac{2\pi}{3}\}$, $j=\sqrt{-1}$. From relations (1) we obtain

$$H_3=\begin{pmatrix}1&1&1\\1&a&a^2\\1&a^2&a\end{pmatrix}=\begin{pmatrix}i_3\\b_3\\b_3^*\end{pmatrix},\quad H_9=\begin{pmatrix}H_3\otimes i_3\\\sqrt{3}I_3\otimes b_3\\\sqrt{3}I_3\otimes b_3^*\end{pmatrix},\quad H_{27}=\begin{pmatrix}H_9\otimes i_3\\3I_9\otimes b_3\\3I_9\otimes b_3^*\end{pmatrix}.$$

Continuing this process we obtain recursive representation of Haar matrices of any order 3^n

$$H_{3^n}=\begin{pmatrix}H_{3^{n-1}}\otimes i_3\\(\sqrt{3})^{n-1}I_{3^{n-1}}\otimes b_3\\(\sqrt{3})^{n-1}I_{3^{n-1}}\otimes b_3^*\end{pmatrix},\quad H_1=1,\quad n=1,2,\dots \tag{7}$$

Now we compute the complexity of generalized Haar transform of order 3^n. First we calculate the complexity of H_3 transform. Let $Z^T=(z_0,z_1,z_2)=(x_0+jy_0,x_1+jy_1,x_2+jy_2)$ be a complex-valued vector of length 3, $a=\exp(j\frac{2\pi}{3})=\cos\frac{2\pi}{3}+j\sin\frac{2\pi}{3}$, $j=\sqrt{-1}$.

The one dimensional forward generalized Haar transform of order 3 can be performed as follows:

$$H_3X=\begin{pmatrix}1&1&1\\1&a&a^2\\1&a^2&a\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix}=\begin{pmatrix}i_3\\b_3\\b_3^*\end{pmatrix}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix}=\begin{pmatrix}v_0\\v_1\\v_2\end{pmatrix}, \tag{8}$$

where

$$v_0=x_0+(x_1+x_2)+j[y_0+(y_1+y_2)],$$
$$v_1=x_0-(x_1+x_2)\cos\tfrac{\pi}{3}-(y_1-y_2)\sin\tfrac{\pi}{3}+j[y_0-(y_1+y_2)\cos\tfrac{\pi}{3}-(x_1+x_2)\sin\tfrac{\pi}{3}],$$
$$v_2=x_0-(x_1+x_2)\cos\tfrac{\pi}{3}+(y_1-y_2)\sin\tfrac{\pi}{3}+j[y_0-(y_1+y_2)\cos\tfrac{\pi}{3}-(x_1-x_2)\sin\tfrac{\pi}{3}].$$

We can see that

$$\begin{aligned}&C^+(i_3\otimes Z)=4,\quad C^+(b_3\otimes Z)=8,\quad C^+(b_3^*\otimes Z)=2,\\&C^\times(i_3\otimes Z)=0,\quad C^\times(b_3\otimes Z)=4,\quad C^\times(b_3^*\otimes Z)=0.\end{aligned} \tag{9}$$

Then the complexity of H_3 transform is: $C^+(H_3)=14$, $C^\times(H_3)=4$.

Now let $Z^T = (z_0, z_1, \ldots, z_{N-1})$ be a complex-valued vector of length $N = 3^n$ $(n > 1)$. Introduce the following notations: P_i are (0,1) column-vectors of length $N/3$ whose only i-th $(i = 0, 1, \ldots, N/3 - 1)$ element equals 1, and $(Z^i)^T = (z_{3i}, z_{3i+1}, z_{3i+2})$. The one dimensional forward generalized Haar transform of order N can be performed as follows:

$$H_{3^n} Z = \begin{pmatrix} (H_{3^{n-1}} \otimes i_3) Z \\ (\sqrt{3}^{n-1} I_{3^{n-1}} \otimes b_3) Z \\ (\sqrt{3}^{n-1} I_{3^{n-1}} \otimes b_3^*) Z \end{pmatrix}$$

Using the above given notations we have

$$\begin{aligned} (H_{3^{n-1}} \otimes i_3) Z &= (H_{3^{n-1}} \otimes i_3)(P_0 \otimes Z^0 + P_1 \otimes Z^1 + \cdots + P_{3^{n-1}-1} \otimes Z^{3^{n-1}-1}) \\ &= H_{3^{n-1}} P_0 \otimes i_3 Z^0 + H_{3^{n-1}} P_1 \otimes i_3 Z^1 + \cdots + H_{3^{n-1}} P_{3^{n-1}-1} \otimes i_3 Z^{3^{n-1}} \\ &= H_{3^{n-1}} P_0 (z_0 + z_1 + z_2) + \cdots + H_{3^{n-1}} P_{3^{n-1}-1} (z_{3^n-3} + z_{3^n-2} + z_{3^n-1}) \\ &= H_{3^{n-1}} \begin{pmatrix} z_0 + z_1 + z_2 \\ z_3 + z_4 + z_5 \\ \vdots \\ z_{3^n-3} + z_{3^n-2} + z_{3^n-1} \end{pmatrix} \end{aligned}$$

Then we can write

$$C^+(H_{3^{n-1}} \otimes i_3) = C^+(H_{3^{n-1}}) + 4 \cdot 3^{n-1}, \quad C^\times(H_{3^{n-1}} \otimes i_3) = C^\times(H_{3^{n-1}} \otimes i_3).$$

Now compute the complexity of $((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3) X$ transform

$$\begin{aligned} ((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3) Z &= ((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3)(P_0 \otimes Z^0 + \cdots + P_{3^{n-1}-1} \otimes Z^{3^{n-1}-1}) \\ &= (\sqrt{3})^{n-1} (P_0 \otimes b_3 Z^0 + \cdots + P_{3^{n-1}-1} \otimes Z_3 X^{3^{n-1}-1}). \end{aligned}$$

From (9) and above given equation we obtain

$$\begin{aligned} C^+((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3) &= 3^{n-1} C^+(b_3) = 8 \cdot 3^{n-1}, \\ C^\times((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3) &= 3^{n-1} C^\times(b_3) = 4 \cdot 3^{n-1}. \end{aligned}$$

Similarly, using again (9) we obtain

$$\begin{aligned} C^+((\sqrt{3})^{n-1} I_{3^{n-1}} \otimes b_3^*) &= 3^{n-1} C^+(b_3^*) = 2 \cdot 3^{n-1}, \\ C^\times((\sqrt{3})^{n-1} I_{N/3} \otimes b_3^*) &= 0. \end{aligned}$$

Finally, the complexity of H_{3^n} transform can be calculated as follows:

$$\begin{aligned} C^+(H_{3^n}) &= 7(3^n - 1), \\ C^\times(H_{3^n}) &= 2(3^n - 1), \quad n = 1, 2, 3, \ldots. \end{aligned} \tag{10}$$

For example, we have $C^+(H_9) = 56$, $C^\times(H_9) = 16$, $C^+(H_{27}) = 182$, $C^\times(H_{27}) = 52$.

5 4^n–Point Generalized Haar Transform

Introduce the following notations: $i_4 = (1, 1, 1, 1)$, $a_1 = (1, j, -1, -j)$, $a_2 = (1, -1, 1, -1)$, where $j = \sqrt{-1}$. From relations (1) we obtain

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} = \begin{pmatrix} i_4 \\ a_1 \\ a_2 \\ a_1^* \end{pmatrix}, \quad H_{16} = \begin{pmatrix} H_4 \otimes i_4 \\ 2I_4 \otimes a_1 \\ 2I_4 \otimes a_2 \\ 2I_4 \otimes a_1^* \end{pmatrix}.$$

Continuing this process we obtain recursive representation of Haar matrices of any order 4^n

$$H_{4^n} = \begin{pmatrix} H_{4^{n-1}} \otimes i_4 \\ 2^{n-1} I_{4^{n-1}} \otimes a_1 \\ 2^{n-1} I_{4^{n-1}} \otimes a_2 \\ 2^{n-1} I_{4^{n-1}} \otimes a_1^* \end{pmatrix}, \quad H_1 = 1, \quad n = 1, 2, \dots. \tag{11}$$

Now we compute the complexity of generalized Haar transform of order 4^n. First we calculate the complexity of H_4 transform. Let $Z^T = (z_0, z_1, z_2, z_3)$ be a complex-valued vector of length 4, $j = \sqrt{-1}$.

The one dimensional forward generalized Haar transform of order 4 can be performed as follows:

$$H_4 Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} i_4 \\ a_1 \\ a_2 \\ a_1^* \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} v_0 &= (x_0 + x_2) + (x_1 + x_3) + j[(y_0 + y_2) + (y_1 + y_3)], \\ v_1 &= (x_0 - x_2) - (y_1 - y_3) + j[(y_0 - y_2) + (x_1 - x_3)], \\ v_2 &= (x_0 + x_2) - (x_1 + x_3) + j[(y_0 + y_2) - (y_1 + y_3)], \\ v_3 &= (x_0 - x_2) + (y_1 - y_3) + j[(y_0 - y_2) - (x_1 - x_3)]. \end{aligned}$$

Then the complexity of H_4 transform is: $C^+(H_4) = 16$, and no multiplications.

Now let $Z^T = (z_0, z_1, \dots, z_{N-1})$ be a complex-valued vector of length $N = 4^n$ $(n > 1)$. Introduce the following notations: P_i are (0,1) column-vectors of length $N/4$ whose only i–th $(i = 1, \dots, N/4)$ element equals 1, and $(Z^i)^T = (z_{4i-4}, z_{4i-3}, z_{4i-2}, z_{4i-1})$. The one dimensional forward generalized Haar transform of order N can be performed as follows:

$$H_{4^n} Z = \begin{pmatrix} (H_{4^{n-1}} \otimes i_4) Z \\ 2^{n-1} I_{4^{n-1}} \otimes a_1) Z \\ 2^{n-1} I_{4^{n-1}} \otimes a_2) Z \\ 2^{n-1} I_{4^{n-1}} \otimes a_1^*) Z \end{pmatrix}$$

Using the above given notations we have

$$\begin{aligned}(H_{4^{n-1}}\otimes i_4)Z &= (H_{4^{n-1}}\otimes i_4)(P_1\otimes Z^1+P_2\otimes Z^2+\cdots+P_{4^{n-1}}\otimes Z^{4^{n-1}})\\ &= H_{4^{n-1}}P_1\otimes i_4Z^1+H_{4^{n-1}}P_2\otimes i_4Z^2+\cdots+H_{4^{n-1}}P_{4^{n-1}}\otimes i_4Z^{4^{n-1}}\\ &= H_{4^{n-1}}P_1(z_0+z_1+z_2+z_3)+\cdots+H_{4^{n-1}}P_{4^{n-1}}(z_{N-4}+z_{N-3}+z_{N-2}+z_{N-1})\\ &= H_{4^{n-1}}\begin{pmatrix} z_0+z_1+z_2+z_3\\ z_4+z_5+z_6+z_7\\ \vdots\\ z_{N-4}+\cdots+z_{N-1}\end{pmatrix}\end{aligned}$$

Then we can write

$$C^+(H_{4^{n-1}}\otimes i_4)=C^+(H_{4^{n-1}})+6\cdot 4^{n-1},$$
$$C^\times(H_{4^{n-1}}\otimes i_4)=C^{shift}(H_{4^{n-1}})=0.$$

Now compute the complexity of $(2^{n-1}I_{4^{n-1}}\otimes a_1)X$ transform

$$\begin{aligned}(2^{n-1}I_{4^{n-1}}\otimes a_1)Z &= (2^{n-1}I_{4^{n-1}}\otimes a_1)(P_1\otimes Z^1+P_2\otimes Z^2+\cdots+P_{4^{n-1}}\otimes Z^{4^{n-1}})\\ &= 2^{n-1}(P_1\otimes a_1Z^1+P_2\otimes a_1Z^2+\cdots+P_{4^{n-1}}\otimes a_1Z^{4^{n-1}}),\end{aligned}$$

from which we obtain

$$C^+(2^{n-1}I_{4^{n-1}}\otimes a_1)=6\cdot 4^{n-1},\quad C^{shift}(2^{n-1}I_{4^{n-1}}\otimes a_1)=4^{n-1}.$$

Similarly, we find

$$C^+(2^{n-1}I_{4^{n-1}}\otimes a_2)=2\cdot 4^{n-1},\qquad C^{shift}(2^{n-1}I_{4^{n-1}}\otimes a_2)=4^{n-1},$$
$$C^+(2^{n-1}I_{4^{n-1}}\otimes a_1^*)=2\cdot 4^{n-1},\qquad C^{shift}(2^{n-1}I_{4^{n-1}}\otimes a_1^*)=4^{n-1}.$$

Finally, the complexity of H_{4^n} transform can be calculated as follows:

$$\begin{aligned}&C^+(H_{4^n})=\tfrac{16(4^n-1)}{3},\quad n=1,2,3,\ldots\\ &C^{shift}(H_{4^n})=3\cdot 4^{n-1},\quad n=2,3,\ldots.\end{aligned}\qquad(13)$$

For example, we have

$$C^+(H_4)=16,\quad C^{shift}(H_4)=0,\quad C^+(H_{16})=80,\quad C^{shift}(H_{16})=12.$$

6 5^n–Point Generalized Haar Transform

Introduce the following notations: $i_4=(1,1,1,1,1)$, $a_1=(1,a,a^2,a^3,a^4)$, $a_2=(1,a^2,a^4,a,a^3)$, $a=\exp(j\frac{2\pi}{5})$, where $j=\sqrt{-1}$. From relations (1) we obtain

$$H_5=\begin{pmatrix}1&1&1&1&1\\1&a&a^2&a^3&a^4\\1&a^2&a^4&a&a^3\\1&a^3&a&a^4&a^2\\1&a^4&a^3&a^2&a\end{pmatrix}=\begin{pmatrix}i_5\\a_1\\a_2\\a_2^*\\a_1^*\end{pmatrix},\quad H_{25}=\begin{pmatrix}H_5\otimes i_5\\ \sqrt{5}I_5\otimes a_1\\ \sqrt{5}I_5\otimes a_2\\ \sqrt{5}I_5\otimes a_2^*\\ \sqrt{5}I_5\otimes a_1^*\end{pmatrix}.$$

Continuing this process we obtain recursive representation of Haar matrices of any order 5^n

$$H_{5^n} = \begin{pmatrix} H_{5^{n-1}} \otimes i_5 \\ (\sqrt{5})^{n-1} I_{5^{n-1}} \otimes a_1 \\ (\sqrt{5})^{n-1} I_{5^{n-1}} \otimes a_2 \\ (\sqrt{5})^{n-1} I_{5^{n-1}} \otimes a_2^* \\ (\sqrt{5})^{n-1} I_{5^{n-1}} \otimes a_1^* \end{pmatrix}, \quad H_1 = 1, \quad n = 1, 2, \ldots \tag{14}$$

Now we compute the complexity of generalized Haar transform of order 5^n. First we calculate the complexity of H_5 transform. Let $Z^T = (x_0 + jy_0, \ldots, x_4 + y_4)$ be a complex-valued vector of length 5, $a = \exp(j\frac{2\pi}{5})$, $j = \sqrt{-1}$, then

$$H_5 Z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & a & a^2 & a^3 & a^4 \\ 1 & a^2 & a^4 & a & a^3 \\ 1 & a^3 & a & a^4 & a^2 \\ 1 & a^4 & a^3 & a^2 & a \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} i_5 \\ a_1 \\ a_2 \\ a_2^* \\ a_1^* \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}
v_0^r &= x_0 + x_1 + x_2 + x_3 + x_4, \\
v_0^i &= y_0 + y_1 + y_2 + y_3 + y_4; \\
v_1^r &= x_0 + (x_1 + x_4)\cos\tfrac{2\pi}{5} - (x_2 + x_3)\cos\tfrac{\pi}{5} - [(y_1 - y_4)\sin\tfrac{2\pi}{5} + (y_2 - y_3)\sin\tfrac{\pi}{5}], \\
v_1^i &= y_0 + (y_1 + y_4)\cos\tfrac{2\pi}{5} - (y_2 + y_3)\cos\tfrac{\pi}{5} + [(x_1 - x_4)\sin\tfrac{2\pi}{5} + (x_2 - x_3)\sin\tfrac{\pi}{5}]; \\
v_4^r &= x_0 + (x_1 + x_4)\cos\tfrac{2\pi}{5} - (x_2 + x_3)\cos\tfrac{\pi}{5} + [(y_1 - y_4)\sin\tfrac{2\pi}{5} + (y_2 - y_3)\sin\tfrac{\pi}{5}], \\
v_4^i &= y_0 + (y_1 + y_4)\cos\tfrac{2\pi}{5} - (y_2 + y_3)\cos\tfrac{\pi}{5} - [(x_1 - x_4)\sin\tfrac{2\pi}{5} + (x_2 - x_3)\sin\tfrac{\pi}{5}]; \\
v_2^r &= x_0 - (x_1 + x_4)\cos\tfrac{\pi}{5} + (x_2 + x_3)\cos\tfrac{2\pi}{5} - [(y_1 - y_4)\sin\tfrac{\pi}{5} - (y_2 - y_3)\sin\tfrac{2\pi}{5}], \\
v_2^i &= y_0 - (y_1 + y_4)\cos\tfrac{\pi}{5} + (y_2 + y_3)\cos\tfrac{2\pi}{5} + [(x_1 - x_4)\sin\tfrac{\pi}{5} - (x_2 - x_3)\sin\tfrac{2\pi}{5}]; \\
v_3^r &= x_0 - (x_1 + x_4)\cos\tfrac{\pi}{5} + (x_2 + x_3)\cos\tfrac{2\pi}{5} + [(y_1 - y_4)\sin\tfrac{\pi}{5} - (y_2 - y_3)\sin\tfrac{2\pi}{5}], \\
v_3^i &= y_0 - (y_1 + y_4)\cos\tfrac{\pi}{5} + (y_2 + y_3)\cos\tfrac{2\pi}{5} - [(x_1 - x_4)\sin\tfrac{\pi}{5} - (x_2 - x_3)\sin\tfrac{2\pi}{5}].
\end{aligned} \tag{16}$$

Now introduce the following notations:

$$\begin{aligned}
&X_1 = x_1 + x_4, \quad X_2 = x_2 + x_3, \quad \overline{X}_1 = x_1 - x_4, \quad \overline{X}_2 = x_2 - x_3, \\
&Y_1 = y_1 + y_4, \quad Y_2 = y_2 + y_3, \quad \overline{Y}_1 = y_1 - y_4, \quad \overline{Y}_2 = y_2 - y_3, \\
&C_1 = X_1\cos\tfrac{2\pi}{5}, \quad C_2 = X_2\cos\tfrac{\pi}{5}, \quad C_3 = Y_1\cos\tfrac{2\pi}{5}, \quad C_4 = Y_2\cos\tfrac{\pi}{5}, \\
&S_1 = \overline{X}_1\sin\tfrac{2\pi}{5}, \quad S_2 = \overline{X}_2\sin\tfrac{\pi}{5}, \quad S_3 = \overline{Y}_1\sin\tfrac{2\pi}{5}, \quad S_4 = \overline{Y}_2\sin\tfrac{\pi}{5}, \\
&T_1 = X_1\cos\tfrac{\pi}{5}, \quad T_2 = X_2\cos\tfrac{2\pi}{5}, \quad T_3 = Y_1\cos\tfrac{\pi}{5}, \quad T_4 = Y_2\cos\tfrac{2\pi}{5}, \\
&R_1 = \overline{Y}_1\sin\tfrac{\pi}{5}, \quad R_2 = \overline{Y}_2\sin\tfrac{2\pi}{5}, \quad R_3 = \overline{X}_1\sin\tfrac{\pi}{5}, \quad R_4 = \overline{X}_2\sin\tfrac{2\pi}{5}.
\end{aligned} \tag{17}$$

Using the notations (17), the equations can be represented as

$$\begin{aligned}
v_0 &= x_0 + X_1 + X_2 + j(y_0 + Y_1 + Y_2);\\
v_1 &= (x_0 + C_1 - C_2) - (S_3 + S_4) + j[(y_0 + C_3 - C_4) + (S_1 + S_2)],\\
v_4 &= (x_0 + C_1 - C_2) + (S_3 + S_4) + j[(y_0 + C_3 - C_4) - (S_1 + S_2)],\\
v_2 &= (x_0 - T_1 + T_2) - (R_1 - R_2) + j[(y_0 - T_3 + T_4) + (R_3 - R_4)],\\
v_3 &= (x_0 - T_1 + T_2) + (R_1 - R_2) + j[(y_0 - T_3 + T_4) - (R_3 - R_4)].
\end{aligned} \tag{18}$$

Now it is not difficult to find that $C^+(i_5) = C^+(a_1) = C^+(a_2) = 8$, $C^+(a_2^*) = C^+(a_1^*) = 2$. $C^\times(i_5) = C^\times(a_1^*) = C^\times(a_2^*) = 0$, $C^\times(a_1) = C^\times(a_2) = 8$. Therefore we obtain

$$C^+(H_5) = 28, \quad C^\times(H_5) = 16.$$

Now let $Z^T = (z_0, z_1, \dots, z_{N-1})$ be a complex-valued vector of length $N = 5^n$ $(n > 1)$. Introduce the following notations: P_i are (0,1) column-vectors of length $N/5$ whose only i–th $(i = 1, \dots, N/5)$ element equals 1, and

$$Z^i = (z_{5i-5}, z_{5i-4}, z_{5i-3}, z_{5i-2}, z_{5i-1})^T.$$

The one dimensional forward generalized Haar transform of order N can be performed as follows:

$$H_N Z = \begin{pmatrix} (H_{N/5} \otimes i_5)Z \\ \sqrt{5}^{n-1} I_{N/5} \otimes a_1)Z \\ \sqrt{5}^{n-1} I_{N/5} \otimes a_2)Z \\ \sqrt{5}^{n-1} I_{N/5} \otimes a_2^*)Z \\ \sqrt{5}^{n-1} I_{N/5} \otimes a_1^*)Z \end{pmatrix}$$

Using the above given notations we have

$$\begin{aligned}
(H_{N/5} \otimes i_5)Z &= (H_{N/5} \otimes i_5)(P_1 \otimes Z^1 + P_2 \otimes Z^2 + \cdots + P_{N/5} \otimes Z^{N/5})\\
&= H_{N/5}P_1 \otimes i_5 Z^1 + H_{N/4}P_2 \otimes i_4 Z^2 + \cdots + H_{N/5}P_{N/5} \otimes i_5 Z^{N/5}\\
&= H_{N/5}P_1(z_0 + z_1 + \cdots + z_4) + \cdots + H_{N/5}P_{N/5}(z_{N-5} + x_{N-4} + \cdots + X_{N-1})\\
&= H_{N/5} \begin{pmatrix} z_0 + z_1 + \cdots + z_4 \\ z_5 + z_6 + \cdots + z_9 \\ \vdots \\ x_{N-5} + \cdots + x_{N-1} \end{pmatrix}
\end{aligned}$$

Then we can write

$$C^+(H_{N/5} \otimes i_5) = C^+(H_{N/5}) + 8N/5, \quad C^\times(H_{N/5} \otimes i_5) = C^\times(H_{N/5}).$$

Now compute the complexity of $((\sqrt{5})^{n-1} I_{N/5} \otimes a_1)Z$ transform

$$\begin{aligned}
((\sqrt{5})^{n-1} I_{N/5} \otimes a_1)Z &= ((\sqrt{5})^{n-1} I_{N/5} \otimes a_1)(P_1 \otimes Z^1 + \cdots + P_{N/5} \otimes Z^{N/5}\\
&= (\sqrt{5})^{n-1}(P_1 \otimes a_1 Z^1 + \cdots + P_{N/5} \otimes a_1 Z^{N/5}),
\end{aligned}$$

from which we obtain

$$C^{+}((\sqrt{5})^{n-1} I_{N/5} \otimes a_1) = \tfrac{N}{5} C^{+}(a_1) = \tfrac{8N}{5},$$
$$C^{\times}((\sqrt{5})^{n-1} I_{N/5} \otimes a_1) = \tfrac{N}{5} + \tfrac{N}{5} C^{\times}(a_1) = \tfrac{9N}{5}.$$

Similarly, we find

$$C^{+}((\sqrt{5})^{n-1} I_{N/5} \otimes a_2) = \tfrac{N}{5} C^{+}(a_2) = \tfrac{8N}{5},$$
$$C^{\times}((\sqrt{5})^{n-1} I_{N/5} \otimes a_2) = \tfrac{N}{5} + \tfrac{N}{5} C^{\times}(a_2) = \tfrac{9N}{5},$$
$$C^{+}((\sqrt{5})^{n-1} I_{N/5} \otimes a_2^{*}) = \tfrac{N}{5} C^{+}(a_2^{*}) = \tfrac{2N}{5},$$
$$C^{\times}((\sqrt{5})^{n-1} I_{N/5} \otimes a_2^{*}) = \tfrac{N}{5} + \tfrac{N}{5} C^{\times}(a_2^{*}) = \tfrac{3N}{5},$$
$$C^{+}((\sqrt{5})^{n-1} I_{N/5} \otimes a_1^{*}) = \tfrac{N}{5} C^{+}(a_1^{*}) = \tfrac{2N}{5},$$
$$C^{\times}((\sqrt{5})^{n-1} I_{N/5} \otimes a_1^{*}) = \tfrac{N}{5} + \tfrac{N}{5} C^{\times}(a_1^{*}) = \tfrac{3N}{5}.$$

Finally, the complexity of H_{5^n} transform can be calculated as follows:

$$C^{+}(H_{5^n}) = 7(5^n - 1), \qquad (19)$$
$$C^{\times}(H_{5^n}) = 6 \cdot 5^n - 14, \quad n = 1, 2, 3, \ldots.$$

The numerical results of the complexities of Generalized Haar transforms are given in the table below

Size	n	Addition	Multiplication	Shift
2^n	n	$2^{n+1} - 2$	$2^n - 2$	0
2	1	2	0	0
4	2	6	2	0
8	3	14	6	0
16	4	30	14	0
3^n	n	$7(3^n - 1)$	$2(3^n - 1$	0
3	1	14	4	0
9	2	56	16	0
27	3	182	52	0
81	4	560	160	0
4^n	n	$\frac{16(4^n-1)}{3}$	0	$3 \cdot 4^{n-1}$
4	1	16	0	0
16	2	80	0	12
64	3	336	0	48
128	4	1370	0	192
5^n	n	$7(5^n - 1)$	$6 \cdot 5^n - 14$	0
5	1	28	16	0
25	2	168	136	0
125	3	868	736	0
625	4	4368	3736	0

References

[1] Ahmed N., Rao K.R. Orthogonal Transforms for Digital Signal Processing. Springer-Verlag, New York, 1975.

[2] Haar Z. Zur Theorie der orthogonalen funktionensysteme. Math. Ann., vol 69, p.331-371, 1914.

[3] Rao K.R., Narasimham, Revuluri K. A Family of discrete Haar transforms. Comput. Electr. Engrg., vol. 2, p.367-368, 1975.

[4] Seberry J., Zhang X.M. Some orthogonal designs and complex Hadamard matrices by using two Hadamard matrices. Austral. J. of Combin. Theory., No.4, p.93-102. 1991.

[5] Corinthios M.J. A new class of general-base matrices and a formalism for optimal parallel/pipelined computer architecture. Electrical and Computer Engineering, 1993. Canadian Conference on 14-17 Sept., vol.2, p.851-856, 1993.

[6] Mingyong Zhou, Zhongkan Liu, Hama H. A resolution-controllable harmonical retrieval approach on the Chrestenson discrete space. IEEE Transactions, Signal Processing, vol.42, Issue 5, p.1281-1284, 1994.

[7] Moraga, C., Poswig J. Properties of the Zhang-Hartley spectrum of patterns. Proceedings of the Twentieth International Symposium, Multiple-Valued Logic, p.62-68. 1990.

[8] Agaian S., Bajadian H. Generalized orthogonal Haar systems: Synthesis, Metric and Computing Properties. Haar Memorial Conference, vol.1, p.97-113, Collog. Math. Soc. Janos Bolyai, 49, North-Holland, Amsterdam, New York, 1987.

[9] Mallat S. A Wavelet Tour of Signal Processing. Academic Press, 2001.

Հաարի ընդհանրացված արագ ձևափոխություններ

Հ. Սարուխանյան

Ամփոփում

Հոդվածում ներկայացված են 4^n, 3^n, և 5^n կարգի Հաարի ընդհանրացված արագ ձևափոխությունների ալգորիթմները: