Mathematical Problems of Computer Science 28, 2007, 51–59.

Quaternionic Fourier Transform for Enhancement and Compression of Color Images

Tigran Manukyan

Institue for Informatics and Automation Problems of NAS of RA
e-mail: tigran_ipia@yahoo.com

Abstract

This paper presents color image processing and analysis using quaternionic Fourier transform (QFT). Algorithm of color image enhancement is described. Image compression using QFT is considered. Results using the suggested method are compared with the results using existing methods.

1 Introduction

Orthogonal transforms such as Fourier, Hadamard, sinusoidal, and wavelet transforms [1]-[3] are widely used in digital image processing problems (compression, filtration, enhancement, and etc.) The mentioned above transforms have the energy compaction property and possess fast algorithms, which determines their intensive usage in various problems of digital image and signal processing.

Gray scale image processing methods are widely explored. Note that, as a rule, color images are processed similar to gray scale images by separate color components Reed, Green, and Blue. For the purpose of color image processing at once, i.e. applying the transform on the color image taken as a whole, we present image pixels in quaternion form and use the QFT in image enhancement and compression problems. The paper is organized as follows: in section 2, quaternion algebra and discrete QFT are briefly described. In section 3, the color image enhancement method and simple compression mode using QFT are described. Then, in section 4, experimental results are presented with their further analysis. And the final 5th section includes the conclusion and direction of further tasks.

2 Quaternion Algebra and Discrete QFT

Quaternions are mathematical objects which act as rotational operators in three-dimensional geometric space. The quaternion algebra was introduced by Sir Hamilton in 1843. The motivation for introducing quaternions was the aim to extend the geometrical meaning of complex numbers in the plane to 3D space through a system of hyper complex numbers. Quaternions are an extension of complex numbers [4]. Instead of one imaginary unit i, we have three different imaginary units i, j, k, that are all square roots of -1, i.e. $i = j = k = \sqrt{-1}$. Each quaternion is a linear combination of $1, i, j, k$, basis quaternions.

A quaternion defined as a hyper-complex number of the form $q = q_0 + q_1 i + q_2 j + q_3 k$, where q_i are real numbers (scalars), and multiplication of i, j, k, units is defined by the following rules

$$
\begin{aligned}
i^2 &= j^2 = k^2 = -1, \\
ij &= k = -ji, \\
jk &= i = -kj, \\
ki &= j = -ik.
\end{aligned}
$$

Thus quaternions form an associative but non-commutative algebra. Let $q = q_0 + q_1 i + q_2 j + q_3 k$ and $r = r_0 + r_1 i + r_2 j + r_3 k$. Some operations on quaternions are given below:

1. $\overline{q} = q_0 + q_1 i + q_2 j + q_3 k$, called conjugate quaternion q.
2. $q = r$ if and only if $q_0 = r_0$, $q_1 = r_1$, $q_2 = r_2$, $q_3 = r_3$.
3. $q \pm r = (q_0 \pm r_0) + (q_1 \pm r_1)i + (q_2 \pm r_2)j + (q_3 \pm r_3)k$.
4. $$
\begin{aligned}
qr &= (q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3) + (q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2)i \\
&+ (q_0 r_2 - q_1 r_3 + q_2 r_0 + q_3 r_1)j + (q_0 r_3 + q_1 r_2 - q_2 r_1 + q_3 r_0)k
\end{aligned}
$$
5. The norm of a quaternion is defined by: $\|q\| = \overline{q}q = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$.
6. The inverse of quaternion defined as: $q^{-1} = \frac{1}{q} = \frac{\overline{q}}{q\overline{q}} = \frac{\overline{q}}{\|q\|}$, $\frac{q}{r} = qr^{-1}$.

The exponential and logarithmic functions of quaternion $\mu = a + bi + cj + dk$ are defined as:

7. $\exp(\mu) = (\cos(p) + \frac{bi+cj+dk}{p}\sin(p))\exp(a)$,
8. $\ln(\mu) = \ln\sqrt{\|\mu\|} + \frac{\arctan(b/a)bi + \arctan(c/a)cj + \arctan(d/a)dk}{p}$, where $p = \sqrt{b^2 + c^2 + d^2}$.

If the scalar part of quaternion is equal to zero, the quaternion is called a pure quaternion. In this case we have $\overline{\mu} = -\mu$, $\mu^2 = -\|\mu\|$. Now define 2D quaternionic Fourier transform [5]. Let $X = (x[m,n])$ be a matrix of size $M \times N$, where $x[m,n]$ is the real, complex or quaternion number. The direct and inverse discrete quaternionic Fourier transform (DDQFT and IDQFT) defined by the following formulas:

$$
F^q[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \exp(-q_1 \frac{2\pi}{M} mu) x[m,n] \exp(-q_2 \frac{2\pi}{N} nv),\ u = \overline{0, M-1},\ v = \overline{0, N-1},
$$

$$
x[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \exp(q_1 \frac{2\pi}{M} mu) F^q[u,v] \exp(q_2 \frac{2\pi}{N} nv),\ m = \overline{0, M-1},\ n = \overline{0, N-1}, \tag{1}
$$

where q_1, q_2 are any quaternions.

In some case the following definitions of QFT are also used

$$F^q[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \exp\Big(-\mu 2\pi(\frac{mu}{M} + \frac{nv}{N})\Big) x[m,n],\ u = \overline{0, M-1},\ v = \overline{0, N-1}.$$

$$x[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \exp\Big(\mu 2\pi(\frac{mu}{M} + \frac{nv}{N})\Big) F^q[u,v],\ m = \overline{0, M-1},\ n = \overline{0, N-1}, \tag{2}$$

where μ is a pure unite quaternion, such as $\mu = \frac{1}{\sqrt{3}}(i + j + k)$.

3 Suggested Method of Color Image Enhancement and Compression

All image enhancement techniques have one major goal: to improve some characteristics/quality of an image so that the processed image is better than the original one. Two major classifications of image enhancement techniques can be defined: spatial domain enhancement and transform (or spectral) domain enhancement.

Spatial domain enhancement techniques deal with the image's direct intensity values. A common non-transform based enhancement technique is a global histogram equalization. Histogram equalization suffers from the problem of being poorly suited for retaining local details due to its global treatment of the image. Transform domain enhancement techniques involve mapping the image intensity data into a given transform domain by using the orthogonal transforms such as the DCT-2, Fourier, and other unitary transforms.

The basic limitations of the transform based image enhancement methods are: a) they introduce certain artifacts; b) they cannot simultaneously enhance all parts of the image very well; c) it is difficult to automate the image enhancement procedure.

It then becomes obvious to ask the following questions: is it possible to apply the existing methods of gray scale image enhancement on color images and is it possible to develop a new method of color image enhancement?

It is of utmost importance to preliminarily determine a suitable method of color image enhancement measure. An important step towards image enhancement approach is the creation of a new measure. This problem becomes more apparent when the enhancement algorithms are parametric and one needs: a) to choose the best parameters; b) to choose the best transform among class of unitary transforms; c) to automate the image enhancement procedures.

Let $X = (x[m,n])$ be a quaternion matrix of color image of size $M \times N$, i.e.

$$x[m,n] = ix^R[m,n] + jx^G[m,n] + kx^B[m,n], \quad m = \overline{0, M-1},\ n = \overline{0, N-1},$$

where $x^R[m,n]$, $x^G[m,n]$, and $x^B[m,n]$ are the value of Reed, Green, and Blue component of color image, respectively.

Below we consider a special case of formula (1) for $q_1 = i$, and $q_2 = j$. In this case the DDQFT and the IDQFT can be rewritten as:

$$F^q[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \exp(-i\frac{2\pi}{M}mu)x[m,n]\exp(-j\frac{2\pi}{N}nv),\ u = \overline{0,M-1},\ v = \overline{0,M-1},$$

$$x[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \exp(i\frac{2\pi}{M}mu)F^q[u,v]\exp(j\frac{2\pi}{N}nv),\ m = \overline{0,M-1},\ n = \overline{0,M-1}. \quad (3)$$

Suggested color image enhancement method can be realized by the following steps:

Step 1. Calculate spectral components of quaternion image according to formula (3).

Step 2. For all u and v define one of the following sets of real numbers:

(a) $C_0[u,v] - const.$;

(b) $C_1[u,v] - C_0[u,v]|F^q[u,v]|^{\alpha-1}$, $\quad 0 \le \alpha < 1$;

(c) $C_2[u,v] = \lg^{\beta}(|F^q[u,v]|^{\lambda+1})$, $\quad 0 \le \beta < 1,\ \ 0 < \lambda < 2$;

(d) $C_3[u,v] = |F^q[u,v]|^{\alpha-1}\lg^{\beta}(|F^q[u,v]|^{\lambda+1})$, $\quad 0 \le \alpha < 1,\ \ 0 \le \beta < 1,\ \ 0 < \lambda < 2$.

Step 3. With one of components C_k modify the spectrum as:

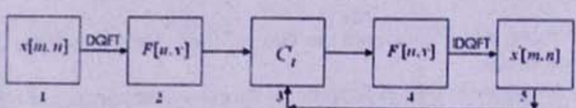
$$F_1^q[u,v] = C_k[u,v]F^q[u,v].$$

Step 4. On the matrix $(F_1^q[u,v])$ realize IDQFT transform (see eq. (3) and define new color image $(x_1[m,n])$.

The quality measure of the resulting image is calculated by the following way [6]. The image is split into non overlapping blocks of size $m \times n$. As a result we obtain $\frac{MN}{mn}$ blocks. Then, for each $r-$th color block we define the maximal and the minimal intensity values Y^r_{max} and Y^r_{min}(the mean of color pixel calculated by formula $(R+G+B)/3$)), and define the value $EME_r = 20\lg\frac{Y^r_{max}}{Y^r_{min}}$. Then, we calculate the average value for all blocks:

$$EME_{\alpha,\beta,\lambda,m,n} = \frac{mn}{MN}\sum_{r=0}^{MN/mn-1} EME_r. \quad (4)$$

Then for a fixed m and n we define the values of parameters α, β, λ when the function $EME_{\alpha,\beta,\lambda,m,n}$ takes the local right-sided minimal value. As a result we receive an image defined for these parameters that is considered to be the enhanced version of the original image. The above given method schematically is presented below:



Further, the color image compression task using the DQFT is considered. The DDQFT is realized on the quaternion matrix $X = x[m,n]$ of a color image of size $M \times N$ using formula (3). Spectral elements that do not exceed the several value of threshold T are considered to be zero, thereby image compression is realized.

4 Experimental Results

Image sized 128x128 is examined below. In enhancement experiments using the algorithm described in the section 3 $m = n = 8$ and α are fixed. The following images were examined:

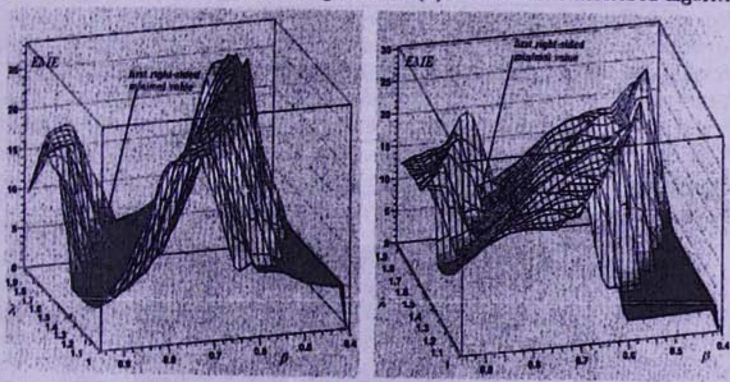
a b c d e

Figure 1. Original images.

The resulting images after using QFT by formula (3) in algorithm described in the Section 3 on the images of Fig.1 are presented below:

a b c d e

Figure 2. Outcome images after using formula (3) in the above described algorithm.



a b

Figure 3. Power spectral density graphics for images a and b from Fig.1. The first right-side minimal value EME for Fig.1(a) and Fig.1(b) achieve for $\beta = 0.8$, $\lambda = 1.85$, $EME = 4.39$, and $\beta = 0.85$, $\lambda = 1.8$, $EME = 6.97$, respectively.

The values of parameters corresponding for enhancement images from Fig. 2 are given in Table 1.

Table 1

Image	β	λ	EME	$PSNR_{red}$	$PSNR_{green}$	$PSNR_{blue}$
2(a)	0.8	1.85	4.39	18.15	16.94	16.52
2(b)	0.8	1.85	6.97	19.6	12.86	13.38
2(c)	0.95	1.8	9.13	10.78	16.63	18.91
2(d)	0.75	1.95	12.27	17.45	25.11	25.1
2(e)	0.95	1.9	6.6	5.98	5.98	5.98

The resulting images after using QFT by formula (2) in algorithm described in the Section 3 on the images of Fig.1 are presented below:

a b c d e

Figure 3. Outcome images after using formula (2) in the above described algorithm.

The values of parameters corresponding for enhancement images from Fig. 2 are given in Table 2.

Table 2

Image	β	λ	EME	$PSNR_{red}$	$PSNR_{green}$	$PSNR_{blue}$
2(a)	0.8	1.85	4.39	18.14	16.94	16.52
2(b)	0.8	1.85	6.99	19.62	12.86	13.37
2(c)	0.95	1.8	9.18	11.17	17.02	19.3
2(d)	0.75	1.95	12.41	13.16	18.72	18.19
2(e)	0.70	1.90	6.60	8.00	8.00	8.00

The resulting images after using conventional Fourier transform algorithm on color images from Fig. 1 with separate components Red, Green, and Blue are presented below:

a b c d e

Figure 4. Outcome enhanced images while using Fourier transform with separate components Red, Green, and Blue.

The resulting parameters while using using Fourier transform on separate components Red, Green, and Blue are given in a Table 3 below:

Table 3

Image	β_r	β_g	β_b	λ_r	λ_g	λ_b	EME_r	EME_g	EME_b	$PSNR_r$	$PSNR_g$	$PSNR_b$
2(a)	0.85	0.8	0.8	1.6	1.85	1.85	3.58	4.01	4.12	14.96	16.75	16.32
2(b)	0.8	0.8	0.8	1.85	1.85	1.85	4.65	10.99	9.42	19.74	7.06	13.34
2(c)	0.95	0.95	0.95	1.85	1.8	1.9	9.38	8.23	6.69	10.92	17.85	19.61
2(d)	0.95	0.95	0.95	1.85	1.8	1.9	13.06	14.85	8.28	13.2	19.25	17.72
2(e)	0.8	0.8	0.8	1.85	1.85	1.85	6.07	6.07	6.07	16.77	16.77	16.77

Now we present the results of color image compression. The original images of size 256x256 are given in a Fig. 5.

a b

Figure 5. Original color images.

The direct Fourier transform is used on separate Red, Green, and Blue components of images from Fig.5. As a result we receive three complex matrices of size 256x256=65536 for each image. Spectral elements with low intensity are discarded from each three matrices. As a result, 14649 spectral elements of image Fig.5(a) and 12414 spectral elements of image Fig.5(b) were retained. For saving a complex number $a + jb$, where a, b are real numbers, we need memory for two real numbers. So then, for retaining of three complex matrices, the quantity of elements form a total of 14649 that requires memory for 29298 real numbers, and for 12414 complex numbers correspondingly - 24828. On the images (a) and (b) from Fig. 5 DDQFT was applied as well. Spectral components with low intensity were discarded and 4883 spectral elements for Fig. 5(a) and 4138 spectral elements for Fig.5(b) were retained, i.e. 4883 quaternions for Fig. 5(a) and 4138 quaternions for Fig. 5(b). For saving 4883 quaternions, memory for 19532 real numbers is required, and for 4138 quaternions the required memory is correspondingly 16552. Then, inverse Fourier transform and IDQFT were applied on retained spectra accordingly. The outcome images are presented on Fig.6(a),(b) and Fig.7(a),(b). They are approximately of the same quality. This can be noticed on calculated PSNR, but the outcome spectrum while using DDQFT on Fig. 5(a) and Fig. 5(b) images occupies 1.5 times less space than the outcome spectrum while using the conventional Fourier transform.

a b

Figure 6. (a) compressed with DFT: PSNR=26.25, memory 117192; (b) compressed with DQFT: PSNR=26.26, memory 78128.

a b

Figure 7. a) compressed with DFT: PSNR=26.77, memory 99312; (b) compressed with DQFT: PSNR=26.79, memory 66208.

5 Conclusion

The color image enhancement algorithm is suggested, as well as image quality measure rate is defined in the paper. Comparison of results of using quaternionic Fourier transform and conventional Fourier transform for color image processing and analysis are presented. The paper is significant by the fact that color image is analysis taken as a whole, i.e. it is not divided into image forming separate color components Red, Green, and Blue. This allows using many of the existing gray scale image processing algorithms for color images. Later on, it is planned to develop and realize fast algorithms of quaternionic Fourier transform for PC and ArmCluster.

References

[1] Ahmed and Rao. Orthogonal Transforms for Digital Signal Processing. Springer-Verlag, New York, 1975.

[2] Yarglagadda Rao R.K., Hershey E.J. Hadamard Matrix Analysis and Synthesis with Applications in Signal/Image Processing, 1997.

[3] Hakob Sarukhanyan, Arthur Petrosian. Construction and Application of Hybrid Wavelet and Other Parametric Orthogonal Transforms, Journal of Mathematical Imaging and Vision, vol. 23, 2005, p.25-46.

[4] Thomas Bulow, "Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images", 1999

[5] S. J. Sangwine and T. A. Ell, "Hypercomplex Fourier Transforms of Color Image."

[6] Todd Anthony Ell. Quaternion-Fourier Transforms for the analysis of Two-dimensional linear time-invariant partial differential systems. IEEE, 1993

[7] Sos S. Agaian, Karen Panetta and Artyom M. Grigoryan, Transform-Based Image Enhancement Algorithms with Performance Measure. IEEE Trans. On Image Processing, vol. 10, No. 3, 2001

Գունավոր պատկերների լավացում և սեղմում Ֆուրյեի քվատերնիոն ձևափոխությամբ

Տ. Մանուկյան

Ամփոփում

Հոդվածը նվիրված է Ֆուրյեի քվատերնիոն ձևափոխության օգնությամբ գունավոր պատկերների մշակման ու անալիզի խնդիրներին: Մշակված է գունավոր պատկերների լավացման ալգորիթմ, դիտարկվել է գունավոր պատկերների սեղմումը Ֆուրյեի քվատերնիոն ձևափոխության միջոցով: Կատարվել է առաջարկված մեթոդի և հայտնի մեթոդներով ստացված արդյունքների համեմատություն: