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Fast DCT-2 Transform via Hadamard Transform

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Abstract

In this paper we present Walsh-Hadamard transform (WHT) based on the fast discrete cosine transform (DCT-2) algorithm. The basic idea of this algorithm is the following: at first we compute the WHT coefficients, then using so called conversion matrix we convert these coefficients to the transform domain coefficients.

1 Introduction

The increasing importance of large vectors processing and parallel computing in many scientific and engineering applications requires new ideas for designing super efficient algorithms of the transforms and their implementations. In the past decade fast orthogonal transforms have been widely used in areas such as data compression, pattern recognition and image reconstruction, interpolation, linear filtering, spectral analysis, watermarking, cryptography and communication systems. The computation of unitary transforms is complicated and time consuming. However it would not be possible to use the orthogonal transforms in signal and image processing applications without effective algorithms calculating them.

A class of Hadamard transforms (such as the Hadamard matrices ordered by Walsh and Paley) plays an imperfect role among these orthogonal transforms. These matrices are known as non-sinusoidal orthogonal transform matrices and have been application in digital signal processing [1] -[14]. Recently, Hadamard transforms and their variations are widely used in audio and video processing [2], [4], [12]. For efficient computation of these transforms fast algorithms were developed [8],[9], [11]. These algorithms require only $N \log_2 N$ addition and subtraction operations,($N = 2^k$, $N = 12 \cdot 2^k$, $N = 4^k$ and some others [1],[11], [12]).

In the past decade fast orthogonal transforms have been widely used in such areas as data compression, pattern recognition and image reconstruction, interpolation, linear filtering, and spectral analysis. The increasing requirements upon the speed and cost in many applications have stimulated the development of new fast unitary transforms such as Fourier, Cosine, Sine, Hartley, Hadamard, Slant transforms. We can observe the considerable interest to many applications of the above transforms.

In this paper we present Walsh-Hadamard transform (WHT) based on the fast discrete cosine transform (DCT-2) algorithm. The basic idea of this algorithm is the following: at first we compute the WHT coefficients, then using so called conversion matrix we convert these coefficients to the transform domain coefficients. These algorithms are useful for development of integer-to-integer discrete orthogonal transform.

2 Fast Implementation of Discrete Orthogonal Transforms via Walsh-Hadamard Transform

An N-point discrete orthogonal transform can be defined as

$$X = F_N x, \tag{1}$$

where $x = (x_0, x_2, \ldots, x_{N-1})$, $X = (X_0, X_2, \ldots, X_{N-1})$ denote the input and output column vectors, respectively, and F_N is an arbitrary discrete orthogonal transform matrix of order N.

We can represent equation (1) in the following form

$$X = F_N x = \frac{1}{N} F_N H_N H_N^T x, \tag{2}$$

where H_N is an Hadamard transform matrix of order $N = 2^n$.

Denote $A_N = \frac{1}{N} F_N H_N$ or $F_N = A_N H_N$ (recall that H_N is a symmetric matrix). Then equation (2) takes the form

$$X = A_N H_N x.$$

In other words, the Hadamard transform coefficients are computed first, then they are used to obtain the coefficients of discrete transform F_N. This is achieved by the transform matrix A_N which is orthonormal and has a block-diagonal structure. We will call A_N a *correction transform*. So, any transform can be decomposed into two orthogonal transforms: a) Hadamard transform, and b) correction transform.

Lemma 2.1 *Let the orthogonal transform matrix $F_N = F_{2^n}$ has the following representation*

$$F_{2^n} = \begin{pmatrix} \widehat{F}_{2^{n-1}} & \widehat{F}_{2^{n-1}} \\ B_{2^{n-1}} & -B_{2^{n-1}} \end{pmatrix}, \tag{3}$$

where $\widehat{\cdot}$ stands for an appropriate permutation, and $B_{2^{n-1}}$ is a $2^{n-1} \times 2^{n-1}$ the submatrix of F_{2^n}. Then

$$\begin{aligned} A_{2^n} &= diag\{2^{n-1} I_2, 2^{n-2} B_2 H_2, \cdots, B_{2^{n-1}} H_{2^{n-1}}\} \\ &= 2^{n-1} I_2 \oplus 2^{n-2} B_2 H_2 \oplus \cdots \oplus B_{2^{n-1}} H_{2^{n-1}}, \end{aligned} \tag{4}$$

i.e. A_N matrix has a block diagonal structure, here $\oplus$ denotes the direct sum of matrices.

Proof. Clearly this is true for $n = 1$. Let assume that the equation (4) is valid for $N = 2^{k-1}$, i.e.

$$A_{2^{k-1}} = 2^{k-2} I_2 \oplus 2^{k-3} B_2 H_2 \oplus \cdots \oplus B_{2^{k-1}} H_{2^{k-1}}, \tag{5}$$

and show that it takes place for $N = 2^k$. From definition of correction transform matrix A_N we have

$$A_{2^k} = F_{2^k} H_{2^k} = \begin{pmatrix} 2\widehat{F}_{2^{k-1}} H_{2^{k-1}} & 0 \\ 0 & 2B_{2^{k-1}} H_{2^{k-1}} \end{pmatrix}. \tag{6}$$

Using definitions of $F_{2^{k-1}}$ and $H_{2^{k-1}}$ once more we can rewrite the equation (6) as

$$\widehat{F}_{2^{k-1}} = \begin{pmatrix} \widehat{F}_{2^{k-2}} & \widehat{F}_{2^{k-2}} \\ B_{2^{k-2}} & -B_{2^{k-2}} \end{pmatrix}, \quad H_{2^{k-1}} = \begin{pmatrix} H_{2^{k-2}} & H_{2^{k-2}} \\ H_{2^{k-2}} & -H_{2^{k-2}} \end{pmatrix},$$

$$A_{2^k} = diag\{4\widehat{F}_{2^{k-2}} H_{2^{k-2}}, 4B_{2^{k-2}} H_{2^{k-2}}, B_{2^{k-1}} H_{2^{k-1}}\}$$

From equation (5), we conclude

$$A_{2^k} = diag\{2^{k-1} I_2, 2^{k-2} B_2 H_2, \cdots, B_{2^{k-2}} H_{2^{k-2}}\}$$

■

For example, Hadamard-based discrete transforms of order 16 can be represented as following (see [15]), where X denotes nonzero elements.

$$\begin{pmatrix}
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X
\end{pmatrix}$$

3 Fast Cosine Transform Implementation

Let C_N be the $(N \otimes N)$ transform matrix of DCT-2, that is

$$C_N = \left\{a_k \cos\frac{k(2n+1)\pi}{2N}\right\}_{k,n=0}^{N-1}, \text{ where } \quad a_0 = \sqrt{2}/2,\ a_k = 1,\ k \neq 0.$$

We can check that C_N is an orthogonal matrix, i.e. $C_N C_N^T = \frac{N}{2} I_N$. Denote the elements of DCT-2 matrix (without normalizing coefficients a_k) by

$$c_{k,n} = \cos\frac{k(2n+1)\pi}{2N}, \quad k, n = 0, 1, \ldots, N-1.$$

One can show, that

$$c_{2k,n} = c_{2k,N-n-1},\ c_{2k+1,n} = -c_{2k+1,N-n-1},\ k,n = \overline{0, N/2-1}. \tag{7}$$

From equation (7) it follows that the matrix C_N can be represented as

$$C_N \equiv \begin{pmatrix} \hat{C}_{N/2} & \hat{C}_{N/2} \\ D_{N/2} & -D_{N/2} \end{pmatrix}.$$

Hence, according to Lemma 2.1, the matrix $A_N = \frac{1}{N} C_N H_N$ has a block diagonal structure, where H_N is a Sylvester-Hadamard matrix of order N. Without losing the generalization, we prove it for the cases $N = 4, 8, 16$.

Case N=4: The discrete cosine transform (DCT) matrix of order 4 has the form (here we use the notation $c_i = cos\frac{i\pi}{8}$):

$$C_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_3 & -c_3 & -c_1 \\ c_2 & -c_2 & -c_2 & c_2 \\ c_3 & c_1 & -c_1 & -c_3 \end{pmatrix}.$$

Using the following permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we obtain

$$\hat{C}_4 - P_1 C_4 P_2 - \begin{pmatrix} 1 & 1 & 1 & 1 \\ c_2 & -c_2 & c_2 & -c_2 \\ c_1 & c_3 & -c_1 & -c_3 \\ c_3 & c_1 & -c_3 & -c_1 \end{pmatrix} = \begin{pmatrix} \hat{C}_2 & \hat{C}_2 \\ D_2 & -D_2 \end{pmatrix}.$$

Therefore, the correction matrix in this case takes the form

$$A_4 = \frac{1}{4} diag\left\{2\hat{C}_2 H_2, 2D_2 H_2\right\} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 \\ 0 & 0 & 2(c_1+c_3) & 2(c_1-c_3) \\ 0 & 0 & 2(c_1+c_3) & -2(c_1-c_3) \end{pmatrix}.$$

Case N=8: The discrete cosine transform (DCT) matrix of order 8 has the form (here we use the notation

$c_k = \cos\frac{k\pi}{16}$):

$$C_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ c_1 & c_3 & c_5 & c_7 & -c_7 & -c_5 & -c_3 & -c_1 \\ c_2 & c_6 & -c_6 & -c_2 & -c_2 & -c_6 & c_6 & c_2 \\ c_3 & -c_7 & -c_1 & -c_5 & c_5 & c_1 & c_7 & -c_3 \\ c_4 & -c_4 & -c_4 & c_4 & c_4 & -c_4 & -c_4 & c_4 \\ c_5 & -c_1 & c_7 & c_3 & -c_3 & -c_7 & c_1 & -c_5 \\ c_6 & -c_2 & c_2 & -c_6 & -c_6 & c_2 & -c_2 & c_6 \\ c_7 & -c_5 & c_3 & -c_1 & c_1 & -c_3 & c_5 & -c_7 \end{pmatrix}.$$

Let

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using above given matrices we obtain the block representation for DCT matrix of order 8:

$$\tilde{C}_8 = P_1 C_8 P_2 = \begin{pmatrix} \hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 \\ B_2 & -B_2 & B_2 & -B_2 \\ & D_4 & & -D_4 \end{pmatrix},$$

where

$$\hat{C}_2 = \begin{pmatrix} 1 & 1 \\ c_4 & -c_4 \end{pmatrix}, B_2 = \begin{pmatrix} c_2 & c_6 \\ c_6 & -c_2 \end{pmatrix}, D_4 = \begin{pmatrix} c_1 & c_3 & c_7 & c_5 \\ c_3 & -c_7 & -c_5 & -c_1 \\ c_5 & -c_1 & c_3 & c_7 \\ c_7 & -c_5 & -c_1 & c_3 \end{pmatrix}.$$

Therefore, the correction matrix can take the following block diagonal form:

$$A_8 = \frac{1}{8} \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8c_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4r_1 & 4r_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4r_2 & 4r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & -b_1 & b_2 & b_3 & -b_4 \\ 0 & 0 & 0 & 0 & -b_4 & b_3 & -b_2 & b_1 \\ 0 & 0 & 0 & 0 & -a_4 & -a_3 & a_2 & a_1 \end{pmatrix},$$

where

$$
\begin{array}{ll}
a_1 = c_1 + c_3 + c_5 + c_7, & a_2 = c_1 - c_3 - c_5 + c_7, \\
a_3 = c_1 + c_3 - c_5 - c_7, & a_4 = c_1 - c_3 + c_5 - c_7, \\
b_1 = c_1 - c_3 + c_5 + c_7, & b_2 = c_1 + c_3 - c_5 + c_7, \\
b_3 = c_1 + c_3 + c_5 - c_7, & b_4 = c_1 - c_3 - c_5 - c_7, \\
r_1 = c_2 + c_6, & r_2 = c_2 - c_6.
\end{array}
$$

Case N=16: Denote $r_k = cos\frac{k\pi}{32}$. Then the Cosine transform matrix of order 16 is given by (in C_{16} we use the notations $n := cos\frac{n\pi}{32}$ and $-n := -cos\frac{n\pi}{32}$)

$$
C_{16} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & -15 & -13 & -11 & -9 & -7 & -5 & -3 & -1 \\
2 & 6 & 10 & 14 & -14 & -10 & -6 & -2 & -2 & -6 & -10 & -14 & 14 & 10 & 6 & 2 \\
3 & 9 & 15 & -11 & -5 & -1 & -7 & -13 & 13 & 7 & 1 & 5 & 11 & -15 & -9 & -3 \\
4 & 12 & -12 & -4 & -4 & -12 & 12 & 4 & 4 & 12 & -12 & -4 & -4 & -12 & 12 & 4 \\
5 & 15 & -7 & -3 & -13 & 9 & 1 & 11 & -11 & -1 & -7 & 13 & 3 & 7 & -15 & -5 \\
6 & -14 & -2 & -10 & 10 & 2 & 14 & -6 & -6 & 14 & 2 & 10 & -10 & -2 & -14 & 6 \\
7 & -11 & -3 & 15 & 1 & 13 & -5 & -9 & 9 & 5 & -13 & -1 & -15 & 3 & 11 & -7 \\
8 & -8 & -8 & 8 & 8 & -8 & -8 & 8 & 8 & -8 & -8 & 8 & 8 & -8 & -8 & 8 \\
9 & -5 & -13 & 1 & -15 & -3 & 11 & 7 & -7 & -11 & 3 & 15 & -1 & 13 & 5 & -9 \\
10 & -2 & 14 & 6 & -6 & -14 & 2 & -10 & -10 & 2 & -14 & -6 & 6 & 14 & -2 & 10 \\
11 & -1 & 9 & 13 & -3 & 7 & 15 & -5 & 5 & -15 & -7 & 3 & -13 & -9 & 1 & -11 \\
12 & -4 & 4 & -12 & -12 & 4 & -4 & 12 & 12 & -4 & 4 & -12 & -12 & 4 & -4 & 12 \\
13 & -7 & 1 & -5 & 11 & 15 & -9 & 3 & -3 & 9 & -15 & -11 & 5 & -1 & 7 & -13 \\
14 & -10 & 6 & -2 & 2 & -6 & 10 & -14 & -14 & 10 & -6 & 2 & -2 & 6 & -10 & 14 \\
15 & -13 & 11 & -9 & 7 & -5 & 3 & -1 & 1 & -3 & 5 & -7 & 9 & -11 & 13 & -15
\end{pmatrix}.
$$

From this matrix we can generate new equivalent matrix by the following operations:

a) Rewrite the rows of the matrix C_{16} in the following order: 0, 2, 4, 6, 8, 10, 14, 1, 3, 5, 7, 9, 11, 13, 15;

b) Rewrite the first 8 rows of the new matrix as: 0, 2, 4, 6, 1, 3, 5, 7;

c) Reorder the columns of this matrix as follows: 0, 1, 3, 2, 4,5, 7, 6, 8, 9, 11, 10, 12, 13, 15, 14.

Finally, the discrete cosine transform (DCT) matrix of order 16 can be represented by the equivalent block matrix as:

$$
C_{16} = \begin{pmatrix}
\hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 & \hat{C}_2 \\
A_2 & -A_2 & A_2 & -A_2 & A_2 & -A_2 & A_2 & -A_2 \\
B_{1,1} & B_{1,2} & -B_{1,1} & -B_{1,2} & B_{1,1} & B_{1,2} & -B_{1,1} & -B_{1,2} \\
B_{2,1} & B_{2,2} & -B_{2,1} & -B_{2,2} & B_{2,1} & B_{2,2} & -B_{2,1} & -B_{2,2} \\
B_{3,1} & B_{3,2} & B_{3,4} & B_{3,3} & -B_{3,1} & -B_{3,2} & -B_{3,4} & -B_{3,3} \\
B_{4,1} & B_{4,2} & B_{4,4} & B_{4,3} & -B_{4,1} & -B_{4,2} & -B_{4,4} & -B_{4,3} \\
B_{5,1} & B_{5,2} & B_{5,4} & B_{5,3} & -B_{5,1} & -B_{5,2} & -B_{5,4} & -B_{5,3} \\
B_{6,1} & B_{6,2} & B_{6,4} & B_{6,3} & -B_{6,1} & -B_{6,2} & -B_{6,4} & -B_{6,3}
\end{pmatrix}, \qquad (8)
$$

where

$$
\hat{C}_2 = \begin{pmatrix} 1 & 1 \\ r_8 & -r_8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} r_4 & r_{12} \\ r_{12} & -r_4 \end{pmatrix},
$$

$$B_{1,1}=\begin{pmatrix} r_2 & r_6 \\ r_6 & -r_{1,4} \end{pmatrix},\quad B_{1,2}=\begin{pmatrix} r_{14} & r_{10} \\ -r_{10} & -r_2 \end{pmatrix},\quad B_{2,1}=\begin{pmatrix} r_{10} & -r_2 \\ r_{14} & -r_{10} \end{pmatrix},\quad B_{2,2}=\begin{pmatrix} r_6 & r_{14} \\ -r_2 & r_6 \end{pmatrix},$$

$$B_{3,1}=\begin{pmatrix} r_1 & r_3 \\ r_3 & r_9 \end{pmatrix},\quad B_{3,2}=\begin{pmatrix} r_7 & r_5 \\ -r_{11} & r_{15} \end{pmatrix},\quad B_{3,3}=\begin{pmatrix} r_9 & r_{11} \\ -r_5 & -r_1 \end{pmatrix},\quad B_{3,4}=\begin{pmatrix} r_{15} & r_{13} \\ -r_{13} & -r_7 \end{pmatrix}$$

$$B_{4,1}=\begin{pmatrix} r_5 & r_{15} \\ r_7 & -r_{11} \end{pmatrix},\quad B_{4,2}=\begin{pmatrix} -r_3 & -r_7 \\ r_{15} & -r_3 \end{pmatrix},\quad B_{4,3}=\begin{pmatrix} -r_{13} & r_9 \\ r_1 & r_{13} \end{pmatrix},\quad B_{4,4}=\begin{pmatrix} r_{11} & r_1 \\ -r_9 & -r_5 \end{pmatrix},$$

$$B_{5,1}=\begin{pmatrix} r_9 & -r_5 \\ r_{11} & -r_1 \end{pmatrix},\quad B_{5,2}=\begin{pmatrix} r_1 & -r_{13} \\ r_{13} & r_9 \end{pmatrix},\quad B_{5,3}=\begin{pmatrix} -r_{15} & -r_3 \\ -r_3 & r_7 \end{pmatrix},\quad B_{5,4}=\begin{pmatrix} r_7 & r_{11} \\ -r_5 & r_{15} \end{pmatrix}.$$

$$B_{6,1}=\begin{pmatrix} r_{13} & -r_7 \\ r_{15} & -r_{13} \end{pmatrix},\quad B_{6,2}=\begin{pmatrix} -r_5 & r_1 \\ -r_9 & r_{11} \end{pmatrix},\quad B_{6,3}=\begin{pmatrix} r_{11} & r_{15} \\ r_7 & -r_5 \end{pmatrix},\quad B_{6,4}=\begin{pmatrix} r_3 & -r_9 \\ -r_1 & r_3 \end{pmatrix}.$$

Now the matrix (8) can be presented as

$$C_{16}=\begin{pmatrix} \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 & \widehat{C}_2 \\ A_2 & -A_2 & A_2 & -A_2 & A_2 & -A_2 & A_2 & -A_2 \\ \multicolumn{2}{c}{D_4} & \multicolumn{2}{c}{-D_4} & \multicolumn{2}{c}{D_4} & \multicolumn{2}{c}{-D_4} \\ \multicolumn{4}{c}{D_8} & \multicolumn{4}{c}{-D_8} \end{pmatrix},$$

where

$$D_4=\begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix},\quad D_8=\begin{pmatrix} B_{3,1} & B_{3,2} & B_{3,4} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,4} & B_{4,3} \\ B_{5,1} & B_{5,2} & B_{5,4} & B_{5,3} \\ B_{6,1} & B_{6,2} & B_{6,4} & B_{6,3} \end{pmatrix}. \tag{9}$$

Now, according to Lemma 2.1, we have

$$A_{16}=\frac{1}{16}diag\{8\widehat{C}_2H_2; 8A_2H_2; 4D_4H_4; 2D_8H_8\}. \tag{10}$$

Introduce the notations:

$$\begin{array}{llll} q_1=r_2+r_{14}, & q_2=r_6+r_{10}, & t_1=r_2-r_{14}, & t_2=r_6-r_{10}, \\ a_1=r_1+r_{15}, & a_2=r_3+r_{13}, & a_3=r_5+r_{11}, & a_4=r_7+r_9, \\ b_1=r_1-r_{15}, & b_2=r_3-r_{13}, & b_3=r_5-r_{11}, & b_4=r_7-r_9. \end{array}$$

Using the equation (9) and the above given notations, we find

$$\widehat{C}_2H_2=\begin{pmatrix} 2 & 0 \\ 0 & 2r_8 \end{pmatrix},\quad D_2H_2=\begin{pmatrix} r_4+r_{12} & r_4-r_{12} \\ -r_4+r_{12} & r_4+r_{12} \end{pmatrix},$$

$$D_4H_4=\begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{2,4} & d_{2,3} & -d_{2,2} & -d_{2,1} \\ -d_{1,4} & -d_{1,3} & d_{1,2} & d_{1,1} \end{pmatrix} \tag{11}$$

where

$$d_{1,1} = q_1 + q_2, \quad d_{1,2} = q_1 - q_2, \quad d_{1,3} = t_1 + t_2, \quad d_{1,4} = t_1 - t_2,$$
$$d_{2,1} = -q_1 + t_2, \quad d_{2,2} = q_1 + t_2, \quad d_{2,3} = t_1 + q_2, \quad d_{2,4} = -t_1 + q_2. \tag{12}$$

The elements of matrix $D_8 H_8$ can be calculated as

$$\begin{aligned}
P_{1,1} &= a_1 + a_2 + a_3 + a_4, & P_{1,2} &= a_1 - a_2 - a_3 + a_4,\\
P_{1,3} &= a_1 + a_2 - a_3 - a_4, & P_{1,4} &= a_1 - a_2 + a_3 - a_4,\\
P_{1,5} &= b_1 + b_2 + b_3 + b_4, & P_{1,6} &= b_1 - b_2 - b_3 + b_4,\\
P_{1,7} &= b_1 + b_2 - b_3 - b_4, & P_{1,8} &= b_1 - b_2 + b_3 - b_4,
\end{aligned}$$

$$\begin{aligned}
P_{2,1} &= -b_1 + b_2 - b_4 - a_3, & P_{2,2} &= b_1 + b_2 + b_4 - a_3,\\
P_{2,3} &= b_1 + b_2 - b_4 + a_3, & P_{2,4} &= -b_1 + b_2 + b_4 + a_3,\\
P_{2,5} &= a_1 + a_2 + a_4 + b_3, & P_{2,6} &= -a_1 + a_2 - a_4 + b_3,\\
P_{2,7} &= -a_1 + a_2 + a_4 - b_3, & P_{2,8} &= a_1 + a_2 - a_4 - b_3,
\end{aligned}$$

$$\begin{aligned}
P_{3,1} &= a_1 - a_2 + a_3 - b_4, & P_{3,2} &= -a_1 - a_2 + a_3 + b_4,\\
P_{3,3} &= a_1 + a_2 + a_3 + b_4, & P_{3,4} &= -a_1 + a_2 + a_3 - b_4,\\
P_{3,5} &= -b_1 - b_2 + b_3 - a_4, & P_{3,6} &= b_1 - b_2 + b_3 + a_4,\\
P_{3,7} &= -b_1 + b_2 + b_3 + a_4, & P_{3,8} &= b_1 + b_2 + b_3 - a_4,
\end{aligned}$$

$$\begin{aligned}
P_{4,1} &= a_1 - a_3 - b_2 + b_4, & P_{4,2} &= a_1 + a_3 + b_2 + b_4,\\
P_{4,3} &= -a_1 - a_3 + b_2 + b_4, & P_{4,4} &= -a_1 + a_3 - b_2 + b_4,\\
P_{4,5} &= -b_1 + b_3 - a_2 + a_4, & P_{4,6} &= -b_1 - b_3 + a_2 + a_4,\\
P_{4,7} &= b_1 + b_3 + a_2 + a_4, & P_{4,8} &= b_1 - b_3 - a_2 + a_4,
\end{aligned}$$

$$\begin{aligned}
P_{5,1} &= P_{4,8}, & P_{5,2} &= P_{4,7}, & P_{5,3} &= P_{4,6}, & P_{5,4} &= P_{4,5},\\
P_{5,5} &= -P_{4,4}, & P_{5,6} &= -P_{4,3}, & P_{5,7} &= -P_{4,2}, & P_{5,8} &= -P_{4,1},
\end{aligned}$$

$$\begin{aligned}
P_{6,1} &= -P_{3,8}, & P_{6,2} &= -P_{3,7}, & P_{6,3} &= -P_{3,6}, & P_{6,4} &= -P_{3,5},\\
P_{6,5} &= P_{3,4}, & P_{6,6} &= P_{3,3}, & P_{6,7} &= P_{3,2}, & P_{6,8} &= P_{3,1},
\end{aligned}$$

$$\begin{aligned}
P_{7,1} &= P_{2,8}, & P_{7,2} &= P_{2,7}, & P_{7,3} &= P_{2,6}, & P_{7,4} &= P_{2,5},\\
P_{7,5} &= -P_{2,4}, & P_{7,6} &= -P_{2,3}, & P_{7,7} &= -P_{2,2}, & P_{7,8} &= -P_{2,1},
\end{aligned}$$

$$\begin{aligned}
P_{8,1} &= -P_{1,8}, & P_{8,2} &= -P_{1,7}, & P_{8,3} &= -P_{1,6}, & P_{8,4} &= -P_{1,5},\\
P_{8,5} &= P_{1,4}, & P_{8,6} &= P_{1,3}, & P_{8,7} &= P_{1,2}, & P_{8,8} &= P_{1,1}.
\end{aligned}$$

Therefor, the matrix $P = D_8 H_8$ is given by

$$P = \begin{pmatrix}
P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} & P_{1,5} & P_{1,6} & P_{1,7} & P_{1,8}\\
P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} & P_{2,5} & P_{2,6} & P_{2,7} & P_{2,8}\\
P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} & P_{3,5} & P_{3,6} & P_{3,7} & P_{3,8}\\
P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} & P_{4,5} & P_{4,6} & P_{4,7} & P_{4,8}\\
P_{4,8} & P_{4,7} & P_{4,6} & P_{4,5} & -P_{4,4} & -P_{4,3} & -P_{4,2} & -P_{4,1}\\
-P_{3,8} & -P_{3,7} & -P_{3,6} & -P_{3,5} & P_{3,4} & P_{3,3} & P_{3,2} & P_{3,1}\\
P_{2,8} & P_{2,7} & P_{2,6} & P_{2,5} & -P_{2,4} & -P_{2,3} & -P_{2,2} & -P_{2,1}\\
-P_{1,8} & -P_{1,7} & -P_{1,6} & -P_{1,5} & P_{1,4} & P_{1,3} & P_{1,2} & P_{1,1}
\end{pmatrix}.$$

Now we want to show that the Cosine transform can be done via fast algorithm. Denote $y = H_{16}x$. Then $z = A_{16}y$. Using equation (10) we find

$$z = diag\left\{8\hat{C}_2H_2\begin{pmatrix}y_0\\y_1\end{pmatrix}; 8D_2H_2\begin{pmatrix}y_2\\y_3\end{pmatrix}; 4D_4H_4\begin{pmatrix}y_4\\\vdots\\y_7\end{pmatrix}; 2D_8H_8\begin{pmatrix}y_8\\\vdots\\y_{15}\end{pmatrix}\right\}.$$

From (11) and (12), we obtain (here $s = \sqrt{2}$)

$$\begin{aligned}
&z_0 = 2y_0, \quad z_1 = sy_1,\\
&z_2 = r_4(y_2 + y_3) + r_{12}(y_2 - y_3),\\
&z_3 = -r_4(y_2 - y_3) + r_{12}(y_2 + y_3),\\
&z_4 = q_1(y_4 + y_5) + q_2(y_4 - y_5) + t_1(y_6 + y_7) + t_2(y_6 - y_7),\\
&z_5 = -q_1(y_4 - y_5) + t_2(y_4 + y_5) + t_1(y_6 - y_7) + q_2(y_6 + y_7),\\
&z_6 = -t_1(y_4 - y_5) + q_2(y_4 + y_5) - q_1(y_6 - y_7) - t_2(y_6 + y_7),\\
&z_7 = -t_1(y_4 + y_5) + t_2(y_4 - y_5) + q_1(y_6 + y_7) - q_2(y_6 - y_7).
\end{aligned}$$

Now using the following notations

$$\begin{aligned}
&n_1 = y_8 + y_9, \quad n_2 = y_{10} + y_{11}, \quad n_3 = y_{12} + y_{13}, \quad n_4 = y_{14} + y_{15},\\
&m_1 = y_8 - y_9, \quad m_2 = y_{10} - y_{11}, \quad m_3 = y_{12} - y_{13}, \quad m_4 = y_{14} - y_{15},
\end{aligned}$$

we obtain

$$\begin{aligned}
z_8 &= a_1(n_1 + n_2) + a_2(m_1 + m_2) + a_3(m_1 - m_2) + a_4(n_1 - n_2)\\
&\quad + b_1(n_3 + n_4) + b_2(m_3 + m_4) + b_3(m_3 - m_4) + b_4(n_3 - n_4),\\
z_9 &= -b_1(m_1 - m_2) + b_2(n_1 + n_2) - a_3(n_1 - n_2) - b_4(m_1 + m_2)\\
&\quad + a_1(m_3 - m_4) + a_2(n_3 + n_4) + b_3(n_3 - n_4) + a_4(m_3 + m_4),\\
z_{10} &= a_1(m_1 + m_2) - a_2(n_1 - n_2) + a_3(n_1 + n_2) - b_4(m_1 - m_2)\\
&\quad - b_1(m_3 + m_4) - b_2(n_3 - n_4) + b_3(n_3 + n_4) - a_4(m_3 - m_4),
\end{aligned}$$

$$\begin{aligned}
z_{11} &= a_1(n_1 - n_2) + b_4(n_1 + n_2) - a_3(m_1 + m_2) - b_2(m_1 - m_2)\\
&\quad - b_1(n_3 - n_4) + a_4(n_3 + n_4) + b_3(m_3 + m_4) - a_2(m_3 - m_4),\\
z_{12} &= b_1(n_1 - n_2) - b_3(m_1 + m_2) - a_2(m_1 - m_2) + a_4(n_1 + n_2)\\
&\quad + a_1(n_3 - n_4) - a_3(m_3 + m_4) + b_2(m_3 - m_4) - b_4(n_3 + n_4),\\
z_{13} &= -b_1(m_1 + m_2) - b_2(n_1 - n_2) - b_3(n_1 + n_2) + a_4(m_1 - m_2)\\
&\quad - a_1(m_3 + m_4) + a_2(n_3 - n_4) + a_3(n_3 + n_4) - b_4(m_3 - m_4),\\
z_{14} &= a_1(m_1 - m_2) + a_2(n_1 + n_2) - a_4(m_1 + m_2) - b_3(n_1 - n_2)\\
&\quad + b_1(m_3 - m_4) - b_2(n_3 + n_4) - b_4(m_3 + m_4) - a_3(n_3 - n_4),\\
z_{15} &= -b_1(n_1 + n_2) + b_2(m_1 + m_2) - b_3(m_1 - m_2) + b_4(n_1 - n_2)\\
&\quad + a_1(n_3 + n_4) - a_2(m_3 + m_4) + a_3(m_3 - m_4) - a_4(n_3 - n_4).
\end{aligned}$$

Algorithm 3.1 *16−point Cosine transform algorithm using Hadamard transform.*

Step 1. *Input column-vector* $x = (x_0, x_1, \ldots, x_{15})$.

Step 2. *Perform* 16−*point Hadamard transform,* $y = H_{16}x$.

Step 3. *Compute the coefficients* z_i, $i = 0, 1, \ldots, 15$.

Step 4. *Perform shift operations (3 bits for* $z_0, \ldots, z_3$*; 2 bits for* $z_4, \ldots, z_7$*; and one bit for* $z_8, \ldots, z_{15}$*.)*

Step 5. *Output the results of* **Step 4.**

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DCT-2 արագ ձևափոխություն Հադամարի ձևափոխությամբ

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Ամփոփում

Հոդվածում ներկայացված է Ուոլշ-Հադամարի ձևափոխության կիրառմամբ DCT-2 ձևափոխության արագ ալգորիթմը: Ալգորիթմի հիմնական իմաստը հետևյալն է՝ նախ հաշվվում են Ուոլշ-Հադամարի ձևափոխության գործակիցները, այնուհետև, այսպես կոչված կոնվերտացիոն մատրիցի օգնությամբ որոշվում են DCT-2 ձևափոխության գործակիցները: