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On Existence of 2-partition of a Tree, Which Obeys the Given Priority

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Abstract

A necessary and sufficient condition is obtained for the problem of such partitioning of the set of vertices of a tree G into two disjoint sets V_1 and V_2, which, for a given function $p : V(G) \to \{-1,0,1\}$ with some special restriction, satisfies the condition $|\lambda(v) \cap V_1| - |\lambda(v) \cap V_2| = p(v) \cdot (|\{v\} \cap V_1| - |\{v\} \cap V_2|)$ for any vertex v of G, where $\lambda(v)$ is the set of all vertices of G adjacent to v.

We consider finite, undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. If $v \in V(G)$ then $ex_G(v)$ denotes the eccentricity of a vertex v in a graph G. For a graph G let $\Delta(G)$ be the greatest degree of a vertex of G. Let $\rho_G(x,y)$ denote the distance between the vertices $x \in V(G)$ and $y \in V(G)$ in a graph G. For $\nu \in V(G)$ let's denote $\lambda(\nu) \equiv \{\omega \in V(G)/(\omega,\nu) \in E(G)\}$.

A function $f : V(G) \to \{0,1\}$ is called 2-partition of a graph G.

A function $p : V(G) \to \{-1,0,1\}$ is called a priority in a graph G if the following condition holds: for $\forall x \in V(G)$ $p(x) = 0$ iff $d_G(x)$ is even.

We'll say that 2-partition f of a graph G obeys the priority p iff for any vertex $v \in V(G)$

$$|\{\omega \in \lambda(v)/f(\omega) = f(v)\}| - |\{\omega \in \lambda(v)/f(\omega) = 1 - f(v)\}| = p(v).$$

Non-defined concepts can be found in [1, 2, 3, 4, 5].

For any $n \in \{0,1\}$, an arbitrary function $g : X_g \to \{0,1\}$ and any set $X \subseteq X_g$, denote:

$$P(X,g,n) \equiv |\{\nu \in X/g(\nu) = n\}| - |\{\nu \in X/g(\nu) = 1-n\}|.$$

Obviously, the definition of 2-partition, obeying the given priority, can be rewritten in the following way.

We'll say that 2-partition f of a graph G obeys the priority p iff for any vertex $v \in V(G)$

$$P(\lambda(v), f, f(v)) = p(v).$$

Let $x \in V(G)$ be an arbitrary vertex of a tree G.

We define the subset $N_i(x)$ of the set $V(G)$, where $0 \le i \le ex_G(x)$, as follows:

$$N_i(x) \equiv \{z \in V(G)/\rho_G(x,z) = i\}.$$

Obviously, for any $u \in N_i(x)$, where $1 \le i \le ex_G(x)$, there exists a single vertex $u^{(-1)} \in N_{i-1}(x)$ satisfying the condition $(u, u^{(-1)}) \in E(G)$.

Let's assume we have some partitioning of the set $V(G)\backslash\{x\}$ into sets $A(x)$, $B(x)$, which satisfies the following conditions:

$$V(G)\backslash\{x\} = A(x) \cup B(x), \quad A(x) \cap B(x) = \emptyset.$$

For $\forall u \in V(G)\backslash N_{ex_G(x)}(x)$ define:

$$a(u) \equiv |N_{\rho_G(x,u)+1}(x) \cap \lambda(u) \cap A(x)|,$$
$$b(u) \equiv |N_{\rho_G(x,u)+1}(x) \cap \lambda(u) \cap B(x)|.$$

Note 1. *From the definitions of functions a, b it follows that if for $\forall i$, $1 \le i \le ex_G(x)$ for all $u \in N_i(x)$ it is already determined whether $u \in A(x)$ or $u \in B(x)$, then for an arbitrary $u \in N_{i-1}(x)$ the values $a(u)$ and $b(u)$ are unambiguously calculated.*

Note 2. $a(x)+b(x) = d_G(x)$; *for $\forall u \in V(G)\backslash(N_{ex_G(x)}(x) \cup \{x\})$ the equality $a(u)+b(u)+1 = d_G(u)$ holds.*

Now assume that p is a priority in the tree G. Let's inductively define sets $A(x)$ and $B(x)$ as follows:

$$N_{ex_G(x)}(x) \cap A(x) \equiv \{v \in N_{ex_G(x)}(x)/p(v) = 1\},$$
$$N_{ex_G(x)}(x) \cap B(x) \equiv \{v \in N_{ex_G(x)}(x)/p(v) = -1\}.$$

Let's assume that for i, $2 \le i \le ex_G(x)$, the partitioning of $N_i(x)$ is already defined:

$$N_i(x) = (N_i(x) \cap A(x)) \cup (N_i(x) \cap B(x)).$$

It follows from the note 1 that for each $u \in N_{i-1}(x)$ the values of functions a, b can be calculated. Let's define the partitioning of $N_{i-1}(x)$ as follows: for $\forall u \in N_{i-1}(x)$

$$u \in \begin{cases} A(x), & \text{if } a(u) - b(u) - p(u) < 0, \\ B(x), & \text{if } a(u) - b(u) - p(u) > 0. \end{cases}$$

(Note that for any $u \in V(G)$ $a(u) - b(u) - p(u) \neq 0$).

Obviously, under the given definition the following condition is true

$$(N_{i-1}(x) \cap A(x)) \cap (N_{i-1}(x) \cap B(x)) = \emptyset.$$

It is easy to see that the sets $A(x)$ and $B(x)$ are unambiguously defined and, moreover,

$$V(G)\backslash\{x\} = A(x) \cup B(x), \quad A(x) \cap B(x) = \emptyset.$$

Note that we have also defined the following functions

$$a : (V(G)\backslash N_{ex_G(x)}(x)) \to Z_+, \quad b : (V(G)\backslash N_{ex_G(x)}(x)) \to Z_+.$$

Further we shall assume, that consideration of any tree G with an arbitrary priority p automatically implies the choice of a vertex $x \in V(G)$, the realization of the partitioning of the set $V(G)\backslash\{x\}$ into sets $A(x)$, $B(x)$ mentioned above and the definition of functions a, b on the set $V(G)\backslash N_{ex_G(x)}(x)$.

Lemma 1. *If G is a tree and f – it's 2-partition, obeying the priority p, then for $\forall u \in V(G)\backslash\{x\}$ following properties hold*

$$u \in A(x) \Rightarrow f(u^{(-1)}) = f(u), \quad u \in B(x) \Rightarrow f(u^{(-1)}) = 1 - f(u).$$

Proof by the reverse induction on $\rho_G(x,u)$. First of all let's prove the lemma for vertices of the set $N_{ex_G(x)}(x)$. Let $u \in N_{ex_G(x)}(x)$ be an arbitrary vertex. Obviously, $d_G(u) = 1$.

Case 1. $u \in A(x)$.

It is clear that $p(u) = 1$. Since f is a 2-partition of a tree G, obeying the priority p and $d_G(u) = 1$, then $f(u^{(-1)}) = f(u)$, which is the statement of the lemma.

Case 2. $u \in B(x)$.

It is clear that $p(u) = -1$. Since f is a 2-partition of the tree G, obeying the priority p, and $d_G(u) = 1$, then $f(u^{(-1)}) = 1 - f(u)$, which is the statement of the lemma. Assume that the lemma holds for all vertices of the set $N_i(x)$, where $2 \le i \le ex_G(x)$. Let's prove the lemma for vertices of the set $N_{i-1}(x)$.

Let $u \in N_{i-1}(x)$ be an arbitrary vertex.

From the inductive assumption it follows that

$$P(\lambda(u) \cap N_i(x), f, f(u)) = a(u) - b(u).$$

Consequently, since f is a 2-partition of the tree G, obeying the priority p, then we obtain

$$\begin{aligned} p(u) = P(\lambda(u), f, f(u)) = P(\lambda(u) \cap N_i(x), f, f(u)) + \\ + P(\lambda(u)\backslash N_i(x), f, f(u)) = a(u) - b(u) + P(\lambda(u)\backslash N_i(x), f, f(u)). \end{aligned} \tag{1}$$

Case 1_{Ind}. $u \in A(x)$.

In this case $a(u) - b(u) - p(u) < 0$. Consequently, taking into account the equality (1), we obtain

$$a(u) - b(u) + P(\lambda(u)\backslash N_i(x), f, f(u)) = p(u) > a(u) - b(u).$$

As a result, $P(\lambda(u)\backslash N_i(x), f, f(u)) > 0$. The obtained inequality, taking into account that $\lambda(u)\backslash N_i(x) = \{u^{(-1)}\}$ and $|P(\lambda(u)\backslash N_i(x), f, f(u))| = 1$, implies $f(u^{(-1)}) = f(u)$, which is the statement of the lemma.

Case 2_{Ind}. $u \in B(x)$.

In this case $a(u) - b(u) - p(u) > 0$.

Consequently, taking into account the equality (1), we obtain

$$a(u) - b(u) + P(\lambda(u)\backslash N_i(x), f, f(u)) = p(u) < a(u) - b(u).$$

As a result, $P(\lambda(u)\backslash N_i(x), f, f(u)) < 0$. The obtained inequality, taking into account that $\lambda(u)\backslash N_i(x) = \{u^{(-1)}\}$ and $|P(\lambda(u)\backslash N_i(x), f, f(u))| = 1$, implies $f(u^{(-1)}) = 1 - f(u)$, which is the statement of the lemma.

Lemma is proved.

The proof of the lemma implies the following

Corollary 1. *If G is a tree and f – it's 2-partition, obeying the priority p, then for any vertex $u \in V(G)\backslash N_{ex_G(x)}(x)$ the following equality holds*

$$p(u) = a(u) - b(u) + P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(x), f, f(u)).$$

Theorem 1. *For a given tree G there exists a 2-partition, obeying the priority p, iff for $\forall u \in V(G)\backslash N_{ex_G(x)}(x)$*

$$|a(u) - b(u) - p(u)| = d_G(u) - a(u) - b(u).$$

Proof.

Necessity. Suppose f is a 2-partition, obeying the priority p.

Let $u \in V(G)\backslash N_{ex_G(x)}(x)$ be an arbitrary vertex. It follows from the corollary 1 and the note 2 that

$$\begin{aligned} &|a(u) - b(u) - p(u)| = \\ &= |a(u) - b(u) - (a(u) - b(u) + P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(x), f, f(u)))| = \\ &= |P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(x), f, f(u))| = d_G(u) - a(u) - b(u). \end{aligned}$$

Sufficiency. Suppose that for any $u \in V(G)\backslash N_{ex_G(x)}(x)$ $|a(u) - b(u) - p(u)| = d_G(u) - a(u) - b(u)$.

Let's inductively define a function $f : V(G) \to \{0, 1\}$. Let's set $f(x) \equiv 1$. Let's assume that for all vertices of the set $N_i(x)$, where $0 \le i \le ex_G(x) - 1$, the function f is already defined. Let's define the function f for vertices of the set $N_{i+1}(x)$. For each vertex $u \in N_i(x)$ let's define the function f for vertices of the set $N_{i+1}(x) \cap \lambda(u)$. First of all let's define the function f on vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap A(x)$ by the following way: for $\forall z \in N_{i+1}(x) \cap \lambda(u) \cap A(x)$ set $f(z) \equiv f(u)$.

Now let's define the function f on vertices of the set $N_{i+1}(x) \cap \lambda(u) \cap B(x)$ by the following way: for $\forall z \in N_{i+1}(x) \cap \lambda(u) \cap B(x)$ set $f(z) \equiv 1 - f(u)$. So we have defined the function f on all vertices of the set $N_{i+1}(x)$. Therefore, the function f is defined on whole $V(G)$. Let's check that 2-partition f of the tree G, defined above, obeys the priority p, indeed.

Let $u \in V(G)$ be an arbitrary vertex.

Case 1. $u = x$.

The condition of the theorem and the note 2 imply $|a(x) - b(x) - p(x)| = 0$.

Consequently, from the equality $\lambda(x) = \lambda(x) \cap N_1(x)$ and from the definition of f we obtain

$$P(\lambda(x), f, f(x)) = P(\lambda(x) \cap N_1(x), f, f(x)) = a(x) - b(x) = p(x).$$

Case 2. $u \neq x$. The condition of the theorem and the note 2 imply $|a(u) - b(u) - p(u)| = 1$. Consequently, from the definition of f we obtain

$$\begin{aligned} &P(\lambda(u), f, f(u)) = P(\lambda(u) \cap N_{\rho_G(x,u)+1}(u), f, f(u)) + \\ &+ P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = \\ &= a(u) - b(u) + P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(u), f, f(u)). \end{aligned} \tag{2}$$

Case 2a). $u \in A(x)$. In this case from the definitions of $A(x)$ and f it follows that $a(u) - b(u) - p(u) < 0$ and $P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = 1$, respectively. Consequently, taking into account $|a(u) - b(u) - p(u)| = 1$, we obtain $a(u) - b(u) - p(u) = -1$. The obtained equality, taking into account the equality (2) and the equality $P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = 1$, implies

$$\begin{aligned} &P(\lambda(u), f, f(u)) = a(u) - b(u) + P(\lambda(u)\backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = \\ &= a(u) - b(u) + 1 = p(u). \end{aligned}$$

Case 2b). $u \in B(x)$.

In this case from the definitions of $B(x)$ and f it follows that $a(u) - b(u) - p(u) > 0$ and $P(\lambda(u) \backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = -1$, respectively. Consequently, taking into account $|a(u) - b(u) - p(u)| = 1$, we obtain $a(u) - b(u) - p(u) = 1$. The obtained equality, taking into account the equality (2) and the equality $P(\lambda(u) \backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = -1$, implies

$$\begin{aligned} P(\lambda(u), f, f(u)) &= a(u) - b(u) + P(\lambda(u) \backslash N_{\rho_G(x,u)+1}(u), f, f(u)) = \\ &= a(u) - b(u) - 1 = p(u). \end{aligned}$$

Theorem is proved.

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Ծառի այնպիսի 2-տրոհման գոյության մասին, որը ենթարկվում է տրված ճախապատվությանը

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Ամփոփում

Ստացված է անհրաժեշտ և բավարար պայման G ծառի գագաթների բազմության V_1 և V_2 չհատվող ենթաբազմությունների այնպիսի տրոհման գոյությունը պարզելու համար, որ տրված հատուկ սահմանափակումներով ֆունկցիայի համար բավարարվի հետևյալ պայմանը $p : V(G) \to \{-1, 0, 1\}$ ծառի յուրաքանչյուր v գագաթի համար $|\lambda(v) \cap V_1| - |\lambda(v) \cap V_2| = p(v) \cdot (|\{v\} \cap V_1| - |\{v\} \cap V_2|)$, որտեղ $\lambda(v)$-ով նշանակված է v-ին կից գագաթների բազմությունը: