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On Cyclically Continuous Edge Colorings of Trees

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Abstract

A cyclically continuous edge coloring of a graph is defined. For an arbitrary tree the existence of this coloring is proved and all possible numbers of colors in such colorings are found.

We consider finite, undirected graphs without loops or multiple edges [1]. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. If $v \in V(G)$ then let $d_G(v)$ denote the degree of a vertex v in a graph G. For a graph G let $\Delta(G)$ be the greatest degree of a vertex of G, $\chi'(G)$ be the chromatic index of G [2]. The set of edges of G incident to a vertex $x \in V(G)$ is denoted by $J_{G,e}(x)$. The set of vertices of G adjacent to a vertex $x \in V(G)$ is denoted by $J_{G,v}(x)$.

Let $\rho(x, y)$ denote the distance between the vertices $x \in V(G)$ and $y \in V(G)$. For a vertex $x_0 \in V(G)$ and $V_0 \subseteq V(G)$ set:

$$\rho(x_0, V_0) = \min_{z \in V_0} \rho(x_0, z).$$

Non-defined terms and concepts can be found in [1, 3].

The set of positive integers is denoted by N, the cardinality of an arbitrary finite set A is denoted by $|A|$. If D is a finite non-empty subset of N then let $l(D)$ and $L(D)$ denote the least and the greatest element of D, respectively. A non-empty finite subset D of N is referred as interval if $l(D) \le t \le L(D)$, $t \in N$ implies that $t \in D$. An interval D is called h-interval if $|D| = h$. An interval D is denoted by $Int(q, h)$ if $l(D) = q$, $|D| = h$.

For $\forall t \in N$ and arbitrary i_1, i_2 satisfying the conditions $1 \le i_1 \le t$, $1 \le i_2 \le t$, define the sets $intcyc_1((i_1, i_2), t)$, $intcyc_1[(i_1, i_2), t]$, $intcyc_2((i_1, i_2), t)$, $intcyc_2[(i_1, i_2), t]$, and the number $dif((i_1, i_2), t)$ as follows:

$$intcyc_1[(i_1, i_2), t] \equiv Int(\min(i_1, i_2), \max(i_1, i_2) - \min(i_1, i_2) + 1),$$
$$intcyc_1((i_1, i_2), t) \equiv intcyc_1[(i_1, i_2), t] \backslash (\{i_1\} \cup \{i_2\}),$$
$$intcyc_2((i_1, i_2), t) \equiv Int(1, t) \backslash intcyc_1[(i_1, i_2), t],$$
$$intcyc_2[(i_1, i_2), t] \equiv Int(1, t) \backslash intcyc_1((i_1, i_2), t),$$
$$dif((i_1, i_2), t) \equiv \min(|intcyc_1[(i_1, i_2), t]|, |intcyc_2[(i_1, i_2), t]|) - 1.$$

A non-empty set $M \subset N$ is called t-cyclic interval if there are i_1, i_2, j_0, t with $1 \le i_1 \le t$, $1 \le i_2 \le t$, $j_0 \in \{1, 2\}$, such that $M = intcyc_{j_0}[(i_1, i_2), t]$.

A function $\varphi : E(G) \to Int(1, t)$ is referred as a proper edge t-coloring of a graph G if

1) for any adjacent edges $e_1 \in E(G), e_2 \in E(G)$ $\varphi(e_1) \neq \varphi(e_2)$,

2) for each i, $1 \leq i \leq t$, there is $e \in E(G)$ such that $\varphi(e) = i$.

If φ is a proper edge t-coloring of a graph G and $E_0 \subseteq E(G)$ then $\varphi_{E_0} \equiv \{\varphi(e)/e \in E_0\}$.

A proper edge t-coloring φ of a graph G is called interval t-coloring of G [4] if for $\forall x \in V(G)$ the set $\varphi_{J_{G,e}(x)}$ is a $d_G(x)$-interval. Let $\mathcal{N}_{\lambda_1,t}$ denote the set of graphs, for which there is an interval t-coloring, and assume:

$$\mathcal{N}_{\lambda_1} \equiv \bigcup_{t \geq 1} \mathcal{N}_{\lambda_1,t}.$$

For $G \in \mathcal{N}_{\lambda_1}$ let $w_{\lambda_1}(G)$ and $W_{\lambda_1}(G)$ be the least and the greatest possible value of t, respectively, for which $G \in \mathcal{N}_{\lambda_1,t}$.

A proper edge t-coloring φ of a graph G is called cyclically continuous t-coloring of a graph G if for $\forall x \in V(G)$ the set $\varphi_{J_{G,e}(x)}$ is a t-cyclic interval. Let $\mathcal{N}_{\lambda_2,t}$ denote the set of graphs, for which there is a cyclically continuous t-coloring, and assume:

$$\mathcal{N}_{\lambda_2} \equiv \bigcup_{t \geq 1} \mathcal{N}_{\lambda_2,t}.$$

For $G \in \mathcal{N}_{\lambda_2}$ let $w_{\lambda_2}(G)$ and $W_{\lambda_2}(G)$ be the least and the greatest possible value of t, respectively, for which $G \in \mathcal{N}_{\lambda_2,t}$.

It is clear that an interval t-coloring of a graph G is a cyclically continuous t-coloring of a graph G. This implies that for $\forall t \in N$ $\mathcal{N}_{\lambda_1,t} \subseteq \mathcal{N}_{\lambda_2,t}$ and $\mathcal{N}_{\lambda_1} \subseteq \mathcal{N}_{\lambda_2}$. It is also clear that for $\forall G \in \mathcal{N}_{\lambda_1}$ the following inequalities are true:

$$\Delta(G) \leq \chi'(G) \leq w_{\lambda_2}(G) \leq w_{\lambda_1}(G) \leq W_{\lambda_1}(G) \leq W_{\lambda_2}(G) \leq |E(G)|.$$

For a tree D, with $V(D) = \{b_1, ..., b_p\}$, $p \geq 1$, let $P(b_i, b_j)$ be the simple path connecting the vertices b_i and b_j, $VP(b_i, b_j)$ and $EP(b_i, b_j)$ be the sets of vertices and edges of this path, respectively, $1 \leq i \leq p$, $1 \leq j \leq p$. Define:

$$intVP(b_i, b_j) \equiv VP(b_i, b_j) \backslash (\{b_i\} \cup \{b_j\});$$

$$\tilde{V}P(b_i, b_j) \equiv VP(b_i, b_j) \cup \Big(\bigcup_{x \in intVP(b_i,b_j)} J_{G,v}(x) \Big);$$

$$TP(b_i, b_j) \equiv \begin{cases} \bigcup\limits_{x \in intVP(b_i,b_j)} J_{G,e}(x) & \text{if } intVP(b_i, b_j) \neq \emptyset, \\ EP(b_i, b_j) & \text{if } intVP(b_i, b_j) = \emptyset; \end{cases}$$
$$1 \leq i \leq p,\ 1 \leq j \leq p.$$

Assume:

$$M(D) \equiv \max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}} |TP(b_i, b_j)|.$$

Theorem 1 [5]. Let D be a tree. Then

1) $D \in \mathcal{N}_{\lambda_1}$,

2) $w_{\lambda_1}(G) = \Delta(G)$,

3) $W_{\lambda_1}(G) = M(D)$,

4) if $w_{\lambda_1}(G) \leq t \leq W_{\lambda_1}(G)$, then $D \in \mathcal{N}_{\lambda_1,t}$.

Lemma 1. If $M_1, ..., M_n$ $(n \geq 2)$ are t-cyclic intervals, and for $\forall j$, $1 \leq j \leq n-1$ $M_j \cap M_{j+1} \neq \emptyset$, then $\bigcup_{i=1}^{n} M_i$ is a t-cyclic interval.

Proof can be easily done by induction on n.

Lemma 2. Let α be a cyclically continuous t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ to a vertex $x_k \in V(G)$, $k \geq 2$. Then $\alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}$ is a t-cyclic interval.

Proof. If $k = 2$ then the statement follows from the definition of the cyclically continuous t-coloring. Now assume that $k \geq 3$. It is clear that the sets $\alpha_{J_{G,\alpha}(x_1)}, ..., \alpha_{J_{G,\alpha}(x_{k-1})}$, are t-cyclic intervals, with

$$\alpha_{J_{G,\alpha}(x_j)} \cap \alpha_{J_{G,\alpha}(x_{j+1})} \neq \emptyset \text{ for } j = 1, ..., k-2,$$

as

$$\alpha(e_{j+1}) \in (\alpha_{J_{G,\alpha}(x_j)} \cap \alpha_{J_{G,\alpha}(x_{j+1})}) \text{ for } j = 1, ..., k-2.$$

Lemma 1 implies that $\alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}$ is a t-cyclic interval. The proof of lemma 2 is completed.

Lemma 3. Let α be a cyclically continuous t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ to a vertex $x_k \in V(G)$, $k \geq 2$. Then at least one of the following statements is true:

1) $intcyc_1((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}$,

2) $intcyc_2((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}$.

Proof. Without loss of generality, we may assume that $dif((\alpha(e_1), \alpha(e_k)), t) \geq 2$.

Let us assume that none of the statements of 1) and 2) is true. Then there are τ_1, τ_2 such that

$$\tau_1 \in intcyc_1((\alpha(e_1), \alpha(e_k)), t),\ \tau_1 \notin \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)},$$
$$\tau_2 \in intcyc_2((\alpha(e_1), \alpha(e_k)), t),\ \tau_2 \notin \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)},$$

therefore $\{\tau_1, \tau_2\} \cap \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)} = \emptyset$.

Lemma 2 implies that $\alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}$ is a t-cyclic interval, with

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)}.$$

It is not hard to see that the relations

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)} \text{ and } \{\tau_1, \tau_2\} \cap \alpha_{\bigcup_{i=1}^{k-1} J_{G,\alpha}(x_i)} = \emptyset$$

are incompatible. The proof of lemma 3 is completed.

Lemma 4. If α is a cyclically continuous t-coloring of a tree D, $V(D) = \{b_1, ..., b_p\}$, $p \geq 1$, then there are vertices $b_i \in V(D)$, $b_j \in V(D)$ such that $Int(1,t) = \alpha_{TP(b_i,b_j)}$.

Proof. Assume the contrary. Suppose that for an arbitrary $b_i \in V(D)$, $b_j \in V(D)$ $\alpha_{TP(b_i,b_j)} \subset Int(1,t)$. Set: $\max\limits_{\substack{1\leq i\leq p\\ 1\leq j\leq p}} \left|\alpha_{TP(b_i,b_j)}\right| \equiv m_0$. It is clear that $m_0 < t$.
Without loss of generality, we may assume that $m_0 \geq 2$. Consider the simple path $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ with $|\alpha_{TP_0}| = m_0$. Clearly, without loss of generality, we may assume that $k \geq 2$.

Lemma 2 implies that there are i', i'', j' with $1 \leq i' \leq t$, $1 \leq i'' \leq t$, $j' \in \{1,2\}$ such that $\alpha_{\bigcup_{i=1}^{k-1} J_{D,e}(x_i)} = intcyc_{j'}[(i', i''), t]$. As $m_0 < t$, there is $\tau_0 \in Int(1,t)$ such that $\tau_0 \notin intcyc_{j'}[(i', i''), t]$.

Consider the edge $e^1 \in E(D)$ for which $\alpha(e^1) = \tau_0$, and assume that $e^1 = (u_0, u_1)$. Clearly, $e^1 \notin TP_0(x_0, x_k)$.

Without loss of generality, we may assume that $\rho_D(u_1, \tilde{V}P_0(x_0, x_k)) < \rho_D(u_0, \tilde{V}P_0(x_0, x_k))$. Let $z_0 \in \tilde{V}P_0(x_0, x_k)$ be a vertex with $\rho_D(u_1, z_0) = \rho_D(u_1, \tilde{V}P_0(x_0, x_k))$. It is not hard to see that $z_0 \in \tilde{V}P_0(x_0, x_k) \backslash intVP_0(x_0, x_k)$ and for $\forall z' \in \tilde{V}P_0(x_0, x_k) \backslash intVP_0(x_0, x_k)$, $z' \neq z_0$ $\rho_D(u_1, z_0) < \rho_D(u_1, z')$.

Case 1. $z_0 = x_0$. Clearly, $\left|\alpha_{TP(u_0,x_k)}\right| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 2. $z_0 = x_k$. This case is considered analogous to case 1.

Case 3. $z_0 \neq x_0$, $z_0 \neq x_k$.

Clearly, there is $\tilde{x} \in intVP_0(x_0, x_k)$ such that $z_0 \in J_{G,v}(\tilde{x})$. Suppose that $\alpha((z_0, \tilde{x})) = \tau'$. Clearly, $i' \neq i''$.

Case 3a. $\tau' = i'$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = i'$ and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1,2\}$ such that $intcyc_{j_1}[(\tau_0, i'), t] \subseteq \alpha_{\bigcup_{x\in intVP(u_0,\tilde{x})} J_{D,e}(x)}$. Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality, we may assume that $\rho_D(z_0, x') < \rho_D(z_0, x'')$. It is not hard to check that $TP(z_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha_{TP(z_0,x'')}$. Lemma 2 implies that $\alpha_{TP(z_0,x'')}$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1,2\}$ such that $\tau_0 \in intcyc_{j_2}((i', i''), t)$ and therefore $intcyc_{j_2}((i', i''), t) \nsubseteq \alpha_{TP(z_0,x'')}$.

This conclusion, the equalities $\alpha((z_0, \tilde{x})) = i'$, $\alpha(\tilde{e}) = i''$ and lemma 3 imply that $intcyc_{3-j_2}[(i', i''), t] \subseteq \alpha_{TP(z_0,x'')}$, hence $\left|\alpha_{TP(u_0,x'')}\right| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 3b. $\tau' = i''$. This case is considered analogous to case 3a by changing the roles of i' and i''.

Case 3c. $\tau' \notin \{i', i''\}$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = \tau'$ and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1,2\}$ such that $intcyc_{j_1}[(\tau_0, \tau'), t] \subseteq \alpha_{\bigcup_{x\in intVP(u_0,\tilde{x})} J_{D,e}(x)}$. This implies that at least one of the following statements is true:

1) $i' \in intcyc_{j_1}[(\tau_0, \tau'), t]$,

2) $i'' \in intcyc_{j_1}[(\tau_0, \tau'), t]$.

Without loss of generality, let us assume that the statement 1) is true. Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality, we may assume that $\rho_D(x_0, x') < \rho_D(x_0, x'')$. It is not hard to check that $TP(x_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha_{TP(x_0,x'')}$. Lemma 2 implies that $\alpha_{TP(x_0,x'')}$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1, 2\}$ such that $\tau_0 \in intcyc_{j_2}((\tau', i''), t)$ and therefore $intcyc_{j_2}((\tau', i''), t) \nsubseteq \alpha_{TP(x_0,x'')}$. This conclusion, the equalities $\alpha((x_0, \tilde{x})) = \tau'$, $\alpha(\tilde{e}) = i''$ and lemma 3 imply that $intcyc_{3-j_2}[(\tau', i''), t] \subseteq \alpha_{TP(x_0,x'')}$, hence $\left|\alpha_{TP(x_0,x'')}\right| \geq m_0 + 1$, which contradicts the choice of P_0. The proof of lemma 4 is completed.

Corollary 1. If α is a cyclically continuous t-coloring of a tree D, then there are vertices $x' \in V(D)$, $x'' \in V(D)$ such that $t \leq |TP(x', x'')|$.

Proof follows from lemma 4 and the inequality $\left|\alpha_{TP(x,y)}\right| \leq |TP(x, y)|$ for arbitrary vertices $x \in V(D)$, $y \in V(D)$.

Theorem 1 and corollary 1 imply:

Theorem 2. Let D be a tree. Then

1) $D \in \mathcal{N}_{\lambda_2}$,
2) $w_{\lambda_2}(G) = \Delta(G)$,
3) $W_{\lambda_2}(G) = M(D)$,
4) if $w_{\lambda_2}(G) \leq t \leq W_{\lambda_2}(G)$, then $D \in \mathcal{N}_{\lambda_2,t}$.

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Ծառերի ցիկլային-անընդհատ կողային ներկումների մասին

Ռ. Ռ. Քամալյան

Ամփոփում

Տրված է գրաֆի ցիկլային-անընդհատ կողային ներկման սահմանումը, կամայական ծառի համար ապացուցված է այդպիսի ներկման գոյությունը և գտնված են մասնակցող գույների քանակի բոլոր հնարավոր արժեքները: