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On Existence of Certain Locally-balanced 2-partition of a Tree

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Abstract

A necessary and sufficient condition is obtained for the problem of such partitioning of the set of vertices of a tree G into two disjoint sets V_1 and V_2, such that for given sets $V_1^0 \subseteq V(G)$ and $V_2^0 \subseteq V(G)$ ($V_1^0 \cap V_2^0 = \emptyset$) it satisfies the conditions $V_1^0 \subseteq V_1$, $V_2^0 \subseteq V_2$ and $||\lambda(v) \cap V_1| - |\lambda(v) \cap V_2|| \leq 1$ for any vertex v of G, where $\lambda(v)$ is the set of all vertices of G adjacent to v.

We consider finite, undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. If $v \in V(G)$ then $ex_G(v)$ denotes the eccentricity of a vertex v in a graph G. For a graph G let $\Delta(G)$ be the greatest degree of a vertex of G. Let $\rho_G(x, y)$ denote the distance between the vertices $x \in V(G)$ and $y \in V(G)$ in a graph G. For $\nu \in V(G)$ let's denote $\lambda(\nu) \equiv \{\omega \in V(G)/(\omega, \nu) \in E(G)\}$.

A function $f : V(G) \rightarrow \{0, 1\}$ is called 2-partition of a graph G.

A function $g : V_g \rightarrow \{0, 1\}$, where $V_g \subseteq V(G)$, is called a partial 2-partition of a graph G. Note that 2-partition of a graph is also a partial 2-partition of a graph.

A partial 2-partition g_1 of a graph G is called an expansion of a partial 2-partition g_2 of a graph G, defined on a set $V_{g_2} \subseteq V(G)$, iff for $\forall v \in V_{g_2}$ $g_1(v) = g_2(v)$.

2-partition f of a graph G is called locally-balanced iff for $\forall \nu \in V(G)$

$$||\{\omega \in \lambda(\nu)/f(\omega) = 1\}| - |\{\omega \in \lambda(\nu)/f(\omega) = 0\}|| \leq 1.$$

Non-defined concepts can be found in [1, 2, 3, 4, 5].

For any partially defined function $f : X \rightarrow Y$ let's denote by $D(f)$ the set of elements of the set X on which the function f is defined.

For an arbitrary function $g : X_g \rightarrow \{0, 1\}$ and any set $X \subseteq X_g$, denote:

$$P(X, g) \equiv |\{\nu \in X/g(\nu) = 1\}| - |\{\nu \in X/g(\nu) = 0\}|.$$

Obviously, the definition of locally-balanced 2-partition can be rewritten in the following way: 2-partition f of a graph G is called locally-balanced iff for $\forall \nu \in V(G)$ $|P(\lambda(\nu), f)| \leq 1$.

Let $x \in V(G)$ be an arbitrary vertex of a tree G.

We define the subset $N_i(x)$ of the set $V(G)$, where $0 \leq i \leq ex_G(x)$, as follows:

$$N_i(x) \equiv \{z \in V(G)/\rho_G(x, z) = i\}.$$

Obviously, for any $u \in N_i(x)$, where $1 \leq i \leq ex_G(x)$, there exists a single vertex $u^{(-1)} \in N_{i-1}(x)$ satisfying the condition $(u, u^{(-1)}) \in E(G)$.

Let $g : V_g \to \{0,1\}$, where $V_g \subseteq V(G)$, be a partial 2-partition of a graph G.

Let's inductively define a partially defined function $z_g : V(G) \to \{0,1\}$.

For all vertices $\nu \in V_g$, set $z_g(\nu) \equiv g(\nu)$.

If $|V(G)| \leq 1$, then the definition of z_g is complete.

Let's assume that $|V(G)| > 1$.

Let z_g on all vertices of the set $(N_{ex_G(x)} \cup N_{ex_G(x)-1}) \backslash V_g$ will not be defined.

Let for i, $0 \leq i < ex_G(x) - 1$, the function z_g on each vertex $\nu \in N_{i+2}(x)$ either is already defined or it is already determined that it will not be defined on ν.

For each vertex $u \in N_i(x) \backslash V_g$, if there exists such a vertex $u^{+1} \in N_{i+1}(x)$, that $(u^{+1})^{-1} = u$ and $|P(\lambda(u^{+1}) \cap N_{i+2}(x) \cap D(z_g), z_g)| - |(\lambda(u^{+1}) \cap N_{i+2}(x)) \backslash D(z_g)| > 0$ (if there are more than one such a vertex, then we'll choose anyone), then set:

$$z_g(u) \equiv \begin{cases} 0, & \text{if } P(\lambda(u^{+1}) \cap N_{i+2}(x) \cap D(z_g), z_g) > 0, \\ 1, & \text{if } P(\lambda(u^{+1}) \cap N_{i+2}(x) \cap D(z_g), z_g) < 0, \end{cases}$$

otherwise the function z_g will not be defined on a vertex u.

(Note that $|P(\lambda(u^{+1}) \cap N_{i+2}(x) \cap D(z_g), z_g)| - |(\lambda(u^{+1}) \cap N_{i+2}(x)) \backslash D(z_g)| > 0$ implies $P(\lambda(u^{+1}) \cap N_{i+2}(x) \cap D(z_g), z_g) \neq 0$).

Let's define the set $V_{[g,x]}$ as follows: $V_{[g,x]} \equiv D(z_g)$.

Let's define a partial 2-partition $[g,x] : V_{[g,x]} \to \{0,1\}$, which is an expansion of g and called a zone of influence of g towards x.

For any $u \in V_{[g,x]}$ set $[g,x](u) \equiv z_g(u)$.

It is clear that $[g,x]$ is an expansion of g.

Further we shall assume, that consideration of any tree G automatically implies the choice of a vertex $x \in V(G)$.

Theorem 1. *For a given tree G there exists a locally-balanced 2-partition, which is an expansion of a partial 2-partition g of the tree G iff there exists a locally-balanced 2-partition of the tree G, which is an expansion of a partial 2-partition $[g,x]$ of the tree G.*

Proof.

Sufficiency. Suppose, that there exists a locally-balanced 2-partition f of a tree G, which is an expansion of the partial 2-partition $[g,x]$.

Since $[g,x]$ is an expansion of g then f also is an expansion of g.

Necessity. Suppose, that there exists a locally-balanced 2-partition f of a tree G, which is an expansion of the partial 2-partition g. Let's show that f is an expansion of the partial 2-partition $[g,x]$, too, i.e. for $\forall u \in V_{[g,x]}$ $f(u) = [g,x](u)$.

The proof will be fulfilled by the reverse induction on $\rho_G(x,u)$.

First of all let's show that for $\forall u \in (N_{ex_G(x)}(x) \cup N_{ex_G(x)-1}(x)) \cap V_{[g,x]}$ $f(u) = [g,x](u)$.

Let u be an arbitrary vertex from $(N_{ex_G(x)}(x) \cup N_{ex_G(x)-1}(x)) \cap V_{[g,x]}$. It follows from the definition of $V_{[g,x]}$ that $u \in (N_{ex_G(x)}(x) \cup N_{ex_G(x)-1}(x)) \cap D(z_g)$. Consequently, it follows from the algorithm of construction of z_g that $u \in V_g$.

Since f and $[g,x]$ both are expansions of g and $u \in V_g$, then $f(u) = g(u)$, $[g,x](u) = g(u)$ and, as a result, $f(u) = [g,x](u)$.

Let's suppose that for $\forall i$, $k < i \leq ex_G(x)$, where $0 \leq k < ex_G(x) - 1$, $u \in N_i(x) \cap V_{[g,x]}$ implies $f(u) = [g,x](u)$.

Let's prove that for $\forall u \in N_k(x) \cap V_{[g,x]}$ $f(u) = [g,x](u)$.

Let u be an arbitrary vertex from $N_k(x) \cap V_{[g,x]}$.

If $u \in V_g$, then since f and $[g,x]$ both are expansions of g, then $f(u) = g(u)$, $[g,x](u) = g(u)$ and, as a result, $f(u) = [g,x](u)$.

Let's suppose that $u \notin V_g$. Assume the contrary: $f(u) \neq [g,x](u)$.

It follows from the definition of $[g,x]$, from the algorithm of construction of z_g and from $u \notin V_g$ that there exists such a vertex $u^{+1} \in N_{k+1}(x)$, that $|P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g)| - |(\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g)| > 0$, moreover,

$$[g,x](u) = \begin{cases} 0, & \text{if } P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g) > 0, \\ 1, & \text{if } P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g) < 0. \end{cases}$$

Consequently, taking into account the inequality $f(u) \neq [g,x](u)$, we obtain:

$$f(u) = \begin{cases} 1, & \text{if } P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g) > 0, \\ 0, & \text{if } P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g) < 0. \end{cases}$$

The obtained equality and the definition of P imply

$$|P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), f) + P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| =$$

$$= |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), f)| + |P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)|$$

Consequently, from the inductive assumption and from the definition of $[g,x]$ we obtain

$$|P(\lambda(u^{+1}), f)| = |P(\lambda(u^{+1}) \cap N_{k+2}(x), f) + P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| =$$

$$= |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), f) + P((\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g), f) + \\ + P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| \geq$$

$$\geq |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), f) + P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| - \\ - |P((\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g), f)| =$$

$$= |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), f)| + |P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| - \\ - |P((\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g), f)| \geq$$

$$\geq |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g)| + |P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| - \\ - |(\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g)|.$$

The obtained inequality, the inequality $|P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g)| - |(\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g)| > 0$ and the equality $\lambda(u^{+1}) \setminus N_{k+2}(x) = \{u\}$ imply the following inequality

$$|P(\lambda(u^{+1}), f)| \geq |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g)| + |P(\lambda(u^{+1}) \setminus N_{k+2}(x), f)| - \\ - |(\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g)| =$$

$$= |P(\lambda(u^{+1}) \cap N_{k+2}(x) \cap D(z_g), z_g)| - |(\lambda(u^{+1}) \cap N_{k+2}(x)) \setminus D(z_g)| + 1 > 1.$$

The obtained contradiction proves that $f(u) = [g,x](u)$.

Theorem is proved.

The proof of the theorem implies

Corollary 1. *If a locally-balanced 2-partition f of a tree G is an expansion of a partial 2-partition g then it is also an expansion of the partial 2-partition $[g, x]$.*

Theorem 2. *For a given tree G there exists a locally-balanced 2-partition, which is an expansion of a partial 2-partition $[g, x]$ iff for $\forall\nu \in V(G)$ the following inequality holds*

$$|P(\lambda(\nu) \cap V_{[g,x]}, [g, x])| - |\lambda(\nu) \backslash V_{[g,x]}| \leq 1.$$

Proof.

Necessity. Suppose there exists a locally-balanced 2-partition of a tree G, which is an expansion of a partial 2-partition $[g, x]$.

Since f is locally-balanced, then for $\forall\nu \in V(G)$ $|P(\lambda(\nu), f)| \leq 1$. Consequently, we obtain for $\forall\nu \in V(G)$

$$1 \geq |P(\lambda(\nu), f)| = |P(\lambda(\nu) \cap V_{[g,x]}, f) + P(\lambda(\nu) \backslash V_{[g,x]}, f)| \geq$$
$$\geq |P(\lambda(\nu) \cap V_{[g,x]}, f)| - |P(\lambda(\nu) \backslash V_{[g,x]}, f)| \geq |P(\lambda(\nu) \cap V_{[g,x]}, f)| - |\lambda(\nu) \backslash V_{[g,x]}|.$$

The obtained inequality, taking into account that f is an expansion of $[g, x]$, implies that for $\forall\nu \in V(G)$ the following inequality holds

$$|P(\lambda(\nu) \cap V_{[g,x]}, [g, x])| - |\lambda(\nu) \backslash V_{[g,x]}| \leq 1.$$

Sufficiency. Suppose that for $\forall\nu \in V(G)$ the following inequality holds

$$|P(\lambda(\nu) \cap V_{[g,x]}, [g, x])| - |\lambda(\nu) \backslash V_{[g,x]}| \leq 1.$$

Let's inductively define a function $f : V(G) \to \{0, 1\}$.

If $x \in V_{[g,x]}$, then set $f(x) \equiv [g, x](x)$, otherwise set $f(x) \equiv 1$.

Let's assume that for all vertices of the set $N_i(x)$, where $0 \leq i \leq ex_G(x) - 1$, the function f is already defined. Let's define the function f on vertices of the set $N_{i+1}(x)$.

For each vertex $u \in N_i(x)$ let's define the function f on vertices of the set $\lambda(u) \cap N_{i+1}(x)$.

First of all let's define the function f on vertices of the set $\lambda(u) \cap N_{i+1}(x) \cap V_{[g,x]}$ in the following way: for any vertex $\nu \in \lambda(u) \cap N_{i+1}(x) \cap V_{[g,x]}$ set $f(\nu) \equiv [g, x](\nu)$.

Obviously, without loss of generality it can be supposed that all vertices of the set $S(i, u, x, g) \equiv (\lambda(u) \cap N_{i+1}(x)) \backslash V_{[g,x]}$, if it is not empty, are numbered: $h_1(u), h_2(u), \ldots, h_{c(u)}(u)$, where $c(u) \equiv |S(i, u, x, g)|$.

Note 1. *For all vertices of the set $\lambda(u) \backslash S(i, u, x, g)$ the function f is defined.*

It follows from the note 1 that the value of $P(\lambda(u) \backslash S(i, u, x, g), f)$ is defined.

Now, let's define the function f on vertices of the set $S(i, u, x, g)$. For any vertex $v \in S(i, u, x, g)$ let's set:

$$f(v) \equiv \begin{cases} 0, & \text{if } v = h_j(u), \text{ where } 1 \leq j \leq P(\lambda(u) \backslash S(i, u, x, g), f), \\ 1, & \text{if } v = h_j(u), \text{ where } 1 \leq j \leq -P(\lambda(u) \backslash S(i, u, x, g), f), \\ 0, & \text{if } v = h_j(u), \text{ where } |P(\lambda(u) \backslash S(i, u, x, g), f)| < j \leq c(u) \\ & \text{and } j - |P(\lambda(u) \backslash S(i, u, x, g), f)| \text{ is an odd number}, \\ 1, & \text{if } v = h_j(u), \text{ where } |P(\lambda(u) \backslash S(i, u, x, g), f)| < j \leq c(u) \\ & \text{and } j - |P(\lambda(u) \backslash S(i, u, x, g), f)| \text{ is an even number}. \end{cases} \quad (1)$$

So we have defined the function f on all vertices of the set $N_{i+1}(x)$.

Therefore, the function f is defined on whole $V(G)$.

It is clear that 2-partitioning f is an expansion of the partial 2-partition $[g, x]$.

Let's check that the function f defined above is a locally-balanced 2-partition of the tree G, indeed.

Let $u \in V(G)$ be an arbitrary vertex.

Note 2. *If the inequality* $|P(\lambda(u)\backslash S(i,u,x,g), f)| - |S(i,u,x,g)| \leq 1$ *is true, then (1) implies* $|P(\lambda(u), f)| \leq 1$.

Case 1. $u \in N_0(x)$.

Obviously, $u = x$.

Since $\lambda(x)\backslash V_{[g,x]} = S(0,x,x,g)$ and f is an expansion of $[g,x]$, then, taking into account the condition of the theorem, we obtain

$$\begin{aligned}&|P(\lambda(x)\backslash S(0,x,x,g), f)| - |S(0,x,x,g)| = \\ &= |P(\lambda(x) \cap V_{[g,x]}, [g,x])| - |\lambda(x)\backslash V_{[g,x]}| \leq 1.\end{aligned}$$

Consequently, taking into account the note 2, we obtain $|P(\lambda(x), f)| \leq 1$.

Case 2. $u \in N_i(x)$, where $1 \leq i < ex_G(x)$.

Case 2a). $u^{-1} \in V_{[g,x]}$.

Since $\lambda(u)\backslash V_{[g,x]} = S(i,u,x,g)$, then in the same way as in the case 1, we obtain $|P(\lambda(u), f)| \leq 1$.

Case 2b). $u^{-1} \notin V_{[g,x]}$.

In this case it follows from the definition of $[g,x]$ that

$$|P(\lambda(u) \cap N_{i+1}(x) \cap V_{[g,x]}, z_g)| - |(\lambda(u) \cap N_{i+1}(x))\backslash V_{[g,x]}| \leq 0.$$

On the other hand, since f is an expansion of $[g,x]$, from the definition of $[g,x]$ we obtain:

$$|P(\lambda(u)\backslash S(i,u,x,g), f)| - |S(i,u,x,g)| =$$

$$\begin{aligned}&= |P((\lambda(u)\backslash S(i,u,x,g)) \cap N_{i+1}(x), f) + P((\lambda(u)\backslash S(i,u,x,g))\backslash N_{i+1}(x), f)| - \\ &- |S(i,u,x,g)| \leq\end{aligned}$$

$$\begin{aligned}&\leq |P((\lambda(u)\backslash S(i,u,x,g)) \cap N_{i+1}(x), f)| + |P((\lambda(u)\backslash S(i,u,x,g))\backslash N_{i+1}(x), f)| - \\ &- |S(i,u,x,g)| =\end{aligned}$$

$$= |P(\lambda(u) \cap N_{i+1}(x) \cap V_{[g,x]}, z_g)| + 1 - |S(i,u,x,g)|.$$

The obtained inequality, taking into account the definition of z_g, implies

$$|P(\lambda(u)\backslash S(i,u,x,g), f)| - |S(i,u,x,g)| \leq 1.$$

Consequently, taking into account the note 2, we obtain $|P(\lambda(u), f)| \leq 1$.

Case 3. $u \in N_{ex_G(x)}(x)$.

Since $\lambda(u) = \{u^{-1}\}$, then $|P(\lambda(u), f)| = 1 \leq 1$.

Theorem is proved.

The theorems 1 and 2 imply

Theorem 3. *For a given tree G there exists a locally-balanced 2-partition, which is an expansion of a partial 2-partition g of the tree G iff for* $\forall v \in V(G)$ *the following inequality holds*

$$||P(\lambda(v) \cap V_{[g,x]}, [g,x])| - |\lambda(v) \setminus V_{[g,x]}|| \leq 1.$$

Note that in the proof of the theorem 2 an algorithm is given, which constructs a locally-balanced 2-partition, which is an expansion of the zone of influence of the given partial 2-partition (if such one exists).

Combining the mentioned algorithm with the algorithm of construction of the zone of influence, we obtain an algorithm for trees, which constructs a locally-balanced 2-partition, which is an expansion of the given partial 2-partition (if such one exists).

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Ծառի լոկալ-հավասարակշռված որոշակի 2-տրոհման գոյության մասին

Ս.Վ. Բալիկյան

Ամփոփում

Ստացված է անհրաժեշտ և բավարար պայման՝ G ծառի գագաթների բազմության V_1 *և* V_2 *չհատվող ենթաբազմությունների այնպիսի տրոհման գոյությունը պարզելու համար, որ տրված* $V_1^0 \subseteq V(G)$ *և* $V_2^0 \subseteq V(G)$ $(V_1^0 \cap V_2^0 = \emptyset)$ *բազմությունների համար բավարարվեն հետևյալ պայմանները.* $V_1^0 \subseteq V_1$, $V_2^0 \subseteq V_2$ *և ծառի յուրաքանչյուր* v *գագաթի համար տեղի ունենա հետևյալ անհավասարությունը* $||\lambda(v) \cap V_1| - |\lambda(v) \cap V_2|| \geq 1$, *որտեղ* $\lambda(v)$*-ով նշանակված է* v*-ին կից գագաթների բազմությունը:*