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# On Optimal Hypothesis Testing for Pair of Stochastically Coupled Objects.

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**Abstract**

The paper is devoted to hypotheses testing for a model consisting of two stochastically coupled objects. It is supposed that $L_1$ possible probability distributions are known for the first object and the second object is distributed according to one of $L_1 \times L_2$ given conditional distributions depending on the distribution index and the current observed state of the first object. The matrix of interdependencies of all possible pairs of the error probability exponents in asymptotically optimal tests of distributions of both objects is studied. The case of two objects which cannot have the same probability distribution from two possible variants was considered by Ahlswede and Haroutunian. This case for three hypotheses and the model of two statistically dependent objects for two hypotheses were examined by Haroutunian and Yessayan.

## 1 Problem Statement and Preliminary Results

In this paper we solve a generalisation of the problem of many hypotheses testing conserning one object [1]. In paper [2] Ahlswede and Haroutunian and in [3] Haroutunian formulated a number of problems on multiple hypotheses testing and indentification. Haroutunian and Hakobyan in [4] examined the models of two independent objects with three and in [5] with $L$ hypotheses. Haroutunian and Yessayan in [6] studied the model of two objects which can have only different distributions from possible three.

Let $X_1$ and $X_2$ be random variables (RVs) taking values in a finite set $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ be the space of all possible distributions on $\mathcal{X}$. There are given $L_1$ probability distrubutions (PD) $G_{l_1} = \{G_{l_1}(x^1),\ x^1 \in \mathcal{X}\}$, $l_1 = \overline{1, L_1}$, from $\mathcal{P}(\mathcal{X})$. The first object characterized by RV $X_1$ can have one of these $L_1$ distributions and the second object dependent on the first, and caracterized by RV $X_2$ can have one of $L_1 \times L_2$ conditional PDs $G_{l_2/l_1} = \{G_{l_2/l_1}(x^2|x^1),\ x^1, x^2 \in \mathcal{X}\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$. Let $(\mathbf{x}_1, \mathbf{x}_2) = ((x_1^1, x_1^2), (x_2^1, x_2^2), ...(x_N^1, x_N^2))$ be a sequence of results of $N$ independent observations of the vector $(X_1, X_2)$, which can have one of $L_1 \times L_2$ joint PDs $G_{l_1,l_2}(x^1, x^2)$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$, where $G_{l_1,l_2}(x^1, x^2) = G_{l_1}(x^1) G_{l_2/l_1}(x^2|x^1)$. The probability of vector $(\mathbf{x}_1, \mathbf{x}_2)$ is defined by PD $G_{l_1,l_2}$

$$G_{l_1,l_2}^N(\mathbf{x}_1, \mathbf{x}_2) = G_{l_1}^N(\mathbf{x}_1) G_{l_2/l_1}^N(\mathbf{x}_2|\mathbf{x}_1) = \prod_{n=1}^{N} G_{l_1}(x_n^1) G_{l_2/l_1}(x_n^2|x_n^1).$$

where $G_{l_1}^N(\mathbf{x}_1) = \prod_{n=1}^{N} G_{l_1}(x^1)$ and $G_{l_2/l_1}^N(\mathbf{x}_2|\mathbf{x}_1) = \prod_{n=1}^{N} G_{l_2/l_1}(x^2|x^1)$.
The test, which we denote by $\Phi^N$, is a procedure of making decision about indices of distributions on the base of $N$ observations of objects. The test $\Phi^N$ may be composed of a pair of tests $\varphi_1^N$ and $\varphi_2^N$ for the separate objects: $\Phi^N = (\varphi_1^N, \varphi_2^N)$. For the object characterized by $X_1$ the non-randomized test $\varphi_1^N(\mathbf{x}_1)$ can be determined by partition of the sample space $\mathcal{X}^N$ on $L_1$ disjoint subsets $\mathcal{A}_{l_1}^N = \{\mathbf{x}_1 : \varphi_1^N(\mathbf{x}_1) = l_1\}$, $l_1 = \overline{1, L_1}$, i.e. the set $\mathcal{A}_{l_1}^N$ consists of vectors $\mathbf{x}_1$ for which the PD $G_{l_1}$ is adopted. The probability $\alpha_{l_1|m_1}^N(\varphi_1^N)$ of the erroneous acceptance of PD $G_{l_1}$ provided that $G_{m_1}$ is true, $l_1, m_1 = \overline{1, L_1}$, $m_1 \neq l_1$, is defined by the set $\mathcal{A}_{l_1}^N$

$$\alpha_{l_1|m_1}^N(\varphi_1^N) \triangleq G_{m_1}^N(\mathcal{A}_{l_1}^N). \tag{1}$$

We define the probability to reject $G_{m_1}$, when it is true, as follows

$$\alpha_{m_1|m_1}^N(\varphi_1^N) \triangleq \sum_{l_1 \neq m_1} \alpha_{l_1|m_1}^N(\varphi_1^N) = G_{m_1}^N(\overline{\mathcal{A}_{m_1}^N}). \tag{2}$$

Denote by $\varphi_1$, $\varphi_2$ and $\Phi$ the infinite sequences of tests. Corresponding error probability exponents $E_{l_1|m_1}(\varphi_1)$ for test $\varphi_1$ called reliabilities are defined as

$$E_{l_1|m_1}(\varphi_1) \triangleq \overline{\lim_{N\to\infty}} - \frac{1}{N} \log \alpha_{l_1|m_1}^N(\varphi_1^N), \;\; m_1, l_1 = \overline{1, L_1}. \tag{3}$$

It follows from (2) and (3) that

$$E_{m_1|m_1}(\varphi_1) = \min_{l_1 \neq m_1} E_{l_1|m_1}(\varphi_1), \;\; l_1, m_1 = \overline{1, L_1}, \; l_1 \neq m_1. \tag{4}$$

For the second object characterized by RV $X_2$ the non-randomized test $\varphi_2^N(\mathbf{x}_2, \mathbf{x}_1, l_1)$ depending on vectors $\mathbf{x}_1, \mathbf{x}_2$ and on the index of the hypothesis $l_1$ adopted for $X_1$, can be given by division of the sample space $\mathcal{X}^N$ on $L_2$ disjoint subsets $\mathcal{A}_{l_2/l_1}^N(\mathbf{x}_1) = \{\mathbf{x}_2 : \varphi_2^N(\mathbf{x}_2, \mathbf{x}_1, l_1) = l_2\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$. The set $\mathcal{A}_{l_2/l_1}^N(\mathbf{x}_1)$ consists of vectors $\mathbf{x}_2$ for which the PD $G_{l_2/l_1}$ is adopted. The probabilities of the erroneous acceptance of PD $G_{l_2/l_1}$ provided that $G_{m_2/m_1}$ is true are the following

$$\alpha_{l_2|l_1,m_1,m_2}^N(\varphi_2^N) \triangleq G_{m_2/l_1}^N(\mathcal{A}_{l_2/l_1}^N(\mathbf{x}_1)|\mathbf{x}_1), \; l_1, m_1 = \overline{1, L_1}, \; l_2, m_2 = \overline{1, L_2}, \; m_2 \neq l_2. \tag{5}$$

The corresponding reliabilities, are defined as

$$E_{l_2|l_1,m_1,m_2}(\varphi_2) \triangleq \overline{\lim_{N\to\infty}} - \frac{1}{N} \log \alpha_{l_2|l_1,m_1,m_2}^N(\varphi_2^N), \; l_1, m_1 = \overline{1, L_1}, \; l_2, m_2 = \overline{1, L_2}, \; m_2 \neq l_2. \tag{6}$$

It is clear from (5) and (6) that

$$E_{m_2|l_1,m_1,m_2}(\varphi_2) = \min_{l_2 \neq m_2} E_{l_2|l_1,m_1,m_2}(\varphi_2), \; l_1, m_1 = \overline{1, L_1}, \; l_2, m_2 = \overline{1, L_2}. \tag{7}$$

The matrices $\mathbf{E}(\varphi_1) = \{E_{l_1|m_1}(\varphi_1), l_1, m_1 = \overline{1, L_1}\}$, $\mathbf{E}(\varphi_2) = \{E_{l_2|l_1,m_1,m_2}(\Phi), l_1, m_1 = \overline{1, L_1}, l_2, m_2 = \overline{1, L_2}\}$ are called the reliability matrices of the sequence of tests $\varphi_1$, $\varphi_2$. For two objects we study the probabilities $\alpha_{l_1,l_2|m_1,m_2}(\Phi^N)$ of the erroneous acceptance by the test $\Phi^N$ of the pair of PDs $G_{l_1}, G_{l_2/l_1}$ (or joint PD $G_{l_1,l_2}$) provided that the pair

$G_{m_1}, G_{m_2/l_1}$(or joint PD $G_{m_1,m_2}$) is true, where $(m_1, m_2) \neq (l_1, l_2)$, $l_1, m_1 = \overline{1, L_1}$, $l_2, m_2 = \overline{1, L_2}$. The probability to reject a true PD $G_{m_1,m_2}$, is defined as follows

$$\alpha^N_{m_1,m_2|m_1,m_2}(\Phi^N) \triangleq \sum_{(l_1,l_2)\neq(m_1,m_2)} \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \quad (8)$$

The reliabilities of the sequence of tests $\Phi$ are the following

$$E_{l_1,l_2|m_1,m_2}(\Phi) \triangleq \overline{\lim_{N\to\infty}} -\frac{1}{N} \log \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \quad (9)$$

From (8) and (9) we have

$$E_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2)\neq(m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \quad (10)$$

We call the matrix $\mathbf{E}(\Phi) = \{E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}\}$ the reliability matrix of the sequence of tests $\Phi$. Our aim is to study the reliability matrix of optimal tests, and the conditions of positivity of all its elements.

**Definition :** We call the sequence of tests $\varphi_1^*$ (or $\varphi_2^*$, or $\Phi^*$) logarithmically asymptotically optimal (LAO) if for given positive values of $L_1 - 1$ (or $L_2 - 1$, or $(L_1 - 1)(L_2 - 1)$) diagonal elements of the corresponding matrix $\mathbf{E}(\varphi_1^*)$ (or $\mathbf{E}(\varphi_2^*)$, or $\mathbf{E}(\Phi^*)$) maximal values to all other elements of it are provided.

The following lemma is an extention of the lemmas from [2] and [4].

**Lemma:** *If the reliabilities $E_{l_1|m_1}$ and $E_{l_2|l_1,m_1,m_2}$ of tests $\varphi_1^N$ and $\varphi_2^N$ are strictly positive, then the following relations hold:*

$$E_{l_1,l_2|m_1,m_2} = E_{l_1|m_1} + E_{l_2|l_1,m_1,m_2},\ \textit{for } m_1 \neq l_1,\ m_2 \neq l_2, \quad (11.a)$$

$$E_{l_1,l_2|m_1,m_2} = E_{l_1|m_1},\ \textit{for } m_1 \neq l_1,\ m_2 = l_2, \quad (11.b)$$

$$E_{l_1,l_2|m_1,m_2} = E_{l_2|l_1,m_1,m_2},\ \textit{for } m_1 = l_1,\ m_2 \neq l_2. \quad (11.c)$$

**Proof:** The following equalities are valid for error probabilities:

$$\alpha^N_{l_1,l_2|m_1,m_2} = \alpha^N_{l_1|m_1} \alpha^N_{l_2|l_1,m_1,m_2},\ \text{for } m_1 \neq l_1,\ m_2 \neq l_2, \quad (12.a)$$

$$\alpha^N_{l_1,l_2|m_1,m_2} = \alpha^N_{l_1|m_1}(1 - \alpha^N_{l_2|l_1,m_1,m_2}),\ \text{for } m_1 \neq l_1,\ m_2 = l_2, \quad (12.b)$$

$$\alpha^N_{l_1,l_2|m_1,m_2} = (1 - \alpha^N_{l_1|m_1})\alpha^N_{l_2|l_1,m_1,m_2},\ \text{for } m_1 = l_1,\ m_2 \neq l_2. \quad (12.c)$$

Thus, in light of (3), (6) and (9), we can obtain (11).

We shall reformulate now the Theorem from [1] for the case of one object with $L_1$ hypotheses. This requires some notions and notations. We define the entropy $H_{Q_{\mathbf{x}_1}}(X_1)$ and the informational divergence $D(Q_{\mathbf{x}_1}||G_{l_1})$, $l_1 = \overline{1, L_1}$, as follows:

$$H_{Q_{\mathbf{x}_1}}(X_1) \triangleq -\sum_{x^1\in\mathcal{X}} Q_{\mathbf{x}_1}(x^1) \log Q_{\mathbf{x}_1}(x^1),$$

$$D(Q_{\mathbf{x}_1}||G_{l_1}) \triangleq \sum_{x^1\in\mathcal{X}} Q_{\mathbf{x}_1}(x^1) \log \frac{Q_{\mathbf{x}_1}(x^1)}{G_{l_1}(x^1)}.$$

For given positive numbers $E_{1|1}, ..., E_{L-1|L-1}$, let us consider the following sets of PDs $Q = \{Q(x^1), x^1 \in \mathcal{X}\}$:

$$\mathcal{R}_{l_1} \triangleq \{Q: \ D(Q||G_{l_1}) \leq E_{l_1|l_1}\}, \ l_1 = \overline{1, L_1 - 1}, \quad (13a)$$

$$\mathcal{R}_{L_1} \triangleq \{Q: \ D(Q||G_{l_1}) > E_{l_1|l_1}, \ l_1 = \overline{1, L_1 - 1}\}, \quad (13b)$$

and the elements of the reliability matrix $\mathbf{E}^*$ of the LAO test:

$$E^*_{l_1|l_1} = E^*_{l_1|l_1}(E_{l_1|l_1}) \triangleq E_{l_1|l_1}, \ l_1 = \overline{1, L_1 - 1}, \quad (14a)$$

$$E^*_{l_1|m_1} = E^*_{l_1|m_1}(E_{l_1|l_1}) \triangleq \inf_{Q \in \mathcal{R}_{l_1}} D(Q||G_{m_1}), \ m_1 = \overline{1, L_1}, \ m_1 \neq l_1, \ l_1 = \overline{1, L_1 - 1}, \quad (14b)$$

$$E^*_{L_1|m_1} = E^*_{L_1|m_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \triangleq \inf_{Q \in \mathcal{R}_{L_1}} D(Q||G_{m_1}), \ m_1 = \overline{1, L_1 - 1}, \quad (14c)$$

$$E^*_{L_1|L_1} = E^*_{L_1|L_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \triangleq \min_{l_1 = \overline{1, L_1 - 1}} E^*_{l_1|L_1}. \quad (14d)$$

**Theorem 1[1]:** *If all distributions $G_{l_1}$, $l_1 = \overline{1, L_1}$, are different in the sense that $D(G_{l_1}||G_{m_1}) > 0$, $l_1 \neq m_1$, and the positive numbers $E_{1|1}, E_{2|2}, ..., E_{L-1|L-1}$ are such that the following inequalities hold*

$$E_{1|1} < \min_{l_1 = \overline{2, L_1}} D(G_{l_1}||G_1), \quad (15)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$E_{m_1|m_1} < \min\left(\min_{l_1 = \overline{m_1 + 1, L_1}} D(G_{l_1}||G_{m_1}), \ \min_{l_1 = \overline{1, m_1 - 1}} E^*_{l_1|m_1}(E_{l_1|l_1})\right), \ m_1 = \overline{2, L_1 - 1},$$

*then there exists a LAO sequence of tests $\varphi_1^*$, the reliability matrix of which $\mathbf{E}^* = \{E_{m|l}(\varphi_1^*)\}$ is defined in (14) and all elements of it are positive.*

*When one of the inequalities (15) is violated, then at least one element of the matrix $\mathbf{E}^*$ is equal to 0.*

## 2 LAO Testing of $L_2$ Hypotheses for the Second (Dependent) Object

We need some notions and estimates from the method of types [7] - [10]. The type of a vector $\mathbf{x_1}$ is a PD

$$Q_{\mathbf{x_1}} = \{Q_{\mathbf{x_1}}(x^1) = \frac{1}{N} N(x^1|\mathbf{x_1}), x^1 \in \mathcal{X}\},$$

where $N(x^1|\mathbf{x^1})$ is the number of repetitions of the symbol $x^1$ in vector $\mathbf{x_1}$. The subset of $\mathcal{P}(\mathcal{X})$ consisting of the possible types of sequences $\mathbf{x_1} \in \mathcal{X}$ is denoted by $\mathcal{P}_N(\mathcal{X})$. The set of all vectors $\mathbf{x_1}$ of the type $Q_{\mathbf{x_1}}$ is denoted by $\mathcal{T}^N_{Q_{\mathbf{x_1}}}(X_1)$, remark that $\mathcal{T}^N_Q(X_1) = \emptyset$ for $Q \notin \mathcal{P}_N(\mathcal{X})$. The following estimates for the set $\mathcal{T}^N_{Q_{\mathbf{x_1}}}(X_1)$ of all vectors of the same type $Q_{\mathbf{x_1}}$ hold

$$(N+1)^{-|\mathcal{X}|} \exp\{NH_{Q_{\mathbf{x_1}}}(X_1)\} \leq |\mathcal{T}^N_{Q_{\mathbf{x_1}}}(X_1)| \leq \exp\{NH_{Q_{\mathbf{x_1}}}(X_1)\}.$$

For a pair of sequences $(\mathbf{x_1}, \mathbf{x_2}) \in \mathcal{X}^N \times \mathcal{X}^N$ let $N(x^1, x^2|\mathbf{x_1}, \mathbf{x_2})$ be the number of occurences of pair $(x^1, x^2) \in \mathcal{X} \times \mathcal{X}$ in the same places of the pair of vectors $(\mathbf{x_1}, \mathbf{x_2})$. The joint type of the pair $(\mathbf{x_1}, \mathbf{x_2})$ is PD $Q_{\mathbf{x_1}, \mathbf{x_2}} = \{Q_{\mathbf{x_1}, \mathbf{x_2}}(x^1, x^2), x^1, x^2 \in \mathcal{X}\}$ defined by

$$Q_{\mathbf{x_1}, \mathbf{x_2}}(x^1, x^2) \triangleq \frac{1}{N} N(x^1, x^2 \mid \mathbf{x_1}, \mathbf{x_2}), \qquad x^1, x^2 \in \mathcal{X}.$$

The conditional type of $\mathbf{x}_2$ for given $\mathbf{x}_1$ is the conditional distribution $V_{\mathbf{x}_1,\mathbf{x}_2} \triangleq \{V_{\mathbf{x}_2,\mathbf{x}_1}(x^2|x^1), x^1, x^2 \in \mathcal{X}\}$ defined by

$$V_{\mathbf{x}_1,\mathbf{x}_2}(x^2|x^1) \triangleq \frac{Q_{\mathbf{x}_1,\mathbf{x}_2}(x^1,x^2)}{Q_{\mathbf{x}_1}(x^1)} = \frac{N(x^1,x^2|\mathbf{x}_1,\mathbf{x}_2)}{N(x^1 \mid \mathbf{x}_1)}, \qquad x^1, x^2 \in \mathcal{X}.$$

The conditional entropy of RV $X_2$ for given $X_1$ is:

$$H_{Q_{\mathbf{x}_1},V_{\mathbf{x}_1,\mathbf{x}_2}}(X_2 \mid X_1) = -\sum_{x^1,x^2} Q_{\mathbf{x}_1}(x^1)V_{\mathbf{x}_1,\mathbf{x}_2}(x^2|x^1)\log V_{\mathbf{x}_1,\mathbf{x}_2}(x^2|x^1).$$

For some conditional PD $V = \{V(x^2|x^1), x^1, x^2 \in \mathcal{X}\}$ the conditional divergences of PD $\{Q_{\mathbf{x}_1}(x^1)V(x^2|x^1), x^1, x^2 \in \mathcal{X}\}$ with respect to PD $\{Q_{\mathbf{x}_1}(x^1)\,G_{l_2/l_1}(x^2|x^1), x^1, x^2 \in \mathcal{X}\}$ for all $l_1, l_2$ is defined as follows

$$D(V||G_{l_2/l_1}|Q_{\mathbf{x}_1}) \triangleq \sum_{x^1,x^2} Q_{\mathbf{x}_1}(x^1)V(x^2|x^1)\log\frac{V(x^2|x^1)}{G_{l_2/l_1}(x^2|x^1)},$$

also

$$D(G_{l_2/l_1}||G_{m_2/l_1}|Q_{\mathbf{x}_1}) \triangleq \sum_{x^1,x^2} Q_{\mathbf{x}_1}(x^1)G_{l_2/l_1}(x^2|x^1)\log\frac{G_{l_2/l_1}(x^2|x^1)}{G_{m_2/l_1}(x^2|x^1)}.$$

The family of vectors $\mathbf{x}_2$ of the conditional type $V_{\mathbf{x}_1,\mathbf{x}_2}$ for given $\mathbf{x}_1$ of the type $Q_{\mathbf{x}_1}$ is denoted by $\mathcal{T}^N_{Q_{\mathbf{x}_1},V_{\mathbf{x}_1,\mathbf{x}_2}}(X_2 \mid \mathbf{x}_1)$ and called $V$-shell of $\mathbf{x}_1$. The set of all possible $V$-shells for $\mathbf{x}_1$ of type $Q_{\mathbf{x}_1}$ is denoted by $\mathcal{V}_N(\mathcal{X}, Q_{\mathbf{x}_1})$. For any conditional type $V_{\mathbf{x}_1,\mathbf{x}_2}$ and $\mathbf{x}_1 \in \mathcal{T}^N_{Q_{\mathbf{x}_1}}(X_1)$ it is known that

$$(N+1)^{-|\mathcal{X}|^2}\exp\{NH_{Q_{\mathbf{x}_1},V_{\mathbf{x}_1,\mathbf{x}_2}}(X_2|X_1)\} \leq |\, \mathcal{T}^N_{Q_{\mathbf{x}_1},V_{\mathbf{x}_1,\mathbf{x}_2}}(X_2 \mid \mathbf{x}_1)\,| \leq$$
$$\leq \exp\{NH_{Q_{\mathbf{x}_1},V_{\mathbf{x}_1,\mathbf{x}_2}}(X_2|X_1)\}. \tag{16}$$

For given positive numbers $E_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2-1}$, for each pair $l_1, m_1 = \overline{1, L_1}$ and for $Q_{\mathbf{x}_1}$ let us define the following regions and values:

$$\mathcal{R}_{l_2/l_1}(Q_{\mathbf{x}_1}) \triangleq \{V:\ D(V||G_{l_2/l_1}|Q_{\mathbf{x}_1}) \leq E_{l_2|l_1,m_1,l_2}\},\ \ l_2 = \overline{1, L_2-1}, \tag{17.a}$$

$$\mathcal{R}_{L_2/l_1}(Q_{\mathbf{x}_1}) \triangleq \{V:\ D(V||G_{l_2/l_1}|Q_{\mathbf{x}_1}) > E_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2-1}\}, \tag{17.b}$$

$$E^*_{l_2|l_1,m_1,l_2} = E^*_{l_2|l_1,m_1,l_2}(E_{l_2|l_1,m_1,l_2}) \triangleq E_{l_2|l_1,m_1,l_2},\ \ l_2 = \overline{1, L_2-1}, \tag{18.a}$$

$$E^*_{l_2|l_1,m_1,m_2} = E^*_{l_2|l_1,m_1,m_2}(E_{l_2|l_1,m_1,l_2}) \triangleq \inf_{V\in\mathcal{R}_{l_2/l_1}(Q_{\mathbf{x}_1})} D(V||G_{m_2/l_1}|Q_{\mathbf{x}_1}),$$
$$m_2 = \overline{1, L_2},\ , m_2 \neq l_2,\ l_2 = \overline{1, L_2-1}, \tag{18.b}$$

$$E^*_{L_2|l_1,m_1,m_2} = E^*_{L_2|l_1,m_1,m_2}(E_{1|l_1,m_1,1}, E_{2|l_1,m_1,2}, \ldots, E_{L-1|l_1,m_1,L-1}) \triangleq$$
$$\triangleq \inf_{V\in\mathcal{R}_{L_2/l_1}(Q_{\mathbf{x}_1})} D(V||G_{m_2/l_1}|Q_{\mathbf{x}_1}),\ m_2 = \overline{1, L_2-1}, \tag{18.c}$$

$$E^*_{L_2|l_1,m_1,L_2} = E^*_{L_2|l_1,m_1,L_2}(E_{1|l_1,m_1,1}, E_{2|l_1,m_1,2}, \ldots, E_{L_2-1|l_1,m_1,L_2-1}) \triangleq$$
$$\triangleq \min_{l_2=\overline{1,L_2-1}} E^*_{l_2|l_1,m_1,L_2}. \tag{18.d}$$

The following theorem is an anolog of Theorem 1 [1] for the dependent object.

**Theorem 2:** *If for given* $\mathbf{x}_1, m_1, l_1$, *with* $m_1, l_1 = \overline{1, L_1}$, *all conditional PDs* $G_{l_2/l_1}$, $l_2 = \overline{1, L_2}$, *are different in the sense that* $D(G_{l_2/l_1} \| G_{m_2/l_1} | Q_{\mathbf{x}_1}) > 0$, $l_2 \neq m_2$, $m_2 = \overline{1, L_2}$, *when the positive numbers* $E_{1|l_1,m_1,1}, E_{2|l_1,m_1,2}, \ldots, E_{L_2-1|l_1,m_1,L_2-1}$ *are such that the following inequalities hold*

$$E_{1|l_1,m_1,1} < \min_{l_2=\overline{2,L}} D(G_{l_2/l_1} \| G_{1/l_1} | Q_{\mathbf{x}_1}),$$

*and for* $m_2 = \overline{2, L_2 - 1}$,

$$E_{m_2|l_1,m_1,m_2} < \min\Big( \min_{l_2=\overline{m_2+1,L_2}} D(G_{l_2/l_1} \| G_{m_2/l_1} | Q_{\mathbf{x}_1}),\ \min_{l_2=\overline{1,m_2-1}} E^*_{l_2|l_1,m_1,m_2}(E_{l_2|l_1,m_1,l_2}) \Big), \tag{19}$$

*then there exists a LAO sequence of tests, the reliability matrix of which* $\mathbf{E}(\varphi_2^*) = \mathbf{E}^*(l_1, m_1, \mathbf{x}_1) = \{E^*_{l_2|l_1,m_1,m_2}\}$ *is defined in* (18) *and all elements of it are positive.*

*When one of the inequalities* (19) *is violated, then at least one element of the matrix* $\mathbf{E}^*(l_1, m_1, \mathbf{x}_1)$ *is equal to* 0.

**Proof:** For $\mathbf{x}_1 \in \mathcal{X}^N$, $\mathbf{x}_2 \in \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)$ the conditional probability $G^N_{m_2/l_1}(\mathbf{x}_2 \mid \mathbf{x}_1)$ can be presented as follows (for brevity in this proof we will write $V$ $(V \in \mathcal{V}_N(\mathcal{X}, Q_{\mathbf{x}_1}))$ instead of $V_{\mathbf{x}_1,\mathbf{x}_2}$ and $Q$ $(Q \in \mathcal{P}_N(\mathcal{X}))$ instead of $Q_{\mathbf{x}_1}$

$$G^N_{m_2/l_1}(\mathbf{x}_2 \mid \mathbf{x}_1) = \prod_{n=1}^{N} G_{m_2/l_1}(x_n^2|x_n^1) = \prod_{x^1,x^2} G_{m_2/l_1}(x^2|x^1)^{N(x^1,x^2|\mathbf{x}_1,\mathbf{x}_2)} =$$

$$= \prod_{x^1,x^2} G_{m_2/l_1}(x^2|x^1)^{NQ(x^1)V(x^2|x^1)} = \exp\{N \sum_{x^1,x^2} (-Q(x^1)V(x^2|x^1) \log \frac{V(x^2|x^1)}{G_{m_2/l_1}(x^2|x^1)} +$$

$$+Q(x^1)V(x^2|x^1) \log V(x^2|x^1))\} = \exp\{-N[D(V \| G_{m_2/l_1}|Q) + H_{Q,V}(X_2 \mid X_1)]\}. \tag{20}$$

We will prove that the sequence of tests $\varphi_2^*$, defined for each $\mathbf{x}_1 \in \mathcal{X}^N$ and $l_1, m_1$ by the following assembly of sets of types

$$\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1) = \bigcup_{V \in \mathcal{R}_{l_2/l_1}(Q)} \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1),\ l_2 = \overline{1, L_2}, \tag{21}$$

is LAO and its matrix $\mathbf{E}^*(l_1, m_1, \mathbf{x}_1)$ is defined in (18). First we show that each $N$-vector $\mathbf{x}_2$ is in one and only in one of $\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1)$, that is

$$\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1) \bigcap \mathcal{B}^{(N)}_{m_2/l_1}(\mathbf{x}_1) = \emptyset,\ l_2 = \overline{1, L_2 - 1},\ m_2 = \overline{l_2 + 1, L_2} \ \text{ and } \ \bigcup_{l_2=1}^{L_2} \mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1) = \mathcal{X}^N.$$

Really, (17.b) and (21) show that

$$\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1) \bigcap \mathcal{B}^{(N)}_{L_2/l_1}(\mathbf{x}_1) = \emptyset,\ l_2 = \overline{1, L_2 - 1},$$

and for $l_2 = \overline{1, L_2 - 2}$, $m_2 = \overline{l_2 + 1, L_2 - 1}$, for each $\mathbf{x}_1 \in \mathcal{X}^N$ let us consider arbitrary $\mathbf{x}_2 \in \mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1)$. It follows from (17.a) and (21) that if $Q \in P_N(\mathcal{X})$ there are $V \in \mathcal{V}_N(\mathcal{X}, Q)$ such that $D(V \| G_{l_2/l_1}|Q) \leq E_{l_2|l_1,m_1,l_2}$ and $\mathbf{x}_2 \in \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)$. From (17) – (19) we have $E_{m_2|l_1,m_1,m_2} < E^*_{l_2|l_1,m_1,m_2}(E_{l_2|l_1,m_1,l_2}) < D(V \| G_{m_2/l_1}|Q)$. From definition (21) we see that $\mathbf{x}_2 \notin \mathcal{B}^{(N)}_{m_2/l_1}(\mathbf{x}_1)$.

Now for $m_2 = \overline{1, L_2 - 1}$, using (5), (16), (17), (18), (20) and (21) we can upper estimate $\alpha^{*N}_{m_2|l_1,m_1,m_2}$ as follows:

$$\alpha^{*N}_{m_2|l_1,m_1,m_2} = G^N_{m_2/l_1}\left(\overline{\mathcal{B}^{(N)}_{m_2/l_1}(\mathbf{x}_1)}|\mathbf{x}_1\right) =$$
$$= G^N_{m_2/l_1}\Big(\bigcup_{V:D(V||G_{m_2/l_1}|Q)>E_{m_2|l_1,m_1,m_2}} \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)|\mathbf{x}_1\Big) \leq$$
$$\leq (N+1)^{|\mathcal{X}|^2} \sup_{V:D(V||G_{m_2/l_1}|Q_{\mathbf{x}_1})>E_{m_2|l_1,m_1,m_2}} G_{m_2/l_1}(\mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)|\mathbf{x}_1) \leq$$
$$\leq (N+1)^{|\mathcal{X}|^2} \sup_{V:D(V||G_{m_2/l_1}|Q)>E_{m_2|l_1,m_1,m_2}} \exp\{-ND(V||G_{m_2/l_1}|Q)\} \leq$$
$$\leq \exp\{-N[\inf_{V:D(V||G_{m_2/l_1}|Q)>E_{m_2|l_1,m_1,m_2}} D(V||G_{m_2/l_1}|Q) - o_N(1)]\} \leq$$
$$\leq \exp\{-N[E_{m_2|l_1,m_1,m_2} - o_N(1)]\},$$

where $o_N(1) \to 0$ with $N \to \infty$.

Then we obtain upper and lower estimates for error probabilities from where we can conclude that the elements of reliability matrix of the test $\varphi_2^*$ are defined in (18):

$$\alpha^{*N}_{l_2|l_1,m_1,m_2} = G^N_{m_2/l_1}(\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1)|\mathbf{x}_1) = G^N_{m_2/l_1}\Big(\bigcup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)|\mathbf{x}_1\Big) \leq$$
$$\leq (N+1)^{|\mathcal{X}|^2} \sup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} G^N_{m_2/l_1}(\mathcal{T}^N_{Q_{\mathbf{x}_1},V}(X_2|\mathbf{x}_1)|\mathbf{x}_1) \leq$$
$$\leq (N+1)^{|\mathcal{X}|^2} \sup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} \exp\{-ND(V||G_{m_2/l_1}|Q)\} = \tag{22}$$
$$= \exp\{-N[\inf_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} D(V||G_{m_2/l_1}|Q) - o_N(1))]\},$$

$$\alpha^{*N}_{l_2|l_1,m_1,m_2} = G^N_{m_2/l_1}(\mathcal{B}^{(N)}_{l_2/l_1}(\mathbf{x}_1)|\mathbf{x}_1) = G^N_{m_2/l_1}\Big(\bigcup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} \mathcal{T}^N_{Q,V}(X_2|\mathbf{x}_1)|\mathbf{x}_1\Big) \geq$$
$$\geq \sup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} G^N_{m_2/l_1}(\mathcal{T}_{Q,V}(X_2|\mathbf{x}_1)|\mathbf{x}_1) \geq$$
$$\geq (N+1)^{-|\mathcal{X}|^2} \sup_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} \exp\{-ND(V||G_{m_2/l_1}|Q)\} = \tag{23}$$
$$= \exp\{-N[\inf_{V:V\in\mathcal{R}_{l_2/l_1}(Q)} D(V||G_{m_2/l_1}|Q) + o_N(1))]\}.$$

Taking into account (22), (23) and the continuity of the functional $D(V||G_{m_2/l_1}|Q)$ we obtain that $\lim\limits_{N\to\infty} -N^{-1}\log\alpha^{*N}_{l_2|l_1,m_1,m_2}$ exists and in correspondence with (18b) equals to $E^*_{l_2|l_1,m_1,m_2}$. Thus $E_{l_2|l_1,m_1,m_2}(\varphi_2^*) = E^*_{l_2|l_1,m_1,m_2}$, $m_2 = \overline{1, L_2}$, $l_2 = \overline{1, L_2}$.

The proof of the first part of the theorem will be finished if we show that the sequence of the tests $\varphi_2^*$ is LAO, that is for given $E_{1|l_1,m_1,1}, \ldots, E_{L_2-1|l_1,m_1,L_2-1}$ and for any sequence of tests $\varphi_2^{**}$ for all $m_2, l_2 = \overline{1, L_2}$, $E^{**}_{l_2|l_1,m_1,m_2} \leq E^*_{l_2|l_1,m_1,m_2}$.

Consider sequence $\varphi_2^{**}$ of tests, which is defined by the sets $\mathcal{D}^{(N)}_{1/l_1}, \mathcal{D}^{(N)}_{2/l_1}, \ldots, \mathcal{D}^{(N)}_{L_2/l_1}$ such that $E^{**}_{l_2|l_1,m_1,m_2} \geq E^*_{l_2|l_1,m_1,m_2}$ for some $l_2, m_2$. For large enough $N$ we can replace this condition with the following inequality

$$\alpha^{**N}_{l_2|l_1,m_1,m_2} \leq \alpha^{*N}_{l_2|l_1,m_1,m_2}. \tag{24}$$

Let us examine the sets $\mathcal{D}_{l_2/l_1}^{(N)} \cap \mathcal{B}_{l_2/l_1}^{(N)}$, $l_2 = \overline{1, L_2 - 1}$. This intersection cannot be empty, because in that case

$$\alpha_{l_2|l_1,m_1,l_2}^{**N} = G_{l_2/l_1}^{N}(\overline{\mathcal{D}}_{l_2/l_1}^{(N)}|\mathbf{x}_1) \geq G_{l_2/l_1}^{N}(\mathcal{B}_{l_2/l_1}^{(N)}|\mathbf{x}_1) \geq$$

$$\geq \sup_{V: D(V||G_{l_2/l_1}|Q) \leq E_{l_2|l_1,m_1,l_2}} G_{l_2/l_1}^{*}(\mathcal{T}_{Q,V}^{N}(X_2|\mathbf{x}_1)) \geq \exp\{-N(E_{l_2|l_1,m_1,l_2} + o_N(1))\},$$

which is at variance with (24). Let us show that $\mathcal{D}_{l_2/l_1}^{(N)} \cap \mathcal{B}_{m_2/l_1}^{(N)} = \emptyset$, $m_2, l_2 = \overline{1, L_2 - 1}$, $l_2 \neq m_2$. If there exists $V$ such that $D(V||G_{m_2/l_1}|Q) \leq E_{m_2|l_1,m_1,m_2}$ and $\mathcal{T}_{V,Q}^{N}(X_2|\mathbf{x}_1) \in \mathcal{D}_{l_2/l_1}^{(N)}$, then

$$\alpha_{l_2|l_1,m_1,m_2}^{**N} = G_{m_2/l_1}^{N}(\mathcal{D}_{l_2/l_1}^{(N)}|\mathbf{x}_1) > G_{m_2/l_1}^{N}(\mathcal{T}_{V,Q}^{N}(X_2|\mathbf{x}_1)) \geq \exp\{-N[E_{m_2|l_1,m_1,m_2} + o_N(1)]\}.$$

When $\emptyset \neq \mathcal{D}_{l_2/l_1}^{(N)} \cap \mathcal{T}_{V,Q}^{N}(X_2|\mathbf{x}_1) \neq \mathcal{T}_{V,Q}^{N}(X_2|\mathbf{x}_1)$, we also obtain that

$$\alpha_{l_2|l_1,m_1,m_2}^{**N} = G_{m_2/l_1}^{N}(\mathcal{D}_{l_2/l_1}^{(N)}|\mathbf{x}_1) > G_{m_2/l_1}^{N}(\mathcal{D}_{l_2/l_1}^{(N)} \bigcap \mathcal{T}_{V,Q}^{N}(X_2|\mathbf{x}_1)|\mathbf{x}_1) \geq$$

$$\geq \exp\{-N(E_{m_2|l_1,m_2,m_2} + O_N(1))\}.$$

Thus we conclude that $E_{l_2|l_1,m_1,m_2}^{**} < E_{m_2|l_1,m_1,m_2}$, which contradicts to (7). Hence we obtain that $\mathcal{D}_{l_2/l_1}^{(N)} \cap \mathcal{B}_{l_2/l_1}^{(N)} = \mathcal{B}_{l_2/l_1}^{(N)}$ for $l_2 = \overline{1, L_2 - 1}$. The following intersection $\mathcal{D}_{l_2/l_1}^{(N)} \cap \mathcal{B}_{L_2/l_1}^{(N)}$ is empty too, because otherwise

$$\alpha_{L_2|l_1,m_1,m_2}^{**N} \geq \alpha_{L_2|l_1,m_1,m_2}^{*N},$$

which contradicts to (24), it means that $\mathcal{D}_{l_2/l_1}^{(N)} = \mathcal{B}_{l_2/l_1}^{(N)}$, for all $l_2 = \overline{1, L_2}$. So we have proved that test $\varphi_2^*$ is the unique LAO test.

The proof of the second assertion of the theorem is simple. If one of the conditions (19) is violated, then it follows from (17) and (18) that at least one of the elements $E_{l_2|l_1,m_1,m_2}$ is equal to 0.

## 3 LAO Testing of $L_1, L_2$ Hypotheses for Two Dependent Objects

Let us define the following subsets of $\mathcal{P}(\mathcal{X})$ for given strictly positive elements $E_{L_1,l_2|l_1,l_2}$, $E_{l_1,L_2|l_1,l_2}$, $l_1 = \overline{1, L_1 - 1}$, $l_2 = \overline{1, L_2 - 1}$:

$$\mathcal{R}_{l_1} \triangleq \{Q : D(Q||G_{l_1}) \leq E_{L_1,l_2|l_1,l_2}\}, \ l_1 = \overline{1, L_1 - 1}, \ l_2 = \overline{1, L_2 - 1},$$

$$\mathcal{R}_{l_2/l_1}(Q_{\mathbf{x}_1}) \triangleq \{V : D(V||G_{l_2/l_1}|Q_{\mathbf{x}_1}) \leq E_{l_1,L_2|l_1,l_2}\}, \ l_1 = \overline{1, L_1 - 1}, \ l_2 = \overline{1, L_2 - 1},$$

$$\mathcal{R}_{L_1} \triangleq \{Q : D(Q||G_{l_1}) > E_{L_1,l_2|l_1,l_2}, \ l_1 = \overline{1, L_1 - 1}, \ l_2 = \overline{1, L_2 - 1}\},$$

$$\mathcal{R}_{L_2/l_1}(Q_{\mathbf{x}_1}) \triangleq \{V : D(V||G_{l_2/l_1}|Q_{\mathbf{x}_1}) > E_{l_1,L_2|l_1,l_2}, \ l_1 = \overline{1, L_1 - 1}, \ l_2 = \overline{1, L_2 - 1}\}.$$

Assume also that

$$E_{l_1,L_2|l_1,l_2}^{*} \triangleq E_{l_1,L_2|l_1,l_2}, \ E_{L_1,l_2|l_1,l_2}^{*} \triangleq E_{L_1,l_2|l_1,l_2}, \ l_1 = \overline{1, L_1 - 1}, \ l_2 = \overline{1, L_2 - 1}, \qquad (25.a)$$

$$E^*_{l_1,l_2|m_1,l_2} \triangleq \inf_{Q:Q\in R_{l_1}} D(Q||G_{m_1}), \quad m_1 \neq l_1 \tag{25.b}$$

$$E^*_{l_1,l_2|l_1,m_2} \triangleq \inf_{V:V\in R_{l_2/l_1}(Q_{x_1})} D(V||G_{m_2/m_1}|Q_{x_1}), \quad m_2 \neq l_2 \tag{25.c}$$

$$E^*_{l_1,l_2|m_1,m_2} \triangleq E^*_{m_1,l_2|m_1,m_2} + E^*_{l_1,m_2|m_1,m_2}, \quad m_i \neq l_i,\ i=1,2, \tag{25.d}$$

$$E^*_{m_1,m_2|m_1,m_2} \triangleq \min_{(l_1,l_2)\neq(m_1,m_2)} E^*_{l_1,l_2|m_1,m_2}. \tag{25.e}$$

**Theorem 3:** *If all distributions $G_{m_1}$, $m_1=\overline{1,L_1}$, are different, that is $D(G_{l_1}||G_{m_1})>0$, $l_1\neq m_1$, $l_1,m_1=\overline{1,L_1}$, and all conditional distributions $G_{l_2/l_1}$, $l_2=\overline{1,L_2}$, are also different for all $l_1=\overline{1,L_1}$, in the sense that $D(G_{l_2/l_1}||G_{m_2/l_1}|Q_{x_1})>0$, $l_2\neq m_2$, then the following two statements are valid.*

*When given elements $E_{L_1,l_2|l_1,l_2}$ and $E_{l_1,L_2|l_1,l_2}$, $l_1=\overline{1,L_1-1}$, $l_2=\overline{1,L_2-1}$, meet the following conditions*

$$0 < E_{L_1,l_2|1,l_2} < \min_{m_1=\overline{2,L_1}} D(G_{m_1}||G_1), \tag{26.a}$$

$$0 < E_{l_1,L_2|l_1,1} < \min_{m_2=\overline{2,L_2}} D(G_{m_2/l_1}||G_{1/l_1}|Q_{x_1}), \tag{26.b}$$

$$0 < E_{L_1,l_2|l_1,l_2} < \min[\min_{m_1=\overline{1,l_1-1}} E^*_{m_1,l_2|l_1,l_2}, \min_{m_1=\overline{l_1+1,L_1}} D(G_{m_1}||G_{l_1})],\ l_1=\overline{2,L_1-1}, \tag{26.c}$$

$$0 < E_{l_1,L_2|l_1,l_2} < \min[\min_{m_2=\overline{1,l_2-1}} E^*_{l_1,m_2|l_1,l_2}, \min_{m_2=\overline{l_2+1,L_2}} D(G_{m_2/l_1}||G_{l_2/l_1}|Q_{x_1})],\ l_2=\overline{2,L_2-1}, \tag{26.d}$$

*then there exists a LAO test sequence $\Phi^*$, the reliability matrix of which $\mathbf{E}(\Phi^*)=\{E_{l_1,l_2|m_1,m_2}(\Phi^*)\}$ is defined in (25) and all elements of it are positive.*

*When even one of the inequalities (26) is violated, then at least one element of the matrix $\mathbf{E}(\Phi^*)$ is equal to 0.*

**Proof:** It is known from [5] that $E_{l_1|l_1}=E_{L_1|l_1}, l_1=\overline{1,L_1-1}$. Analogously we can deduce that

$$E^*_{l_2|l_1,m_1,l_2} = E^*_{L_2|l_1,m_1,l_2},\ l_2=\overline{1,L_2-1}. \tag{27}$$

Applying theorem of Kuhn-Tucker in (18.b) we can show that the elements $E^*_{l_2|l_1,m_1,l_2}$, $l_2=\overline{1,L_2-1}$ can be determined by elements $E^*_{l_2|l_1,m_1,m_2}$, $m_2\neq l_2$, $l_2=\overline{1,L_2}$,

$$E^*_{l_2|l_1,m_1,l_2}(E^*_{l_2|l_1,m_1,m_2}) \triangleq \inf_{V:D(V||G_{m_2/l_1}|Q_{x_1})\leq E^*_{l_2|l_1,m_1,m_2}} D(V||G_{l_2/l_1}|Q_{x_1}).$$

From (19) it is clear that $E^*_{m_2|l_1,m_1,m_2}$ can be equal only to one of $E^*_{l_2|l_1,m_1,m_2}$, $l_2=\overline{m_2+1.L_2}$. Assume that (27) is not correct, that is $E^*_{m_2|l_1,m_1,m_2}=E^*_{l_2|l_1,m_1,m_2}$, $l_2=\overline{m_2+1,L_2-1}$. From (18.b) it follows that

$$E^*_{l_2|l_1,m_1,l_2}(E^*_{l_2|l_1,m_1,m_2}) \triangleq \inf_{V:D(V||G_{m_2/l_1}|Q_{x_1})\leq E^*_{l_2|l_1,m_1,m_2}} D(V||G_{l_2/l_1}|Q_{x_1}) =$$

$$= \inf_{V:D(V||G_{m_2/l_1}|Q_{x_1})\leq E^*_{m_2|l_1,m_1,m_2}} D(V||G_{l_2/l_1}|Q^*_{x_1}) = E^*_{m_2|l_1,m_1,l_2},$$

$$m_2, l_2 = \overline{1,L_2-1}, m_2 < l_2,$$

but from conditions (19) it follows that $E^{*}_{l_2|l_1,m_1,l_2} < E^{*}_{m_2|l_1,m_1,l_2}$ for $m_2 = \overline{1, l_2 - 1}$. Our assumption is not true, thus (27) is valid. Hence we can rewrite the inequalities (15) and (19) as follows:

$$0 < E_{L_1|1} < \min_{m_1=\overline{2,L_1}} D(G_{m_1}||G_1), \tag{28.a}$$

$$0 < E_{L_2|l_1,m_1,1} < \min_{m_2=\overline{2,L_2}} D(G_{m_2/l_1}||G_{1/l_1}|Q_{\mathbf{x}_1}), \tag{28.b}$$

$$0 < E_{L_1|l_1} < \min[\min_{m_1=\overline{1,l_1-1}} E^{*}_{m_1|l_1}, \min_{m_1=\overline{l_1+1,L_1}} D(G_{m_1}||G_{l_1})],\ l_1 = \overline{2, L_1 - 1}; \tag{28.c}$$

$$0 < E_{L_2|l_1,m_1,l_2} < \min[\min_{m_2=\overline{1,l_2-1}} E^{*}_{m_2|l_1,m_1,l_2}, \min_{m_2=\overline{l_2+1,L_2}} D(G_{m_2/l_1}||G_{l_2/l_1}|Q_{\mathbf{x}_1})],\ l_2 = \overline{2, L_2 - 1}. \tag{28.d}$$

According to Theorem 1 and Theorem 2 there exist LAO sequences of tests $\varphi_1^*$ and $\varphi_2^*$, for the first and the second objects, such that the elements of the matrices $\mathbf{E}^*$ and $\mathbf{E}^*(l_1, m_1, \mathbf{x_1})$ are determined in (14) and (18). The inequalities (28.a) , (28.c) are equivalent to the inequalities (15) and (28.b), (28.d) are equivalent to the inequalities (19). Then using Lemma we deduce that the reliability matrix $\mathbf{E}(\Phi^*)$ can be determined in (25) . When one of inequalities of (26.a) and (26.c) ((26.b) and (26.d)) is violated then using (11.b) and (25.b) ( (11.c) and (25.c)) we see that some elements of matrix must be equal to 0.

**Corollary**: When the objects are dependent only statistically, that is the $L_1$ PDs are known for the first objects, and the second object depending on distribution index of the first can be distributed according to one of given conditional distributions, then $G_{m_2/l_1}(x^2|x^1) = G_{m_2/l_1}(x^2)$, $m_2 = \overline{1, L_2}$, $l_1 = \overline{1, L_1}$. Thus the first object characterized by RV $X_1$ can have one of given $L_1$ PDs $G_{l_1} = \{G_{l_1}(x^1),\ x^1 \in \mathcal{X}\}$, $l_1 = \overline{1, L_1}$, from $\mathcal{P}(\mathcal{X})$ and the second object characterized by RV $X_2$ can have one of $L_1 \times L_2$ conditional PDs $G_{l_2/l_1} = \{G_{l_2/l_1}(x^2),\ x^2 \in \mathcal{X}\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$ and it follows that $G^N_{m_2/l_1}(\mathbf{x_2}|\mathbf{x_1}) = G^N_{m_2/l_1}(\mathbf{x_2})$, $m_2 = \overline{1, L_2}$, $l_1 = \overline{1, L_1}$. Hence sequence of tests $\varphi_2^*$ is depended on vector $\mathbf{x_2}$ and on the index of the hypothesis $l_1$ adopted for $X_1$ and we obtain the generalization of the case examined in [11].

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## Ստոխաստիկորեն կախված օբյեկտների զույգի նկատմամբ վարկածների օպտիմալ ստուգման մասին

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**Ամփոփում**

Լուծված է ստոխաստիկորեն կախյալ երկու օբյեկտների նկատմամբ վարկածների լոգարիթմորեն ասիմպտոտորեն օպտիմալ ստուգման խնդիրը: Առաջին օբյեկտը կարող է բաշխված լինել տրված $L_1$ հավանականային բաշխումներից մեկով, իսկ երկրորդը՝ կախված առաջինի բաշխումից և դիտարկվող պահին նրա վիճակից, տրված $L_1 \times L_2$պայմանական հավանականային բաշխումներից մեկով: Ուսումնասիրվել է օբյեկտների նկատմամբ վարկածների տեստավորման սխալների հավանականությունների ցուցիչների (հուսալիությունների) փոխկախվածությունների մատրիցը: