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Strong Normalization for First-order Logic

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Abstract

In this paper we discuss strong normalization for the $\to\forall$ - fragment of first-order logic. The use of the method of collapsing types to transfer the result concerning strong normalization (that is, any derivation r is strongly normalizable) from implicational logic to first-order logic is illustrated (ref [1]). The considered result is improved by a complement, which states that for any derivation r and its collapse r^c we need the same number of one-step reductions (the $\to_1$ rule) to bring them to their normal forms.

Our basic logic calculus is the $\to\forall$ - fragment of minimal natural deduction for first-order logic over simply typed lambda-terms. This restriction regarding the minimal fragment does not mean a loss in generality, since the full classical first-order logic can be embedded in this system by adding stability axiom. The method of collapsing types developed in [2] is used to get some results concerning the strong normalization of derivations in first-order logic.

1 Preliminaries

Let us fix our language. Assume that we have a countable infinite set of function symbols $f, g, h \dots$ and predicate symbols $P, Q, R \dots$, each of arities ≥ 0. Terms (object terms) $d, e, \dots$ are defined inductively from object variables $x, y, z \dots$ by the following rules:

1. object variable x is a term,
2. if $\vec{d}$ is a list of terms, then $f\vec{d}$ is a term,
3. terms are defined only by rules 1 and 2.

Atomic formulas are $\bot$ (falsity) and $P\vec{d}$, where P is a predicate symbol and $\vec{d}$ is a list of terms.

Formulas are built from atomic formulas by implication $\varphi \to \psi$ and universal quantification $\forall x\varphi$.

Derivations $r^\varphi, t^\varphi, s^\varphi, q^\varphi \dots$ are built from assumption variables $u^\varphi, v^\varphi, w^\varphi \dots$ by the introduction and elimination rules for $\to$ and $\forall$:

$\to^+$ - implication introduction - $(\lambda u^\varphi r^\psi)^{\varphi\to\psi}$;

$\to^-$ - implication elimination - $(t^{\varphi\to\psi}s^\varphi)^\psi$;

$\forall^+$ - universal quantification introduction - $(\lambda x r^{\varphi})^{\forall x \varphi}$, with the usual variable condition, i.e. no assumption variable u^{ψ} free in r^{φ} has x free in its type ψ;

$\forall^-$ - universal quantification elimination - $(t^{\forall x \varphi} d)^{\varphi_x[d]}$.

In the case of *classical logic*: for any predicate symbol P the term $stab_P : \forall \vec{x}. \neg\neg P\vec{x} \to P\vec{x}$ is a derivation. We also use the notation $r : \varphi$ instead of r^{φ}.

From now on we will use the word *term* for derivation terms (until there is no confusion with the notion of object terms) and *type* for formulas.

As we have mentioned the $\to \forall$ - fragment of minimal logic contains full classical first-order logic. As in [1] this can be seen as follows:

1. Associate with any formula φ in the language of classical first-order logic a finite list $\vec{\varphi}$ of formulas in our $\to \forall$ - fragment, by induction on φ:

$$\begin{aligned}
P\vec{d} &\mapsto P\vec{d} \\
\neg\varphi &\mapsto \vec{\varphi} \to \bot \\
\varphi \to \psi &\mapsto \vec{\varphi} \to \psi_1, \ldots, \vec{\varphi} \to \psi_n \\
\varphi \wedge \psi &\mapsto \vec{\varphi}, \vec{\psi} \\
\varphi \vee \psi &\mapsto (\vec{\varphi} \to \bot), (\vec{\psi} \to \bot) \to \bot \\
\forall x \varphi &\mapsto \forall x \varphi_1, \ldots, \forall x \varphi_m \\
\exists x \varphi &\mapsto \forall x (\vec{\varphi} \to \bot) \to \bot
\end{aligned}$$

   where we write $\vec{\varphi} \to \psi$ for $(\varphi_1 \to (\varphi_2 \to \cdots (\varphi_m \to \psi) \cdots))$.

2. In any model $\mathbf{M}$, where $\bot$ is interpreted by falsity, we clearly have that a formula φ in the language of full first-order logic holds under an assignment α iff all formulas in the assigned sequence $\vec{\varphi}$ hold under α (in our $\to \forall$ - fragment of minimal logic).

3. Our derivation calculus for the $\to \forall$ - fragment is complete in the following sense:

   a formula φ is derivable from stability assumptions $\forall \vec{x}. \neg\neg P\vec{x} \to P\vec{x}$ for all predicate symbols P in φ iff φ is valid in any model under any assignment.

2 Strong normalization

It was shown in [1] that for pure implicational logic any term can be reduced to a normal form (w.r.t. $\to_1$ conversion, the one step reduction using β-conversion rule) and this form is uniquely determined. Moreover, it was shown that any reduction sequence terminates, i.e. any term is strongly normalizable. Derivation is said to be in normal form if no more reduction is possible to perform. Here we use the method of collapsing types (ref [2]) to transfer the result (concerning strong normalization, obtained in [1]) from implicational logic to first-order logic.

It must be mentioned that the general β-conversion rule is extended to first-order logic. In particular, we have

$(\lambda u^{\varphi} t^{\psi}) s^{\varphi}$ converts into $(\to_1)$ $t_u[s]$,
where t, s are derivations, u is an assumption variable; and

$(\lambda x r^{\varphi}) d$ converts into $(\to_1)$ $r^{\varphi_x[d]}$,
where x is an object variable, d is an object term and r is a derivation.

For any formula φ of first-order logic we define its *collapse* φ^c by

$$
\begin{array}{rcll}
(P\vec{d})^c & \equiv & P & \text{(c1)} \\
(\varphi \to \psi)^c & \equiv & \varphi^c \to \psi^c & \text{(c2)} \\
(\forall x \varphi)^c & \equiv & \top \to \psi^c & \text{(c3)}
\end{array}
$$

where $\top :\equiv \bot \to \bot$ (i.e. $\top$ means truth). Though, $\bot$ is an atomic formula it behaves like predicate symbols, i.e. $(\bot)^c \equiv \bot$, therefore $(\top)^c \equiv \top$.

For any derivation r^{ψ} in first-order logic we can now define its *collapse* $(r^{\psi})^c$. It is obvious from this definition that for any derivation r^{ψ} in first-order logic with free assumption variables $u_1^{\varphi_1}, \ldots, u_m^{\varphi_m}$ the *collapse* $(r^{\psi})^c$ is a derivation $(r^c)^{\varphi^c}$ in implicational logic with free assumption variables $u_1^{\varphi_1^c}, \ldots, u_m^{\varphi_m^c}$.

$$
\begin{array}{rcll}
(u^{\varphi})^c & \equiv & u^{\varphi^c} & \text{(c4)} \\
(\lambda u^{\varphi} r)^c & \equiv & \lambda u^{\varphi^c} r^c & \text{(c5)} \\
(t^{\varphi \to \psi} s^{\varphi})^c & \equiv & t^c s^c & \text{(c6)} \\
(\lambda x r)^c & \equiv & \lambda u^{\top} r^c & \text{(c7)} \\
(t^{\forall x \varphi} d)^c & \equiv & t^c (\lambda z^{\bot} z^{\bot})^{\top} & \text{(c8)}
\end{array}
$$

Note that for any derivation r^{ψ}, assumption variable u^{φ} and derivation s^{φ} we have that $r^c[s^c]$ is a derivation in implicational logic (where the substitution of s^c is done for the assumption variable u^{φ^c}), which is the collapse of $r[s]$. Also for any derivation r^{ψ}, object variable x and object term d we have that $r_x[d]$ is a derivation of $\psi_x[d]$ with collapse $(r_x[d])^c \equiv r^c$.

Lemma 1. *If $r \to_1 r'$ in first-order logic, then $r^c \to_1 (r')^c$ in implicational logic.*
Proof. The lemma can be proved easily by induction on the generation of $r \to_1 r'$ (ref [1]).

From lemma 1 and the theorem, which states that any term in implicational logic is strongly normalizable, the following main result was obtained in [1]:

Theorem 1. *Any derivation r in first-order logic is strongly normalizable.*

Indeed, since the collapse r^c of the term r is a term in implicational logic and any term in implicational logic is strongly normalizable, i.e. any reduction sequence starting from r^c terminates, then from lemma 1 we conclude that any reduction sequence starting from r also terminates. Otherwise, if there is a reduction sequence starting from r, which does not terminate, then from lemma 1 the corresponding reduction sequence starting from r^c will not terminate as well, which contradicts the fact, that r^c is strongly normalizable. From this it follows that r is strongly normalizable.

But it is still conceivable that r terminates (in terms of reduction sequence) before r^c, i.e. the reduction sequence of r^c is longer. Our aim is to show that it is impossible, and both of the terms do the same number of one-step reductions.

First of all it is necessary to emphasize that it is not so obvious, since there is no bijective correspondence between a derivation in first-order logic and its *collapse*.

Note. Although, to any derivation in first-order logic we identically associate collapse, it is not a must for the inverse also to be true. The following instances show the accuracy of the preceding note.

Example 1. Assume the collapse is $t^c = \lambda u^\top r^c$. Then there are two possible forms of derivation t (ambiguity):

1. on the one hand, since $\top^c \equiv \top$, then $t^c = \lambda u^\top r^c = \lambda u^{\top^c} r^c = (\lambda u^\top r)^c$ according to (c5); so, $t = \lambda u^\top r$;
2. on the other hand, $t^c = \lambda u^\top r^c = (\lambda x_{obj} r)^c$ according to (c7); so, $t = \lambda x_{obj} r$.

We write x_{obj} instead of x to indicate the fact that x is an object variable. This notion is extended on object terms too, e.g. d_{obj} instead of d. For the convinience, sometimes the obj pattern will be omitted, but implied.

Example 2. Assume $\varphi^c = \top \to P$, where P is any predicate symbol. Then there are two possible forms of a formula φ (ambiguity):

1. on the one hand, since $(\bot \to \bot)^c = \top^c \equiv \top = (\bot \to \bot)$, then $\varphi^c = \top \to P = (\bot \to \bot) \to P = (\bot \to \bot)^c \to P^c = ((\bot \to \bot) \to P)^c$ according to (c1) and (c2); so, $\varphi = (\bot \to \bot) \to P = \top \to P$;
2. on the other hand, $\varphi^c = \top \to P = \top \to P^c = (\forall x_{obj} P)^c$ according to (c3); so, $\varphi = \forall x_{obj} P$.

Now we reformulate the theorem 1:

Theorem 2. *Any derivation r in first-order logic is strongly normalizable. Moreover, the length of reduction sequence to obtain normal form of r is equal to the the length of reduction sequence to obtain normal form of r^c, in other words, the same number of one-step reductions is needed to bring r and r^c to their normal forms.*

Proof. The first part of the theorem is plain due to the theorem 1. It remains to prove that r^c terminates as soon as r terminates. Assume that $r \to^* r'$ and r' is the normal form of r; that is r terminates and the last term of normalization reduction sequence is r'. Here $\to^*$ denotes transitive and reflexive closure of $\to_1$. From lemma 1 we obtain that $r^c \to^* (r')^c$ as well. Now it should be proved that $(r')^c$ cannot be normalized further, i.e. it terminates.

Let us suppose the opposite and come to contradiction. It means that there exists a term r''_c such that $(r')^c \to_1 r''_c$. So we have the next structure-view:

$$\begin{array}{lllll} r & \to^* & r' & & \text{- terminates} \\ \Downarrow_c & & \Downarrow_c & & \\ r^c & \to^* & (r')^c & \to_1 & r''_c \end{array}$$

Therefore, we conclude that $(r')^c$ has a form

$$(r')^c = t^L_c((\lambda u_c t_c)s_c)t^R_c$$

hence

$$r''_c = t^L_c(t_{c_u}[s_c])t^R_c.$$

Let us denote by t_c^M the middle part of $(r')^c$

$$t_c^M \equiv (\lambda u_c t_c)s_c.$$

More exactly $(r')^c$ has one of the two following forms:

(a)

$$[t_c^L((\lambda u_c t_c)s_c)]t_c^R = (t_c^L\, t_c^M)t_c^R.$$

(b)

$$t_c^L[((\lambda u_c t_c)s_c)t_c^R] = t_c^L(t_c^M t_c^R).$$

Note. There is no need to consider the case that $(r')^c$ could have been extended to the left and to the right, like $(r')^c = t_c^{LL}t_c^L\, t_c^M t_c^R t_c^{RR}$. It does not change the technique of the proof.

Remark. By $\tau(s)$ we denote the type of derivation s, e.g. $\tau(s^{\varphi\to\upsilon}) = \varphi \to \upsilon$. Let us consider the term $t^c((\lambda z^\perp z^\perp)^\top)^c$ in case when $\tau(t^c) = \top \to \varphi^c = \top^c \to \varphi^c$. It is obvious that $((\lambda z^\perp z^\perp)^\top)^c = (\lambda z^\perp z^\perp)^\top$. According to (c6) and (c8) there are two possible forms of term r which collapse is $r^c = t^c((\lambda z^\perp z^\perp)^\top)^c$:

1. on the one hand $r = t^{\top\to\varphi}(\lambda z^\perp z^\perp)^\top$;

2. on the other hand $r = t^{\forall x_{obj}\varphi}d_{obj}$.

Inter alia, this remark can be viewed as one more example, which shows the accuracy of the note about inverse problem mentioned above.

For the form (a): $(r')^c = (t_c^L\, t_c^M)t_c^R$

We consider two cases depending on the form of t_c^R.

Case (a-1). $t_c^R = (\lambda z^\perp z^\perp)^\top = ((\lambda z^\perp z^\perp)^\top)^c$.

Let us denote: $q^c \equiv t_c^L((\lambda u_c t_c)s_c)$, hence $(r')^c = q^c((\lambda z^\perp z^\perp)^\top)^c$.
From the remark mentioned above we obtain that either

$$r' = q^{\top\to\varphi}(\lambda z^\perp z^\perp)^\top$$

or

$$r' = q^{\forall x\varphi}d.$$

Case (a-1-1). $r' = q^{\top\to\varphi}(\lambda z^\perp z^\perp)^\top$ and $q^c = t_c^L((\lambda u_c t_c)s_c) = t_c^L\, t_c^M$. $\tau(q^c) = \top \to \varphi^c$.

Since $t_c^M = (\lambda u_c t_c)s_c \neq (\lambda z^\perp z^\perp)^\top$ then according to (c4)-(c8) we conclude that there is only one possible form for q^c, that is-(c6). It follows that $\exists\, t_L, t_M$ terms, which satisfy these equations: $t_c^L = (t_L)^c$ and $t_c^M = (t_M)^c$, hence $q^c = (t_L)^c(t_M)^c$. Let us denote: $(t_{ML})^c \equiv \lambda u_c t_c$, so we have $(t_M)^c = (\lambda u_c t_c)s_c = (t_{ML})^c s_c$. Depending on the form s_c ($= (\lambda z^\perp z^\perp)^\top$ or not) we get either $t_M = t_{ML}^{\forall x\upsilon}s$, where $s = e_{obj}$, or $t_M = t_{ML}^{\upsilon\to\omega}s$, where $s^c = s_c$ (s_c is a derivation term). As we have $(t_{ML})^c \equiv \lambda u_c t_c$, then according to (c5) and (c7) there are two possible forms of term t_{ML} which collapse is $\lambda u_c t_c$: $t_{ML} = \lambda x_{obj} l$, if $\tau(\hat{u}_c) = \top$ or $t_{ML} = \lambda ut$, if $\tau(u_c) \neq \top$, where $u^c = u_c$ and $t^c = t_c$. Therefore, $t_c^M = (t_M)^c = [(\lambda x_{obj} l)e_{obj}]^c$ or $t_c^M = (t_M)^c = [(\lambda ut)s]^c$, which means that in both cases the term r' contains subterm

$(\lambda x_{obj} t) e_{obj}$ or $(\lambda u t) s$ respectively, i.e. we could have performed one more $\to_1$ reduction for r', which contradicts our theorem condition that r' terminates.

Case (a-1-2). $r' = q^{\forall x \varphi} d$ and $q^c = t_c^L((\lambda u_c t_c) s_c) = t_c^L\, t_c^M$, $\tau(q^c) = \top \to \varphi^c$.

This case is similar to the case (a-1-1).

Case (a-2). $t_c^R \neq (\lambda z^\perp z^\perp)^\top$.

According to (c4)-(c8) we conclude that there is only one possible form for $(r')^c$, that is (c6): it follows that $r' = q^{\varphi \to \psi} t_R$, where $(t_R)^c = t_c^R$ and $q^c = t_c^L\, t_c^M$, hence we come to the case (a-1-1) when $\tau(q^c) = \varphi^c \to \psi^c$.

<u>For the form (b): $(r')^c = t_c^L(t_c^M t_c^R)$</u>

Since $t_c^M t_c^R$ does not have the form $(\lambda z^\perp z^\perp)^\top$, it follows that according to (c4)-(c8) there is only one possible form for $(r')^c$, that is (c6). Hence, $r' = t_L\, q$, where $(t_L)^c = t_c^L$ and $q^c = t_c^M t_c^R = ((\lambda u_c t_c) s_c) t_c^R$. According to (c1)-(c8) q^c may have one of the two following forms: (c6) or (c8). Depending on the form of $t_c^R (= (\lambda z^\perp z^\perp)^\top$ or not) we get either $q = t_M d_{obj}$, where $(t_M)^c = t_c^M$ or $q = t_M t_R$, where $(t_M)^c = t_c^M$ and $(t_R)^c = t_c^R$ respectively. In both cases we have t_M which satisfies the equation $(t_M)^c = (\lambda u_c t_c) s_c$. The rest is similar to the case (a-1-1).

All the cases have been considered, hence the theorem is proved by the method of contradiction.

References

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Խիստ նորմալիզացիա առաջին կարգի տրամաբանության համար

Տ. Մ. Գալոյան

Ամփոփում

Աշխատանքում ուսումնասիրվել է խիստ նորմալիզացման խնդիրը առաջին կարգի տրամաբանության → ∀ կտորի համար: Դիտարկվել է փլուզվող տիպերի մերոդի կիրառումը, որի միջոցով [1]-ում խիստ նորմալիզացիային վերաբերվող արդյունքը (այն է՝ կամայական թերմ խիստ նորմալիզացվող է), ստացված միայն տրամաբանության → կտորի համար, տարածվում է նաև առաջին կարգի տրամաբանության վրա: Ստացվել է առավել ամփոփ արդյունք, որն ի հավելումն պնդում է նաև՝ ցանկացած r արտածման և դրա համապատասխան r^c փլուզման համար անհրաժեշտ է նույն քանակությամբ միաքայլ ռեդուկցիա, որպեսզի դրանք բերվեն նորմալ ձևի: