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On a Generalization of Interval Edge Colorings of Graphs

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Abstract

An interval edge (t,h)–coloring ($h \in Z_+$) of a graph G is a proper coloring α of edges of G with colors $1,2,\ldots,t$ such that at least one edge of G is colored by i, $i=1,2,\ldots,t$ and the colors of edges incident with each vertex v satisfy the condition

$$d_G(v)-1 \le \max S(v,\alpha) - \min S(v,\alpha) \le d_G(v)+h-1,$$

where $d_G(v)$ is the degree of a vertex v and $S(v,\alpha)$ is the set of colors of edges incident with v. In this paper we investigate some properties of interval edge (t,h)–colorings.

1. Introduction.

The graph coloring problems play a crucial role in Discrete Mathematics. The reason for that are the fact of existence of many problems in Discrete Mathematics which can be formulated as graph coloring problems (factorization problems, problems of Ramsey theory, etc.) and the tight relationship between graph coloring problems and scheduling of various timetables. One of the aspects of the problems of scheduling theory is the construction of timetables without "gaps". For studying the coloring problems corresponding to ones of constructing a timetable without a "gap", a definition of interval edge coloring of a graph was introduced [1]. But in real problems the requirement of absence of "gaps", usually, is replaced by a more weak condition, that is one of existence of no more than one or two "gaps". Therefore, it is expedient to consider not only the interval edge colorings but also the colorings which are "close" to them.

The goal of this investigation is the study of a generalization of interval edge colorings of graphs, corresponding to the problems of existence and construction of timetables with no more than h "gaps" ($h \in Z_+$).

All graphs considered in this paper are undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of a vertex of G - by $\Delta(G)$, and the chromatic index of G - by $\chi'(G)$. If α is a proper edge coloring of the graph G, then $\alpha(e)$ denotes the color of an edge $e \in E(G)$ in the coloring α. For a proper edge coloring α of the graph G and for any $v \in V(G)$ we denote by $S(v, \alpha)$ the set of colors of edges incident with v.

Let $R \subseteq V(G)$, $h \in Z_+$.An interval edge $(R, t, h)-$coloring of a graph G is a proper coloring α of edges of G with colors $1, 2, \dots, t$ such that at least one edge of G is colored by i, $i = 1, 2, \dots, t$ and the colors of edges incident with each vertex $v \in R$ satisfy the condition

$$d_G(v) - 1 \leq \max S(v, \alpha) - \min S(v, \alpha) \leq d_G(v) + h - 1.$$

An interval edge $(t, h)-$coloring of G is an interval edge $(V(G), t, h)-$coloring of G.

For $h \in Z_+$ and $t \geq 1$ let $\mathcal{N}_t^h$ denote the set of graphs which have an interval edge $(t, h)-$coloring, and assume: $\mathcal{N}^h = \bigcup_{t \geq 1} \mathcal{N}_t^h$ ($\mathcal{N}^0 = \mathcal{N}$ [2]). Note that $\mathcal{N}^0 \subseteq \mathcal{N}^1 \subseteq \mathcal{N}^2 \subseteq \cdots$. For a graph $G \in \mathcal{N}^h$ the least and greatest values of t, for which $G \in \mathcal{N}_t^h$, are denoted by $w(G, h)$ and $W(G, h)$, respectively ($w(G, 0) = w(G)$, $W(G, 0) = W(G)$ [2]).

Non-defined conceptions and notations can be found in [3, 4, 5].

2. Some upper bounds for $W(G, h)$.

Theorem 1. Let G be a connected graph and $G \in \mathcal{N}^h$, where $h \in Z_+$.Then

$$W(G, h) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) + h - 1),$$

where $\mathbf{P}$ is the set of all simple paths of G.

Proof. Consider an interval edge $(W(G, h), h)-$coloring α of G. Let $\alpha(e) = 1$, $\alpha(e') = W(G, h)$, $e = (u_1, u_2)$, $e' = (v_1, v_2)$. Without loss of generality we may assume that a simple path P joining e with e' joins u_1 with v_1, where

$$P = (x_1, e_1, x_2, e_2, \dots, x_i, e_i, x_{i+1}, \dots, x_k, e_k, x_{k+1}), \; x_1 = u_1, \; x_{k+1} = v_1.$$

Note that

$$\begin{gathered}
\alpha(e_1) \leq d_G(x_1) + h, \\
\alpha(e_2) \leq \alpha(e_1) + d_G(x_2) + h - 1, \\
\dots\dots\dots\dots\dots\dots \\
\alpha(e_i) \leq \alpha(e_{i-1}) + d_G(x_i) + h - 1, \\
\dots\dots\dots\dots\dots\dots \\
W(G, h) = \alpha(e') = \alpha((x_{k+1}, v_2)) \leq \alpha(e_k) + d_G(x_{k+1}) + h - 1.
\end{gathered}$$

Adding these inequalities we obtain

$$W(G, h) \leq \sum_{i=1}^{k+1} d_G(x_i) - k + (k+1) h.$$

From that we have

$$W(G, h) \leq 1 + \sum_{i=1}^{k+1} (d_G(x_i) + h - 1) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) + h - 1).$$

The proof is complete.

Corollary. Let G be a connected graph and $G \in \mathcal{N}^h$, where $h \in Z_+$. Then

$$W(G,h) \leq (d(G)+1)(\Delta(G)+h-1)+1,$$

where $d(G)$ is the diameter of G.

Remark 1. Note that the bound of theorem 1 is sharp for trees [6] in case $h=0$.

Theorem 2. Let G be a connected bipartite graph and $G \in \mathcal{N}^h$, where $h \in Z_+$. Then

$$W(G,h) \leq d(G)(\Delta(G)+h-1)+1,$$

where $d(G)$ is the diameter of G.

Proof. Consider an interval edge $(W(G,h),h)$−coloring α of G. Let $\alpha(e)=1$, $\alpha(e')=W(G,h)$, $e=(u_1,u_2)$, $e'=(v_1,v_2)$. Let P_{ij} be the shortest path joining u_i with v_j, $i=1,2$, $j=1,2$. Let P' be the shortest path among P_{ij}, $i=1,2$, $j=1,2$. Without loss of generality we may assume that P' joins u_1 with v_1. Now consider the path P_{21}. If the length of P_{21} is equal to the length of P' then G has an odd cycle but this is impossible because G is bipartite. Consequently P_{21} is longer than P'. Therefore the length of P' is not greater than $d(G)-1$. Let

$$P' = (x_1, e_1, x_2, e_2, \dots, x_i, e_i, x_{i+1}, \dots, x_k, e_k, x_{k+1}), \text{ where } x_1 = u_1,\ x_{k+1} = v_1.$$

Note that

$$\alpha((x_i,x_{i+1})) \leq 1 + \sum_{j=1}^{i} (d_G(x_j)+h-1),\ i=1,2,\dots,k.$$

From that and $k \leq d(G)-1$ we obtain

$$W(G,h) = \alpha(e') = \alpha((x_{k+1},v_2)) \leq 1 + \sum_{i=1}^{k+1} (d_G(x_i)+h-1) \leq 1 + d(G)(\Delta(G)+h-1).$$

The proof is complete.

Remark 2. Note that the bound of theorem 2 is sharp for complete bipartite graphs $K_{n,n}$ [6] in case $h=0$.

3. Exact values of $w(G,1)$ and $W(G,1)$.

In the following we will determine $w(G,1)$ and $W(G,1)$ for some classes of graphs.

Lemma. Let G be a regular graph.

1) If $h \in N$ then $G \in \mathcal{N}^h$ and $w(G,h) = \chi'(G)$.

2) If $w(G,1) \leq t \leq W(G,1)$ then $G \in \mathcal{N}_t^1$.

Proof. From the result of [4] it follows that 1) holds.

Let us prove 2).

It is clear that $w(G,1) = \chi'(G)$. If $t = \chi'(G)+1$ then interval edge $(\chi'(G)+1,1)$−coloring can be obtained from interval edge $(\chi'(G),1)$−coloring by recoloring one edge of color 1 with color $\chi'(G)+1$. If $t \geq \chi'(G)+2$ then interval edge $(t-1,1)$−coloring can be obtained from interval edge $(t,1)$ –coloring α by recoloring each edge (u,v) having color t with color $\min(\min S(u,\alpha), \min S(v,\alpha)) - 1$.

Proposition 1. For $n \geq 3$

1) $C_n \in \mathcal{N}^1$,

2) $w(C_n, 1) = \chi'(C_n)$,

3) $W(C_n, 1) = n$,

4) if $w(C_n, 1) \leq t \leq W(C_n, 1)$ then $C_n \in \mathcal{N}^1_t$.

Proof. 1) and 2) follow from lemma.

Let us prove 3).

It is clear that $W(C_n, 1) \leq |E(C_n)| = n$.

Now we show that $W(C_n, 1) \geq n$.

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_1, v_n)\}$.

Define a coloring α of the edges of the graph C_n in the following way:

1) for $i = 1, ..., \left\lceil \frac{n}{2} \right\rceil$ $\alpha((v_i, v_{i+1})) = 2i - 1$;

2) for $j = \left\lceil \frac{n}{2} \right\rceil + 1, ..., n-1$ $\alpha((v_j, v_{j+1})) = 2(n-j+1)$;

3) $\alpha((v_1, v_n)) = 2$.

It is not difficult to see that α is an interval edge $(n,1)$–coloring of the graph C_n.

4) follows from lemma.

The proof is complete.

Proposition 2. For any $n \in N$

1) $K_{n,n} \in \mathcal{N}^1$,

2) $w(K_{n,n}, 1) = n$,

3) $W(K_{n,n}, 1) = \begin{cases} n, & \text{if } n = 1, \\ 2n, & \text{if } n = 2, \\ 2n+1, & \text{if } n \geq 3, \end{cases}$

4) if $w(K_{n,n}, 1) \leq t \leq W(K_{n,n}, 1)$ then $K_{n,n} \in \mathcal{N}^1_t$.

Proof. 1) and 2) follow from lemma.

Let us prove 3).

Clearly, 3) is true for the case $n \leq 2$.

Assume that $n \geq 3$.

Since $d(K_{n,n}) = 2$ and $\Delta(K_{n,n}) = n$ then from theorem 2 we have $W(K_{n,n}, 1) \leq 2n + 1$.

Now we show that for $n \geq 3$ $W(K_{n,n}, 1) \geq 2n + 1$.

Let $V(K_{n,n}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and

$E(K_{n,n}) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq n\}$.

Define a coloring α of the edges of the graph $K_{n,n}$ in the following way:

1) $\alpha((u_1, v_1)) = 1$, $\alpha((u_1, v_2)) = 2$, $\alpha((u_2, v_1)) = 3$;

2) for $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, $4 \leq i + j \leq 2n - 2$ $\alpha((u_i, v_j)) = i + j$;

3) $\alpha((u_n, v_{n-1})) = 2n - 1$, $\alpha((u_{n-1}, v_n)) = 2n$, $\alpha((u_n, v_n)) = 2n + 1$.

It is not difficult to see that α is an interval edge $(2n+1, 1)$–coloring of the graph $K_{n,n}$.

4) follows from lemma.

The proof is complete.

Remark 3. From proposition 2 it follows that the bound of theorem 2 is sharp for complete bipartite graphs $K_{n,n}$ in case $h = 1$.

Proposition 3. For $n \geq 2$

1) $K_n \in \mathcal{N}^1$,

2) $w(K_n, 1) = \chi'(K_n)$,

3) $W(K_n, 1) = 2n - 3$,
4) if $w(K_n, 1) \leq t \leq W(K_n, 1)$ then $K_n \in \mathcal{N}_t^1$.

Proof. 1) and 2) follow from lemma.
Let us prove 3).
First we show that for $n \geq 2$ $W(K_n, 1) \geq 2n - 3$.
Let $V(K_n) = \{v_1, v_2, ..., v_n\}$ and $E(K_n) = \{(v_i, v_j) \mid v_i \in V(K_n),\ v_j \in V(K_n),\ i < j\}$.
Now we define an edge coloring α of the graph K_n.
For $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$, where $i \neq j$, we set:

$$\alpha((v_i, v_j)) = i + j - 2.$$

It is not difficult to see that α is an interval edge $(2n-3, 1)$–coloring of the graph K_n.
From corollary we have $W(K_n, 1) \leq 2n - 1$.
It is easy to see that $W(K_n, 1) \leq 2n - 2$.
Now we show that $W(K_n, 1) \leq 2n - 3$.
Clearly, the statement is true for the case $n \leq 4$.
Assume that $n \geq 5$.
Suppose that α is an interval edge $(2n-2, 1)$–coloring of the graph K_n.
Let $v_{i_0} \in V(K_n)$ and $\min S(v_{i_0}, \alpha) = 1$.
Case 1: $\max S(v_{i_0}, \alpha) = n - 1$.
Clearly, without loss of generality we may assume that

$$\alpha((v_{i_0}, v_{i_1})) < \alpha((v_{i_0}, v_{i_2})) < \cdots < \alpha((v_{i_0}, v_{i_k})) < \cdots < \alpha\left(\left(v_{i_0}, v_{i_{n-1}}\right)\right),$$

where $\alpha((v_{i_0}, v_{i_k})) = k$, $k = 1, 2, \ldots, n-1$.
It is easy to see that for $k = 1, 2, \ldots, n-2$ $\max S(v_{i_k}, \alpha) \leq 2n - 3$.Therefore $\max S\left(v_{i_{n-1}}, \alpha\right) \leq 2n - 3$ and the proof of case 1 is complete.
Case 2: $\max S(v_{i_0}, \alpha) = n$.
(i) $(n-1) \in \{1, 2, \ldots, n\} \setminus S(v_{i_0}, \alpha)$.
Without loss of generality we may assume that

$$\alpha((v_{i_0}, v_{i_1})) < \alpha((v_{i_0}, v_{i_2})) < \cdots < \alpha((v_{i_0}, v_{i_k})) < \cdots < \alpha\left(\left(v_{i_0}, v_{i_{n-1}}\right)\right),$$

where $\alpha((v_{i_0}, v_{i_k})) = k$, $k = 1, 2, \ldots, n-2$, and $\alpha\left(\left(v_{i_0}, v_{i_{n-1}}\right)\right) = n$.
It is easy to see that for $k = 1, 2, \ldots, n-2$ $\max S(v_{i_k}, \alpha) \leq 2n - 3$.Therefore $\max S\left(v_{i_{n-1}}, \alpha\right) \leq 2n - 3$ and the proof of case 2(i) is complete.
(ii) $(n-1) \notin \{1, 2, \ldots, n\} \setminus S(v_{i_0}, \alpha)$.
Without loss of generality we may assume that

$$\alpha((v_{i_0}, v_{i_1})) < \cdots < \alpha\left(\left(v_{i_0}, v_{i_{l-1}}\right)\right) < \alpha((v_{i_0}, v_{i_l})) < \cdots < \alpha\left(\left(v_{i_0}, v_{i_{n-1}}\right)\right),$$

where $l \in \{2, 3, \ldots, n-2\}$ and $\alpha((v_{i_0}, v_{i_k})) = k$, $k = 1, \ldots, l-1$, $\alpha((v_{i_0}, v_{i_k})) = k + 1$, $k = l, \ldots, n-1$.
It is easy to see that $\alpha\left(\left(v_{i_{n-2}}, v_{i_{n-1}}\right)\right) = 2n-2$ and $\alpha\left(\left(v_{i_1}, v_{i_{n-2}}\right)\right) = n$, $\alpha\left(\left(v_{i_1}, v_{i_{n-1}}\right)\right) = n-1$.Let v_{i_m} be a vertex, which is incident with edge of color 2, $m \neq 0, 1, n-1, n$. Consider the edges $\left(v_{i_m}, v_{i_{n-2}}\right)$ and $\left(v_{i_m}, v_{i_{n-1}}\right)$. Clearly, $\max S(v_{i_m}, \alpha) \leq n + 1$.Therefore either $\max S\left(v_{i_{n-2}}, \alpha\right) \leq 2n - 3$ or $\max S\left(v_{i_{n-1}}, \alpha\right) \leq 2n - 3$, which is a contradiction.

4) follows from lemma.

The proof is complete.

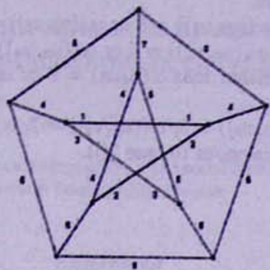

Figure 1.

Proposition 4. If P is the Petersen graph. Then:

1) $P \in \mathcal{N}^1$;

2) $w(P,1) = 4$;

3) $W(P,1) = 8$;

4) if $w(P,1) \leq t \leq W(P,1)$ then $P \in \mathcal{N}_t^1$.

Proof. 1) and 2) follow from lemma.

Let us prove 3).

From Figure 1. we have $W(P,1) \geq 8$.

Since $d(P) = 2$ and $\Delta(P) = 3$ then from corollary we have $W(P,1) \leq 10$.

Now we show that $W(P,1) \leq 8$.

It is easy to see that $W(P,1) \leq 9$.

Suppose that α is an interval edge $(9,1)$ −coloring of the graph P.

Let $V(P) = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ and

$E(P) = \{(u_1,u_2), (u_2,u_3), (u_3,u_4), (u_4,u_5), (u_1,u_5), (v_1,v_3), (v_1,v_4), (v_2,v_4), (v_2,v_5), (v_3,v_5), (u_1,v_1), (u_2,v_2), (u_3,v_3), (u_4,v_4), (u_5,v_5)\}$.

Without loss of generality we may assume that $\alpha((u_1,u_2)) = 9$.

Case 1: $\alpha((u_4,v_4)) = 1$.

Clearly, either $\alpha((u_1,u_5)) = 7$, $\alpha((u_4,u_5)) = 4$, $\alpha((u_2,u_3)) = 6$, $\alpha((u_3,u_4)) = 3$ or $\alpha((u_1,u_5)) = 6$, $\alpha((u_4,u_5)) = 3$, $\alpha((u_2,u_3)) = 7$, $\alpha((u_3,u_4)) = 4$.

(i) $\alpha((u_1,u_5)) = 7$, $\alpha((u_4,u_5)) = 4$, $\alpha((u_2,u_3)) = 6$, $\alpha((u_3,u_4)) = 3$.

It is not difficult to check that $\alpha((u_2,v_2)) = 7$, $\alpha((v_2,v_4)) = 4$, $\alpha((u_1,v_1)) = 6$ and $\alpha((v_1,v_4)) = 3$. Hence we have

1) either $\alpha((v_1,v_3)) = 4$, $\alpha((u_3,v_3)) = 5$ or $\alpha((v_1,v_3)) = 5$, $\alpha((u_3,v_3)) = 4$,

2) either $\alpha((v_2,v_5)) = 5$, $\alpha((u_5,v_5)) = 6$ or $\alpha((v_2,v_5)) = 6$, $\alpha((u_5,v_5)) = 5$.

From 1) and 2) we have either $\max S(v_5,\alpha) = 6$ or $\min S(v_3,\alpha) = 4$ and the proof of case 1(i) is complete.

(ii) $\alpha((u_1,u_5)) = 6$, $\alpha((u_4,u_5)) = 3$, $\alpha((u_2,u_3)) = 7$, $\alpha((u_3,u_4)) = 4$.

This case is considered analogous to case 1(i).

Case 2: $\alpha((v_3,v_5)) = 1$.

Clearly, either $\alpha((u_1,u_5)) = 7$, $\alpha((u_5,v_5)) = 4$, $\alpha((u_2,v_2)) = 6$, $\alpha((v_2,v_5)) = 3$ or $\alpha((u_1,u_5)) = 6$, $\alpha((u_5,v_5)) = 3$, $\alpha((u_2,v_2)) = 7$, $\alpha((v_2,v_5)) = 4$.

(i) $\alpha((u_1, u_5)) = 7$, $\alpha((u_5, v_5)) = 4$, $\alpha((u_2, v_2)) = 6$, $\alpha((v_2, v_5)) = 3$.
It is not difficult to check that $\alpha((u_2, u_3)) = 7$, $\alpha((u_3, v_3)) = 4$, $\alpha((u_1, v_1)) = 6$ and $\alpha((v_1, v_3)) = 3$. Hence we have

1) either $\alpha((v_1, v_4)) = 4$, $\alpha((v_2, v_4)) = 5$ or $\alpha((v_1, v_4)) = 5$, $\alpha((v_2, v_4)) = 4$,

2) either $\alpha((u_3, u_4)) = 6$, $\alpha((u_4, u_5)) = 5$ or $\alpha((u_3, u_4)) = 5$, $\alpha((u_4, u_5)) = 6$.

From 1) and 2) we have either $\max S(u_4, \alpha) = 6$ or $\min S(v_4, \alpha) = 4$ and the proof of case 2(i) is complete.

(ii) $\alpha((u_1, u_5)) = 6$, $\alpha((u_5, v_5)) = 3$, $\alpha((u_2, v_2)) = 7$, $\alpha((v_2, v_5)) = 4$.
This case is considered analogous to case 2(i).

4) follows from lemma.

The proof is complete.

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Գրաֆների միջակայքային կողային ներկումների ընդհանրացման մասին

Պ. Ա. Պետրոսյան և Հ. Զ. Առաքելյան

Ամփոփում

G գրաֆի կողերի ճիշտ α ներկումը $1, 2, ..., t$ գույներով կանվանենք միջակայքային կողային (t, h)-ներկում ($h \in Z_+$), եթե ամեն մի i գույնով, $i = 1, 2, ..., t$ ներկված է առնվազն մեկ կող և յուրաքանչյուր գագաթին կից կողերի գույները բավարարում են հետևյալ պայմանին՝ $d_G(v) - 1 \leq \max S(v, \alpha) - \min S(v, \alpha) \leq d_G(v) + h - 1$, որտեղ $d_G(v)$-ն v գագաթի աստիճանն է G-ում, իսկ $S(v, \alpha)$ v գագաթին կից կողերի գույների բազմությունն է: Այս աշխատանքում հետազոտվում են միջակայքային կողային (t, h) ներկումների որոշ հատկություններ: