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On Simultaneous 2-locally-balanced 2-partition for Two Forests with Same Vertices

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Abstract

The existence of a partition of the common set of the vertices of two forests into two subsets, when difference of their capacities in the neighbourhood of each vertex of each forest is not greater than 2 is proved, and an example, which shows that improvement of the specified constant is impossible is brought.

In this paper we continue researches started in [1, 2] devoted to locally-balanced partitions of a graph. We consider undirected graphs and multigraphs without loops. The set of vertices of the multigraph G is denoted by $V(G)$, and the set of edges of G-by $E(G)$, and the maximum degree of the vertices of G-by $\Delta(G)$. The eccentricity of a vertex $v \in V(G)$ is denoted by $ex_G(v)$. Non-defined concepts can be found in [3]. For $v \in V(G)$ we shall define sets $\gamma_G(v) = \{w \in V(G)/(w,v) \in E(G)\}$ and $\eta_G(v) = \{e \in E(G)/v \text{ incident to } e\}$. A function $f : M \to \{0,1\}$ is called 2-partition of a finite set M. If f is a 2-partition of a finite set M, then for $\forall M_0 \subseteq M$ we define the number $b_f(M_0)$ as follows:

$$b_f(M_0) = ||\{m \in M_0/f(m) = 1\}| - |\{m \in M_0/f(m) = 0\}||.$$

Let $G_1$and G_2 are undirected graphs without loops with $V(G_1) = V(G_2) \equiv V$. The 2-partition f of the set V is called simultaneous k-locally-balanced ($k \in Z, k \geq 0$) 2-partition of the graphs $G_1$and G_2 if:

$$\max_{i=1,2} \max_{v \in V} b_f(\gamma_{G_i}(v)) = k.$$

Let D be a tree, and let $v_1(D) \in V(D)$ be an arbitrarily chosen vertex. For $i = 0, 1, ..., ex_D(v_1(D))$ we define a subset $S_i \subseteq V(D)$ as follows:

$$S_i \equiv \{w \in V(G)/\rho(w, v_1(G)) = i\}.$$

For $i = 1, 2, ..., ex_D(v_1(D))$ and $u \in S_{i-1}$ let's define $S_i(u) \equiv \{w \in S_i/(w,u) \in E(D)\}$. We define a family of subsets $X(D)$ of the set $V(D)$ as follows:

$$X(D) \equiv \{S_i(u)/1 \leq i \leq ex_D(v_1(D)), u \in S_{i-1}, S_i(u) \neq \emptyset\} \cup \{S_0\}.$$

In the further we shall assume, that the consideration of any tree D automatically implies the choice of the vertex $v_1(D)$.

Let G is a forest, and $D_1, D_2, ..., D_{k(G)}$ are its connected components. Define a family of subsets $X(G)$ of the set $V(G)$ as follows:

$$X(G) \equiv \bigcup_{i=1}^{k(G)} X(D_i).$$

Let $G_1$and G_2 are two forests with $V(G_1) = V(G_2) \equiv V$. Define a bipartite multigraph $H(G_1, G_2) = (V_1(H(G_1, G_2)), V_2(H(G_1, G_2)), E(H(G_1, G_2)))$ as follows:

$V_1(H(G_1, G_2)) = X(G_1)$,

$V_2(H(G_1, G_2)) = X(G_2)$,

$E(H(G_1, G_2)) = \bigcup_{v \in V} \{(u, w)_v / u \in V_1(H(G_1, G_2)), w \in V_2(H(G_1, G_2)), v \in u \cap w\}$,

where $E(H(G_1, G_2))$ is understood as multiset containing different elements like $(u, w)_{v_1}$and $(u, w)_{v_2}$ with $v_1 \neq v_2$ in a case $|u \cap w| > 1$.

It is not hard to see that for $\forall v \in V$

$$|\{(u, w)_v / u \in V_1(H(G_1, G_2)), w \in V_2(H(G_1, G_2)), v \in u \cap w\}| = 1.$$

Taking into account that $G_1$and G_2 are forests we can conclude from the construction of the multigraph $H(G_1, G_2)$ that there exists a one-to-one correspondence $\xi : V \to E(H(G_1, G_2))$.

From the results of [4] it follows that there exists a 2-partition φ of the set $E(H(G_1, G_2))$, at which for $\forall v \in V_1(H(G_1, G_2)) \cup V_2(H(G_1, G_2))$

$$b\varphi(\eta_{H(G_1, G_2)}(v)) \leq 1.$$

Theorem: If $G_1$and G_2 are forests with $V(G_1) = V(G_2) \equiv V$, then there exists a simultaneous 2-locally-balanced 2-partition of $G_1$and G_2.

Proof: Define a 2-partition F of the set V as follows: for $\forall v \in V \quad F(v) \equiv \varphi(\xi(v))$. We shall be convinced that F is a simultaneous 2-locally-balanced 2-partition of the forests $G_1$and G_2. From the construction of the sets $X(G_1)$ and $X(G_2)$ it follows that for $\forall v \in V$ $\exists A(v) \in X(G_1)$ and $\exists B(v) \in X(G_2)$ such that $|\gamma_{G_1}(v) \setminus A(v)| \leq 1$ and $|\gamma_{G_2}(v) \setminus B(v)| \leq 1$. Therefore it follows that $b_F(\gamma_{G_1}(v)) \leq b_F(A(v)) + 1 = b_\varphi(\{e \in E(H(G_1, G_2))/\xi^{-1}(e) \in A(v)\}) + 1 = b_\varphi(\eta_{H(G_1, G_2)}(v)) + 1 \leq 2$. Similarly, $b_F(\gamma_{G_2}(v)) \leq 2$.

Theorem is proved.

In the end we bring an example, which explains that not for arbitrary two forests $G_1$and G_2 with $V(G_1) = V(G_2) \equiv V$ there exists a 2-partition f of the set V, which is a simultaneous k-locally-balanced 2-partition of the forests $G_1$and G_2 for $k \leq 1$.

Example: Define trees $G_1$and G_2 as follows:

$G_1 = (\{v_1, v_2, v_3, v_4, v_5\}, \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\})$,

$G_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{(v_1, v_5), (v_5, v_4), (v_3, v_2), (v_2, v_1)\})$.

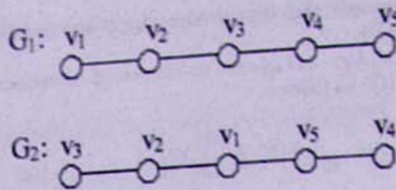

Figure 1.

Let's assume that there exists a 2-partition f of the set $\{v_1, v_2, v_3, v_4, v_5\}$, which is a simultaneous k-locally-balanced 2-partition of the trees $G_1$and G_2 for $k \leq 1$. Without restriction of a generality we can suppose, that $f(v_1) = 0$. From $\gamma_{G_1}(v_2) = \{v_1, v_3\}$ and $\gamma_{G_1}(v_4) = \{v_3, v_5\}$ we can conclude that $f(v_3) = 1$ and $f(v_5) = 0$. Hence, from $\gamma_{G_2}(v_5) = \{v_1, v_4\}$ and $\gamma_{G_2}(v_1) = \{v_2, v_5\}$ we can conclude that $f(v_4) = 1$ and $f(v_2) = 1$. But it means that $b_f(\gamma_{G_1}(v_3)) = 2$, which contradicts the property of f.

References

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Գագաթների համընկնող բազմություններով երկու անտառների միաժամանակյա 2-լոկալ-հավասարակշռված 2-տրոհման մասին

Հ. Գ. Թանանյան և Ռ. Ռ. Քամալյան

Ամփոփում

Ապացուցված է, որ գագաթների համընկնող բազմություններ ունեցող երկու անտառների համար գոյություն ունի նրանց գագաթների բազմության այնպիսի տրոհում երկու ենթաբազմությունների, որի դեպքում յուրաքանչյուր անտառի յուրաքանչյուր գագաթի շրջակայքում այդ երկու ենթաբազմությունների տարրերի քանակների տարբերությունը չի գերազանցում 2-ը, և այդ հաստատումը փոքրացնել հնարավոր չէ: