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On Edge-Disjoint Pairs of Matchings*

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Abstract

For a given graph consider the pairs of edge-disjoint matchings whose union contains as many edges as possible, and consider the relation of the cardinality of a maximum matching to the cardinality of the largest matching among such pairs. We show that 5/4 is a tight upper bound for this relation.

We will consider finite, undirected graphs without multiple edges or loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The cardinality of a maximum matching of a graph G is denoted by $\beta(G)$.

For a graph G define $B_2(G)$ as follows:

$$B_2(G) \equiv \{(H, H')/H, H' \text{ are edge-disjoint matchings of } G\},$$

and set:

$$\lambda(G) \equiv \max\{|H| + |H'|/(H, H') \in B_2(G)\}.$$

Assume:

$$\alpha(G) \equiv \max\{|H|, |H'|/\ (H, H') \in B_2(G) \text{ and } |H| + |H'| = \lambda(G)\},$$
$$M_2(G) \equiv \{(H, H')/(H, H') \in B_2(G), |H| + |H'| = \lambda(G), |H| = \alpha(G)\}.$$

It is clear, that for every graph G the inequality $\alpha(G) \leq \beta(G)$ holds. An intriguing sufficient condition for the equality $\alpha(G) = \beta(G)$ is obtained in [4], which due to [2] is equivalent to the following: for every matching covered tree G, the equality $\alpha(G) = \beta(G)$ holds. Note that a graph G is referred to be matching covered [3, 5] if its every edge belongs to a maximum matching of G.

The aim of this paper is to obtain a tight upper bound for $\frac{\beta(G)}{\alpha(G)}$. We prove that $\frac{5}{4}$ is an upper bound for $\frac{\beta(G)}{\alpha(G)}$, and exhibit a family of graphs, which shows that $\frac{5}{4}$ can not be replaced by a smaller constant. Non defined terms and concepts can be found in [1, 3, 6].

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Lemma. Let G be a graph. Choose a maximum matching M of G and a pair $(H, H') \in M_2(G)$ satisfying the condition:

$|M \cap H|$ is maximized, and then among those matchings choose one satisfying the condition:

$|M \cap H'|$ is maximized.

Define the sets M_A and H_A as ones of edges which lie on $M-H$ alternating paths of odd length and belong to M and H, respectively, and let X be the set of $M_A - H'$ alternating paths. Then every edge of H' either belongs to M or lies on a path from X, and

$$|H'| = |X| + |H_A| + \beta(G) - \alpha(G).$$

Proof. By the choice of M and (H, H'), we have:

(a) there are no $M_A - H'$ alternating cycles; and

(b) each edge of $M_A \backslash H'$ is adjacent to two edges of H', therefore

(c) there are no $M_A - H'$ alternating paths of even length.

Hence, X consists of $M_A - H'$ alternating paths of odd length, which start and end with an edge of H'. This implies that every edge of H' either belongs to M or lies on a path from X, and

$$|H'| = |X| + |M_A|.$$

Since

$$|M_A| = |H_A| + \beta(G) - \alpha(G),$$

we get

$$|H'| = |X| + |H_A| + \beta(G) - \alpha(G).$$

Proof of the **Lemma** is complete.

Theorem. For every graph G inequalities $\frac{\beta(G)}{\alpha(G)} \leq \frac{\beta(G)}{\lambda(G)-\alpha(G)} \leq \frac{5}{4}$ hold.

Proof. Let M and (H, H') be as in the **Lemma**. There are $\beta(G) - \alpha(G)$ $M-H$ alternating paths of odd length. Moreover, if $w_1, (w_1, w_2), w_2, (w_2, w_3), w_3, ..., w_{2l-1}, (w_{2l-1}, w_{2l}), w_{2l}$ is such a path, then, due to $|H| = \alpha(G)$ and the choice of M and (H, H'), we have:

$l \geq 3$;

$\{(w_1, w_2), (w_{2l-1}, w_{2l})\} \subseteq H'$, therefore $|M \cap H'| \geq 2(\beta(G) - \alpha(G))$;

there is i, $2 \leq i \leq l-1$ such that $(w_{2i-1}, w_{2i}) \notin M \backslash (H \cup H')$, therefore

$$|M \backslash (H \cup H')| \geq \beta(G) - \alpha(G),$$

every edge of $M \backslash (H \cup H')$ is adjacent to two edges of H', therefore every edge of H that lies on the path is adjacent to two edges of H', too.

Since $l \geq 3$, we have:

$$|H_A| \geq 2(\beta(G) - \alpha(G)).$$

The choice of M and the pair (H, H') implies that there are no $M-H$ alternating cycles or alternating paths of even length. Consider the $H-H'$ alternating paths starting from vertices w_1, w_{2l} and containing edges $(w_1, w_2) \in H'$ and $(w_{2l-1}, w_{2l}) \in H'$, respectively. Note that they are of even length, therefore the last edges of those paths belong to H and do not belong to $M-H$ alternating paths (see (b) of the proof of the lemma), hence they belong to $M \cap H$; moreover, all edges $(w_1, w_2), (w_{2l-1}, w_{2l})$ (all end edges of $M-H$ alternating paths) lie on different $H-H'$ alternating paths. Hence:

$$|M \cap H| = |M \setminus M_A| = |H \setminus H_A| \geq 2(\beta(G) - \alpha(G)).$$

On the other hand, every end edge of such a path is adjacent to one of H', which, due to **Lemma**, is an end edge of a path from X. Therefore

$$2(\beta(G) - \alpha(G)) \leq |M \cap H| \leq 2|X|,$$

or

$$\beta(G) - \alpha(G) \leq |X|.$$

These inequalities imply:

$$|H'| = |X| + |H_A| + \beta(G) - \alpha(G) \geq 4(\beta(G) - \alpha(G)).$$

Proof of the **Theorem** is complete.

Remark 1. We have given a proof of the **Theorem** which is based on the structural **Lemma**. It is not hard to see that the **Theorem** can also be proved directly, without using the **Lemma**. Note that the $H-H'$ alternating paths which start from vertices w_1, w_{2l}, are of a length at least four, hence

$$|H'| \geq 4(\beta(G) - \alpha(G)).$$

Remark 2. There are infinitely many graphs G for which

$$\frac{\beta(G)}{\alpha(G)} = \frac{5}{4}.$$

In order to construct one, just take an arbitrary graph F containing a perfect matching. Attach to every vertex v of F two paths of the length two, as it is shown in the figure below:

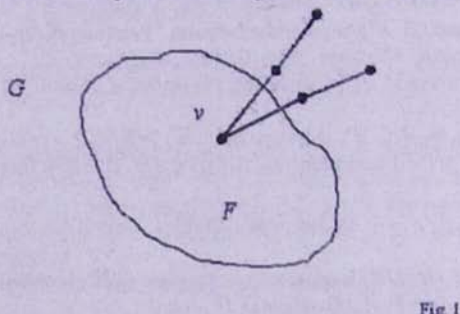

Figure 1.

Let G be the resulting graph. Note that:

$$\beta(G) = \frac{|V(F)|}{2} + 2|V(F)| = \frac{5|V(F)|}{2}.$$

Let us show that for every pair of disjoint matchings (H, H') satisfying $|H| + |H'| = \lambda(G)$ and $e \in E(F)$ we have $e \notin H \cup H'$. On the opposite assumption, consider an edge $e \in E(F)$ and a pair (H, H') with $|H| + |H'| = \lambda(G)$ and $e \in H \cup H'$. Note that without loss of generality, we may always assume that H and H' contain the edges shown in the figure below:

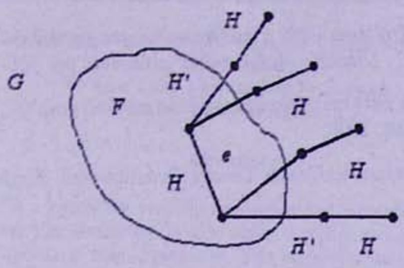

Fig 2

Figure 2.

Now consider new pair of disjoint matchings (H_1, H_1') obtained from (H, H') in the following way:

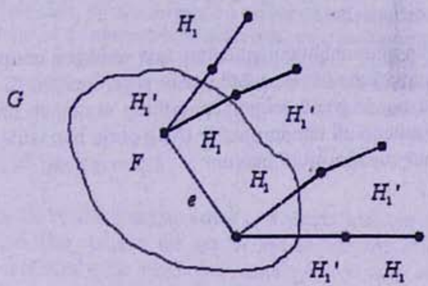

Fig 3

Figure 3.

Note that $|H_1| + |H_1'| = 1 + |H| + |H'| > \lambda(G)$, which contradicts the choice of (H, H'), therefore $e \notin H \cup H'$ and $\lambda(G) = 4|V(F)|$, $\alpha(G) = 2|V(F)|$, hence

$$\frac{\beta(G)}{\alpha(C)} = \frac{5}{4}.$$

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Գրաֆում ընդհանուր կող չունեցող զուգակցումների զույգերի մասին

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Ամփոփում

Տրված գրաֆի համար դիտարկենք ընդհանուր կող չունեցող զուգակցումների զույգերը, որոնց միավորումը պարունակում է ամենաշատ թվով կողեր, և դիտարկենք գրաֆի մաքսիմալ զուգակցման հզորության հարաբերությունը այդպիսի զույգերում ամենաշատ կողեր պարունակող զուգակցման հզորությանը: Մենք ցույց ենք տվել, որ 5/4-ը ճիշտ վերին գնահատական է այս հարաբերության համար: