SOME CRITERIA OF COMPLETENESS FOR AXIOMATIC SYSTEMS IN THREE-VALUED LOGIC[1]

I. D. Zaslavsky

The properties of completeness for axiomatic systems based on J.Lukasiewicz's three-valued logic (i.e. so-called „Luk-completeness") are investigated. For every axiomatic system Ω based on J.Lukasiewicz's logic its classical image Ω^+ is introduced; for every axiomatic system Ω based on classical logic its image $Luk(\Omega)$ in the framework of J.Lukasiewicz's logic is introduced. It is proved that Ω^+ (correspondingly, Ω) is complete in the classical sense if and only if Ω (correspondingly, $Luk(\Omega)$) is Luk-complete. These criteria are used for the investigation of Luk-completeness of some arithmetical systems in the signatures $\{0,',=\}$ and $\{0,',<,=\}$.

§1. INTRODUCTION. Formal axiomatic systems based on J.Lukasiewicz's three-valued logic [4] are considered in [9], [11]-[13]. In particular, the property of Luk-completeness of axiomatic systems (i.e. completeness from the point of view of J.Lukasiewicz's logic) was investigated in [12] and [13]. This property may be described as follows: an axiomatic system Ω in a given language is said to be Luk-complete if for every closed formula in this language the following statement holds: either B, or $\neg B$, or $(B \supset \neg B)\&(\neg B \supset B)$ is a corollary of Ω. It is proved in [13] that, for example, the three-valued analogue of M.Presburger's arithmetical system is Luk-complete in the mentioned sense (though it is not complete in the classical sense).

Below two general criteria of Luk-completeness are considered. For every axiomatic system Ω based on J.Lukasiewicz's three-valued logic its classical image Ω^+ will be defined as in [13]. Similarly, for every axiomatic system Ω in the framework of classical logic its Luk-image $Luk(\Omega)$ will be introduced. It will be proved (theorem 2.1) that a classical axiomatic system Ω is complete in the classical sense if and only if $Luk(\Omega)$ is Luk-complete.

[1] This research is partially supported by the grant INTAS 2001 - 447.

Similarly, an axiomatic system Ω based on J.Lukasiewicz's logic is Luk-complete if and only if Ω^+ is complete in the classical sense (theorem 2.2). The proofs will be given in the sections 3 and 4. Theorem 2.2 will be used in the section 5 for the investigation of Luk-completeness for some arithmetical systems HLUS and HLUL in the signatures, correspondingly $\{0,', =\}$ and $\{0,', <, =\}$.

Let us note that there is no isomorphism between axiomatic systems considered in classical logic and in J.Lukasiewicz's logic. The transformations of axiomatic systems mentioned above are not reverse one to another. The system $(Luk(\Omega))^+$ is in general not equivalent to Ω; similarly, $Luk(\Omega^+)$ is in general not equivalent to Ω. The signatures of Ω and $Luk(\Omega)$ are the same, but the signatures of Ω and Ω^+ are different. The relations between mentioned systems need to be additionally investigated.

The formulations of theorem 2.1 and of some part of theorem 2.2 were given in [14].

§2. Let us give definitions for some notions and notations used below; most of them are given in [13] and we shall recall these definitions here.

We shall use the language of first order predicate calculus ([1]-[3], [5], [13]) based on the logical operations $\&, \vee, \supset, \neg, \forall, \exists$, and containing enumerable sets of predicate and functional symbols; the symbols of constants T („true"), F („false"), U („undefined") are also introduced. The notions of term and formula are given in a usual way ([1]-[3], [5]) as well as main notions connected with them. By LP (correspondingly, L*P) we denote the set of all predicate formulas (correspondingly, the set of all predicate formulas, where the symbols T, F, U are not contained). *Weak implication* $(A \supseteq B)$, *weak negation* $\neg^\circ A$, *equivalency* $A \sim B$ (where A and B are any formulas) are defined, correspondingly, as $(A \supset (A \supset B)), (A \supset \neg A), (A \supset B)\&(B \supset A)$. A *signature* Z is any set of predicate and functional symbols. By L(Z) (correspondingly, L*Z)), where Z is a signature, we denote the set of all predicate formulas belonging to LP (correspondingly, L*P) and containing only such predicate and functional symbols which belong to Z. By $Subst(A, x, s)$, where A is a formula, x is a variable, s is a term, we denote the formula obtained from A by the substitution of s for all free occurrences of x in A; we consider only admissible substitutions (in the usual sense). By $(\forall\forall)$ (correspondingly, $(\exists\exists)$) we denote any group of universal (correspondingly, existential) quantifiers; by $(\forall\forall\forall)(A)$ (correspondingly, $(\exists\exists\exists)(A)$), where A is a formula, we denote the closure of A by universal (correspondingly, existential) quantifiers.

A three-valued n-dimensional predicate on a non-empty set M is defined as any mapping of n-th Cartesian degree of M into the set $\{0, 1, 2\}$, where the numbers 0,1,2 are considered as numerical codes of logical values: 2 for „true", 0 for „false", 1 for „undefined". A Luk-interpretation on M of an n-dimensional predicate symbol is any n-dimensional three-valued predicate on M; a Luk-interpretation on M of an n-dimensional functional symbol is any mapping of n-th Cartesian degree of M into M. A *Luk-assignment* on M for the signature Z is defined as the set of Luk-interpretations on M for all predicate and functional symbols belonging to Z.

Logical connectives and constants are interpreted according to the rules of J.Lukasiewicz's logic ([4]; see also [3], [7], [13]). The logical values of T, F, U are correspondingly, 2, 0, 1.The logical values of $(A\&B), (A \vee B), (A \supset B), \neg A$ are correspondingly, $\min(x, y)$, $\max(x, y)$, 2-$\max(0, x - y)$, 2-x, where x and y are logical values of, correspondingly, A and B. The quantifier $\forall x$ (correspondingly, $\exists x$) is interpreted as a transformation of three-valued predicates; namely, if $p(y, x_1, x_2, ..., x_{n-1})$ is an n-dimensional three-valued predicate on M, then the predicate $\forall y p(y, x_1, x_2, ..., x_{n-1})$ (correspondingly, $\exists y p(y, x_1, x_2, ..., x_{n-1})$) is defined as an $(n-1)$-dimensional predicate on M such that its logical value in the point $(x_1, x_2, ..., x_{n-1})$ is the minimum (correspondingly, maximum) of values $p(y, x_1, x_2, ..., x_{n-1})$ for all $y \in$ M.

The *interpretation* of a given formula A concerning a Luk-assignment δ on M is defined in a natural way as three-valued predicate obtained by replacing of every predicate and functional symbol in A by its interpretation given in δ. *Axiomatic* system in a given signature Z is defined as any enumerable set of closed formulas $(A_1, A_2, ...)$ (probably, finite or empty) in the language L(Z). A Luk-assignment δ on Z is said to be *Luk-model* of an axiomatic system $(A_1, A_2, ...)$ in Z if the interpretations of all axioms A_i concerning δ are equal to 2 („true"). An axiomatic system is said to be *Luk-consistent* if it has a Luk-model; in the opposite case it is said to be *Luk-inconsistent*. A formula B in a signature Z is said to be *Luk-corollary* of a given axiomatic system Ω in Z if the interpretation of B concerning every Luk-model of Ω is a three-valued predicate having a logical value 2 („true") in every point. The *Luk-theory* based on Ω is defined as the set of all its closed Luk-corollaries. A formula B in Z is said to be identically *Luk-true* if its interpretation concerning every Luk-assignment on Z is a three-value predicate everywhere equal to 2. A Luk-consistent axiomatic system $\Omega = (A_1, A_2, ...)$ in a signature Z is said to be *Luk-complete* if for every closed formula B in L(Z) the following condition holds: either B, or $\neg B$, or $(B \supset \neg B)\&(\neg B \supset B)$ is a Luk-corollary of Ω.

The predicate calculus HLU in the language LP is defined as in [13] (cf. also [9], [11]) by the rule of inference modus ponens

$$\frac{A \; (A \supset B)}{B}$$

and by the following logical axiom schemes and axioms (where A, B, C are any formulas in LP; x is any variable; D is any formula in LP which does not contain free occurrences of x, s is any term):

(HLU$_1$) $(\forall\forall)(A \supset (B \supset A))$;

(HLU$_2$) $(\forall\forall)((A \supset (B \supset C)) \supset (B \supset (A \supset C)))$;

(HLU$_3$) $(\forall\forall)((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$;

(HLU$_4$) $(\forall\forall)((A \supset (A \supset B)) \supset ((\neg B \supset (\neg B \supset \neg A)) \supset (A \supset B)))$;

(HLU$_5$) $(\forall\forall)((A \supset B) \supset (\neg B \supset \neg A))$;

(HLU$_6$) $(\forall\forall)(A \supset \neg\neg A)$;

(HLU$_7$) $(\forall\forall)(\neg\neg A \supset A)$;

(HLU$_8$) $(\forall\forall)((A\&B)3A)$;

(HLU$_9$) $(\forall\forall)((A\&B) \supset B)$;

(HLU$_{10}$) $(\forall\forall)((C \supset A) \supset ((C \supset B) \supset (C \supset (A\&B))))$;

(HLU$_{11}$) $(\forall\forall)(A \supset (A \vee B))$;

(HLU$_{12}$) $(\forall\forall)(B \supset (A \vee B))$;

(HLU$_{13}$) $(\forall\forall)((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)))$;

(HLU$_{14}$) $(\forall\forall)(\forall x(A) \supset Subst(A, x, s))$;

(HLU$_{15}$) $(\forall\forall)(D \supset \forall x(D))$;

(HLU$_{16}$) $(\forall\forall)(\forall x(A \supset B) \supset (\forall x(A) \supset \forall x(B)))$;

(HLU$_{17}$) $(\forall\forall)(Subst(A, x, s) \supset \exists x(A))$;

(HLU$_{18}$) $(\forall\forall)(\exists x(D) \supset D)$;

(HLU$_{19}$) $(\forall\forall)(\forall x(A \supset B) \supset (\exists x(A) \supset \exists x(B)))$;

(HLU$_{20}$) $(\forall\forall)(\forall x(A \supset (A \supset B)) \supset (\exists x(A) \supset (\exists x(A) \supset \exists x(B))))$;

(HLU$_{21}$) $(\forall\forall)(((A \supset \neg A) \supset \neg(A \supset \neg A)) \supset A)$;

(HLU$_{22}$) T;

(HLU$_{23}$) $\neg F$;

(HLU$_{24}$) $(U \supset \neg U)$;

(HLU$_{25}$) $(\neg U \supset U)$.

The predicate calculus HK in L*P is defined as in [13] by the rule modus ponens and by the axiom schemes HK$_1$ - HK$_{20}$, where the schemes HK$_1$ - HK$_3$ and HK$_5$ - HK$_2$0 coincide, correspondingly, with HLU$_1$ - HLU$_3$ and HLU$_5$ - HLU$_{20}$; the axiom scheme HK$_4$ has the following form:

(HK$_4$) $(\forall\forall)((A \supset (A \supset B)) \supset (A \supset B))$.

The deducibility of a formula B from formulas $A_1, A_2, ..., A_n$ in the calculi HLU and HK will be denoted, correspondingly, as $A_1, A_2, ..., A_n \vdash_{HLU} B$ and $A_1, A_2, ..., A_n \vdash_{HK} B$. Sometimes we shall write $\Sigma \vdash_{HLU} B$ or $\Sigma \vdash_{HK} B$, where Σ is an infinite set of formulas; such a notation will mean that the formula B is deducible from some finite subset of Σ in the corresponding calculus.

The calculus HLU is an extension of the calculi HS, HL, HSU considered in [6]-[9], [11], [13]. Let us recall some deductive properties of HLU (see [13]). The deduction theorem for the calculus HLU is given in the following forms (see [7], p.106; [13], p.40):

(1) if $A_1, A_2, ..., A_n, B \vdash_{HLU} C$ then $A_1, A_2, ..., A_n \vdash_{HLU} B \supseteq C$;

(2) if $A_1, A_2, ..., A_n, B \vdash_{HLU} C$ and $A_1, A_2, ..., A_n, \neg C \vdash_{HLU} \neg B$ then $A_1, A_2, ..., A_n \vdash_{HLU} B \supset C$.

The classical form of the deduction theorem (if $A_1, A_2, ..., A_n, B \vdash C$ then $A_1, A_2, ..., A_n \vdash B \supset C$) is in general not valid in HLU. The law of double negation, De Morgan's laws, the law of contraposition (as well as the law affirming that „a contradiction implies all") are valid in HLU:

$\vdash_{HLU} \neg\neg A \sim A$; $\vdash_{HLU} \neg(A \& B) \sim (\neg A \vee \neg B)$; $\vdash_{HLU} \neg(A \vee B) \sim (\neg A \& \neg B)$;

$\vdash_{HLU} \neg\forall x(A) \sim \exists x(\neg A)$; $\vdash_{HLU} \neg\exists x(A) \sim \forall x(\neg A)$; $\vdash_{HLU} (A \supset B) \sim (\neg B \supset \neg A)$

The rules of introduction and elimination of logical connectives & and $\vee$ similar to the corresponding rules in the classical predicate calculus ([3], pp.98-101) are also valid in HLU.

The mentioned deducibilities and theorems are valid also for HL, HS, HSU which are subcalculi of HLU ([6], [7]-[9], [11], [13]). Let us note that HL is a subcalculus of HLU in the language L*P; it is defined by the rule modus ponens and by axiom schemes HL_1 - HL_{21} which are formulated, correspondingly, as HLU_1 - HLU_{21}. The calculi HS and HSU describe a constructive (intuitionistic) variant of J.Lukasiewicz's logic, i.e. so-called „Symmetric constructive logic" ([6], [7], [10], [12], [13]). The calculus HSU is different from HLU only in the following point: the axiom scheme HLU_{21} is absent in HSU (and it is in general not valid in HSU). The calculus HS is a subcalculus of HSU in the language L*P; it is defined by the rule modus ponens and by the axiom schemes HS_1 - HS_{20} which are formulated, correspondingly, as HLU_1 - HLU_{20}. Though the law of double negation is valid in HLU; HSU, HL, HS, however the law of double weak negation

$$\vdash \neg^{\circ}\neg^{\circ}A \supset A$$

(which is actually equivalent to the scheme HLU_{21}) is valid in HLU and HL but not in HSU and HS. Let us note that $\neg^{\circ}\neg^{\circ}A$ and A are in general not equivalent in HLU; we have only

$$\vdash_{HLU} (A \supseteq \neg^{\circ}\neg^{\circ}A)\&(\neg^{\circ}\neg^{\circ}A \supseteq A).$$

The law of excluded middle ($\vdash A \vee \neg A$) and the law of contradiction (($\vdash \neg(A\&\neg A)$) are in general not valid neither in HLU, nor in its subcalculi HL, HSU, HS.

A formula A in a signature Z is said to be *HLU-corollary* or *HK-corollary* of an axiomatic system Ω in Z if, correspondingly, $\Omega \vdash_{HLU} A$ or $\Omega \vdash_{HK} A$. The following property of completeness is established in [13] for the calculus HSU: a formula B in a signature Z is a Luk-corollary of an axiomatic system Ω in Z if and only if it is an HLU-corollary of Ω ([13], theorems 3.1 and 3.2). Besides, an axiomatic system Ω in a signature Z is Luk-inconsistent if and only if every formula in L(Z) is an HLU-corollary of Ω. The calculus HK is equivalent to the classical predicate calculus (for example, to the form of this calculus given in [1]), so analogous well-known statements are valid also for HK, for axiomatic systems and their models considered in the framework of the classical logic ([1]-[3], [5]).

We shall use the operations $+, -, \theta$ defined in [6], [7], [13]. Let us recall their definitions. The operations $+$ and $-$ are based on a correspondence between predicate symbols p and pairs of predicate symbols (p^+, p^-); we suppose that this correspondence is introduced in such a way that the following conditions hold: (1) for every predicate symbol p there exists a single pair (p^+, p^-) corresponding to p; (2) in every pair (p^+, p^-) the symbols p^+ and p^- are different; (3) the dimensions of p^+ and p^- are equal to the dimension of p; (4) the pairs (p^+, p^-) corresponding to different p have no common elements; (5) axiomatic systems considered in the framework of J.Lukasiewicz's logic (as well as their signatures) contain no predicate symbol having the form p^+ or p^-. The formulas A^+ and A^- obtained from a formula $A \in$LP by the operations + and are defined inductively by the following rules (where D_0 is some fixed closed predicate formula in L*P):

1) If A is an elementary formula having the form $p(s_1, s_2, ..., s_n)$, then A^+ is $p^+(s_1, s_2, ..., s_n)$, A^- is $p^-(s_1, s_2, ..., s_n)$.

2) $(B\&C)^+$ is $(B^+\&C^+)$; $(B\&C)^-$ is $(B^- \vee C^-)$.

3) $(B \vee C)^+$ is $(B^+ \vee C^+)$; $(B \vee C)^-$ is $(B^-\&C^-)$.

4) $(\neg B)^+$ is B^- ; $(\neg B)^-$ is B^+.

5) $(B \supset C)^+$ is $((B^+ \supset C^+)\&(C^- \supset B^-))$; $(B \supset C)^-$ is $(B^+\&C^-)$.

6) $(\forall x(B))^+$ is $\forall x(B^+)$; $(\forall x(B))^-$ is $\exists x(B^-)$.

7) $(\exists x(B))^+$ is $\exists x(B^+)$; $(\exists x(B))^-$ is $\forall x(B^-)$.

8) T^+ is $(D_0 \supset D_0)$; T^- is $\neg(D_0 \supset D_0)$.

9) F^+ is $\neg(D_0 \supset D_0)$; F^- is $(D_0 \supset D_0)$.

10) U^+ is $\neg(D_0 \supset D_0)$; U^- is $\neg(D_0 \supset D_0)$.

Clearly $A^+ \in$ L*P , $A^- \in$ L*P for every $A \in$ LP.

If a formula A expresses a three-valued predicate considered in the framework of J.Lukasiewicz's logic, then the predicates expressed by A^+ and A^- will be considered as classical two-valued predicates.

The formula $\theta(A)$ obtained by the operation θ from a formula $A \in$ L*P is defined inductively by the following rules:

(1) If A is an elementary formula, then $\theta(A)$ is $\neg\neg^\circ A$.

(2) $\theta(A\&B)$ is $\neg\neg^\circ(\neg(A)\&\neg(B))$.

(3) $\theta(A \vee B)$ is $\theta(A) \vee \theta(B)$.

(4) $\theta(A \supset B)$ is $\neg\neg^\circ(\theta(A) \supset \theta(B))$.

(5) $\theta(\neg A)$ is $\neg\neg^\circ\neg(\theta(A))$.

(6) $\theta(\forall x(A))$ is $\neg\neg^\circ\forall x(\theta(A))$.

(7) $\theta(\exists x(A))$ is $\exists x(\theta(A))$.

Clearly, $\theta(A) \in$ L*P for every $A \in$ L*P.

If p is any n-dimensional predicate symbol, then the formula

$$\forall x_1 \forall x_2 ... \forall x_n \neg(p^+(x_1, x_2, ..., x_n) \& p^-(x_1, x_2, ..., x_n))$$

will be denoted by $Dis(p)$; this formula expresses the following condition: the domains of truth for the predicates p^+ and p^- are disjoint. If p is any n-dimensional predicate symbol, then the formula

$$\forall x_1 \forall x_2 ... \forall x_n (p(x_1, x_2, ..., x_n) \vee \neg p(x_1, x_2, ..., x_n))$$

will be denoted by $Exm(p)$ („the law of excluded middle for p").

Let $\Omega = (A_1, A_2, ...)$ be an axiomatic system in a signature Z considered in the framework of J.Lukasiewicz's logic. The *classical image* Ω^+ of the system Ω is defined as an axiomatic system in the framework of the classical logic consisting of the axioms A_i^+ for all A_i belonging to Ω, and $Dis(p)$ for all $p \in$Z. The system Ω^+ is considered in the signature Z^+ (and in the language $L^*(Z^+)$) consisting of the symbols p^+ and p^- for all $p \in$Z and of all functional symbols belonging to Z (cf. [13]).

Let $\Omega = (A_1, A_2, ...)$ be an axiomatic system in a signature Z (and in the language $L^*(Z)$) considered in the framework of the classical logic. The *Luk-image* Luk(Ω) of Ω is defined as an axiomatic system in Z considered in the framework of J.Lukasiewicz's logic and consisting of axioms $\theta(A_i)$ for all A_i belonging to Ω and $Exm(p)$ for all $p \in$Z.

Theorem 2.1. An axiomatic system Ω in the framework of classical logic is complete in the classical sense if and only if the system Luk(Ω) is Luk-complete.

Theorem 2.2. An axiomatic system Ω in the framework of J.Lukasiewicz's logic is Luk-complete if and only if the system O^+ is complete in the classical sense.

The proofs will be given below in the sections 3 and 4.

§3. In this section we shall give a proof of theorem 2.1. We shall use the operations ρ, O_n considered in [13] and the operations $[]_\neg$ and $[]_{\neg\neg}$ considered in [7]. Let us recall their definitions.

The operations ρ and O_n are defined as follows. For every natural number n and for every formula $A \in$LP the formula $\rho(n, A) \in$L*P is defined as the conjunction of formulas $Dis(p)$ for all p contained in A or having indexes less or equal to n in the sequence of all predicate symbols belonging to LP. The formula $O_n(A) \in$L*P, where $A \in$LP, is defined as $\rho(n, A) \in A^+$. The operation O_n corresponds to the operation Ω introduced in [7];

namely, $\Omega(n, A)$ in [7] has the same sense as $O_n(A)$.

The operations $[]_\neg$ and $[]_{\neg\neg}$ are defined as the corresponding operations introduced in [7] (see [7], p. 137 and p. 146). The formulas $[A]\neg$ and $[A]_\neg\neg$ are defined for every formula $A \in L^*P$ such that all predicate symbols contained in A have the form p^+ and p^-. These formulas are obtained by the following transformations: the formula $[A]_\neg$ is obtained from A by replacing every elementary formula $p^+(s_1, s_2, ..., s_n)$ in A by $p(s_1, s_2, ..., s_n)$, and every $p^-(s_1, s_2, ..., s_n)$ by $\neg p(s_1, s_2, ..., s_n)$. The formula $[A]_{\neg\neg}$ is obtained from A by replacing every elementary formula $p^+(s_1, s_2, ..., s_n)$ in A by $\neg\neg^\circ p(s_1, s_2, ..., s_n)$, and every $p^-(s_1, s_2, ..., s_n)$ by $\neg\neg^\circ\neg p(s_1, s_2, ..., s_n)$.

The proof of theorem 2.1 will be based on the following lemmas 3.1-3.10.

Lemma 3.1. If A is any formula, $A \in$LP, and n is any natural number, then the sets of free variables in $A^+, A^-, O_n(A), [\theta(A^+)]_{\neg\neg}, [\theta(A^-)]_{\neg\neg}$ are the same as the set of free variables in A; if A is any formula, $A \in$L*P, then the sets of free variables in $\theta(A), [A^+]_\neg, [A^-]\neg$ are the same as the set of free variables in A.

The proof is easily obtained by induction on the process of constructing of the formula A.

Corollary. If a formula $A \in$LP (correspondingly, $A \in$L*P) is closed, then the formulas $A^+, A^-, O_n(A)$ for any n, $[\theta(A^+)]_{\neg\neg}, [\theta(A^-)]_{\neg\neg}$ (correspondingly, $\theta(A), [A^+]_\neg, [A^-]\neg$) are closed.

Lemma 3.2. For any formula $A \in$LP

$$\vdash_{HK} \neg[A^+ \& A^-]_\neg$$

Proof: If A is an elementary formula having the form $p(s_1, s_2, ..., s_n)$, then, obviously

$$\vdash_{HK} \neg(p(s_1, s_2, ..., s_n) \& \neg p(s_1, s_2, ..., s_n)).$$

The proof in general case is easily obtained by induction on the process of constructing of the formula A, using the definitions of A^+ and A^-.

Corollary. If $A \in$LP, n is any natural number, then $\vdash_{HK} [\rho(n, A)]_\neg$.

Lemma 3.3. For every formula $A \in$LP

$$\vdash_{HK} [(A \supseteq B)^+]_\neg \sim ([A^+]_\neg \supset [B^+]_\neg)$$

Proof: Using the definitions of the operations $+, -, []_\neg$, we obtain that $[(A \supseteq B)^+]$ is

$$([A^+]_\neg \supset (([A]^!_\neg \supset [B^+]_\neg) \& ([B^-]_\neg \supset [A^-]\neg))) \& (([A^+]_\neg \& [B^-]_\neg) \supset [A^-]_\neg).$$

But if P, Q, R, S are any formulas in LP such that

$$\vdash_{HK} \neg(P\&Q),\ \vdash_{HK} \neg(R\&S),$$

then it is easily seen that

$$\vdash_{HK} ((P \supset ((P \supset R)\&(S \supset Q)))\&((P\&S) \supset Q)) \sim (P \supset R).$$

This completes the proof, taking into account the preceding lemma.

Lemma 3.4. For any formulas A and B belonging to LP

(a) $\vdash_{HLU} \neg\neg^{\circ}A \supset A$;

(b) $\vdash_{HLU} A \supseteq \neg\neg^{\circ}A$;

(c) $\neg\neg^{\circ}A \vdash_{HLU} A$;

(d) $A \vdash_{HLU} \neg\neg^{\circ}A$;

(e) $\vdash_{HLU} \neg\neg^{\circ}A \sim \neg\neg^{\circ}\neg\neg^{\circ}A$

(f) $\vdash_{HLU} \neg\neg^{\circ}(A \vee B) \sim (\neg\neg^{\circ}A \vee \neg\neg^{\circ}B)$;

(g) $\vdash_{HLU} \neg\neg^{\circ}\exists x(A) \sim \exists x(\neg\neg^{\circ}A)$;

(h) $\vdash_{HLU} \neg A \supset \neg^{\circ}A$;

(i) $A \vdash_{HLU} \neg^{\circ}A \sim \neg A$;

(j) $A \vdash_{HLU} \neg^{\circ}A \sim \neg A$;

(k) $A \vee \neg A \vdash_{HLU} \neg^{\circ}A \sim \neg A$;

(l) $A \vee \neg A \vdash_{HLU} \neg^{\circ}\neg A \sim A$;

(m) $\vdash_{HLU} \neg\neg^{\circ}(\neg\neg^{\circ}A\&\neg\neg^{\circ}B) \sim \neg\neg^{\circ}(A\&B)$.

Proof: The statements (a)-(g) are proved in [7] (see [7], lemma 9.2, p. 126) for the calculus HS; these proofs are valid also for HLU. The statements (h)-(m) are easily proved using the deductive properties of HLU mentioned above.

Lemma 3.5. For every formula $A \in$L*P

$$\vdash_{HK} A$$

if and only if

$$\vdash_{HLU} \theta(A).$$

(So, θ is embedding operation from HK to HLU).

The proof is given in [7] (see [7], theorems 9.1 and 10.4, p.125, p.154, pp. 126-129, 137-141, 155-156; cf. also [13], lemma 8.1, p.57) for the calculus HL; similar statement for HLU is obtained using the fact that HLU is a conservative extension of HL ([13], corollary of lemma 6.6, p.51).

Lemma 3.6. For every formula $A \in$LP and for every natural number n

$$\vdash_{HLU} A$$

if and only if

$$\vdash_{HK} O_n(A).$$

(So, O_n is an embedding operation from HLU to HK).

The proof is given in [7] (see [7], theorems 9.3 and 10.6, p.125, p.154, pp.131-137, 146-

150, 155-156; see also [13], lemma 6.6, p.51) for the calculus HL; the proof for additional cases connected with logical constants is easily obtained.

Lemma 3.7. For every formula $A \in L^{*}P$

$$\vdash_{HK} [\theta(A)^{+}]_{\neg} \sim A;$$
$$\vdash_{HK} [\theta(A)^{-}]_{\neg} \sin \neg A.$$

The proof is given in [7] (see [7], lemma 9.10, pp. 138-140; cf. also [13], lemma 8.11, p.60) for the constructive (intuitionistic) predicate calculus; then it holds also for HK.

Lemma 3.8. For every formula $A \in LP$

$$\vdash_{HLU} [\theta(A)^{+}]_{\neg\neg} \sim \neg\neg^{\circ} A;$$
$$\vdash_{HLU} [\theta(A)^{-}]_{\neg\neg} \sin \neg\neg^{\circ}\neg A.$$

The proof is given in [7] (see [7], lemma 9.14, pp. 146-149; cf. also [13], lemma 8.13, pp. 60-61) for the calculus HS; then it holds also for HL; the proof for additional cases in LP connected with logical constants is easily obtained.

Lemma 3.9. For any formulas $A_1, A_2, ..., A_n$ belonging to $L^{*}P$

$$\vdash_{HLU} \theta(A_1 \& A_2 \& ... \& A_n) \sim \neg\neg^{\circ}(\theta(A_1) \& \theta(A_2) \& ... \& \theta(A_n)).$$

Proof: If $n = 1$ then the deducibility

$$\vdash_{HLU} \theta(A_1) \sim \neg\neg^{\circ}\theta(A_1)$$

is obtained by the induction on the process of constructing the formula A_1, using the points (e), (f), (g) in lemma 3.4. If $n = 2$ then the required deducibility is obtained directly from the definition of the operation θ. If $n > 2$ then the required deducibility is obtained by the induction on n, using the points (e) and (m) in lemma 3.4.

Lemma 3.10. If Ω is an axiomatic system considered in the framework of classical logic then Ω is consistent in the classical sense if and only if Luk(Ω) is Luk-consistent.

Proof: We shall prove that Ω is inconsistent in the classical sense if and only if Luk(Ω) is Luk-inconsistent.

Let us suppose that $\Omega = (A_1, A_2, ...)$ is inconsistent in the classical sense. Then for some n

$$A_1, A_2, ..., A_n \vdash_{HK} \neg(D_0 \supset D_0),$$

where D_0 is a fixed closed formula, $D_0 \in L^{*}P$. So, we have

$$\vdash_{HK} (A_1 \& A_2 \& ... \& A_n) \supset \neg(D_0 \supset D_0),$$

or, by lemma 3.5

$$\vdash_{HLU} \theta((A_1\&A_2\&...\&A_n) \supset \neg(D_0 \supset D_0)),$$

Using the definition of θ, lemma 3.9, and the points (c) and (d) in lemma 3.4, we obtain

$$\theta(A_1), \theta(A_2), ..., \theta(A_n) \vdash_{HLU} \theta(\neg(D_0 \supset D_0)).$$

But the formula $\theta(\neg(D_0 \supset D_0))$ is

$$\neg\neg^{\circ}\neg\neg\neg^{\circ}(\theta(D_0) \supset \theta(D_0))$$

or, which is equivalent in HLU,

$$\neg\neg^{\circ}\neg^{\circ}(\theta(D_0) \supset \theta(D_0))$$

However any formula having the form $(A \supset A)$ is deducible in HLU, hence, using the point (i) in lemma 3.4, we conclude that the formula $\theta(\neg(D_0 \supset D_0))$ is equivalent in HLU to

$$\theta(\theta(D_0) \supset \theta(D_0))$$

hence

$$\theta(A_1), \theta(A_2), ..., \theta(A_n) \vdash_{HLU} \neg(\theta(D_0 \supset \theta(D_0)),$$

or

$$\text{Luk}(\Omega) \vdash_{HLU} \neg(\theta(D_0 \supset \theta(D_0)).$$

From other side, obviously,

$$\text{Luk}(\Omega) \vdash_{HLU} \theta(D_0 \supset \theta(D_0).$$

So, Luk(Ω) is Luk-inconsistent.

Now let us suppose that Luk(Ω?) is Luk-inconsistent. Then for some n and m

$$\theta(A_1), \theta(A_2), ..., \theta(A_n), Exm(p_1), Exm(p_2), ..., Exm(p_m) \vdash_{HLU} \neg(D_0 \supset D_0),$$

where D_0 is a fixed closed formula in L*P. Using the form (1) of the deduction theorem for HLU mentioned above we obtain that the following formula

$$(\theta(A_1)\&\theta(A_2)\&...\&\theta(A_n)\&Exm(p_1)\&Exm(p_2)\&...\&Exm(p_m)) \supseteq \neg(D_0 \supset D_0)$$

is deducible in HLU. Let us denote this formula by Q. So, $\vdash_{HLU} Q$. Using lemma 3.6, we conclude that for every natural number l

$$\vdash_{HK} O_l(Q),$$

that is

$$\vdash_{HK} (\rho(l, Q) \supset Q^+),$$

Now, using the theorem on substitution of formulas for predicate symbols in formulas of the classical predicate calculus, we can conclude that the transformation $[]_\neg$ preserves the deducibility of formulas in HK; so, we have

$$\vdash_{HK} [\rho(l,Q) \supset Q^+]_\neg,$$

that is

$$\vdash_{HK} [\rho(l,Q)]_\neg \supset [Q^+]_\neg,$$

or, using the corollary of lemma 3.2,

$$\vdash_{HK} [Q^+]_\neg,$$

that is

$$\vdash_{HK} [(((\theta(A_1) \& \theta(A_2) \& \ldots \& \theta(A_n) \& Exm(p_1) \& Exm(p_2) \& \ldots \& Exm(p_m)) \supseteq \neg(D_0 \supset D_0))^+]_\neg,$$

Using lemma 3.3 and 3.7 we can conclude that

$$\vdash_{HK} (A_1 \& A_2 \& \ldots \& A_n \& [Exm(p_1)^+]_\neg \& [Exm(p_2)^+]_\neg \& \ldots \& [Exm(p_m)^+]_\neg) \supseteq \neg[(D_0 \supset \dot{D_0})^+]_\neg$$

or

$$A_1, A_2, \ldots, A_n, [Exm(p_1)^+]_\neg, [Exm(p_2)^+]_\neg, \ldots, [Exm(p_m)^+]_\neg) \vdash_{HK} \neg[(D_0 \supset D_0)^+]_\neg$$

But $[Exm(p_i)^+]_\neg$ is $Exm(p_i)$; so, obviously,

$$\vdash_{HK} [Exm(p_i)^+]_\neg$$

for $1 \leq i \leq m$. Besides, $[(\neg(D_0 \supset D_0))^+]_\neg$ is $[D_0^+ \& D_0^-]_\neg$, and we obtain

$$A_1, A_2, \ldots, A_n \vdash_{HK} [D_0^+ \& D_0^-]_\neg.$$

From other side, using lemma 3.2, we have

$$A_1, A_2, \ldots, A_n \vdash_{HK} \neg[D_0^+ \& D_0^-]_\neg.$$

So, the system Ω is inconsistent in the classical sense. This completes the proof.

Proof of theorem 2.1: Let $\Omega = (A_1, A_2, \ldots)$ be an axiomatic system in a signature Z (and in the language L*(Z)) considered in the framework of classical logic. Let us suppose that Ω is complete in the classical sense (hence it is consistent in the classical sense). Let B be any closed formula in the language L(Z). We shall prove that either Luk(Ω) $\vdash_{HLU} B$, or Luk(Ω) $\vdash_{HLU} \neg B$, or Luk(Ω) $\vdash_{HLU} (B \supset \neg B) \& (\neg B \supset B)$ (this means that Luk(Ω) is Luk-complete).

Let us consider the formulas $[B^+]_\neg$ and $[B^-]_\neg$. They are closed by the corollary of lemma 3.1. We have, for some n (because $\Omega = (A_1, A_2, \ldots)$ is complete in the classical sense):

$$A_1, A_2, \ldots, A_n \vdash_{HK} \sigma[B^+]_\neg,$$
$$A_1, A_2, \ldots, A_n \vdash_{HK} \tau[B^-]_\neg,$$

where σ and τ are either empty, or negation symbols. They cannot be together empty, otherwise we could obtain, using lemma 3.2,

$$A_1, A_2, ..., A_n \vdash_{HK} [B^+ \& B^-]_\neg,$$

$$A_1, A_2, ..., A_n \vdash_{HK} \neg[B^+ \& B^-]_\neg,$$

and Ω would be inconsistent in the classical sense, but it is not so.

Further, we have

$$\vdash_{HK} (A_1 \& A_2 \& ... \& A_n) \supset \sigma[B^+]_\neg;$$

$$\vdash_{HK} (A_1 \& A_2 \& ... \& A_n) \supset \tau[B^-]_\neg;$$

hence, by lemma 3.4,

$$\vdash_{HLU} \theta((A_1 \& A_2 \& ... \& A_n) \supset \sigma[B^+]_\neg);$$

$$\vdash_{HLU} \theta((A_1 \& A_2 \& ... \& A_n) \supset \tau[B^-]_\neg).$$

Using the definition of the operation θ, we obtain

$$\vdash_{HLU} \neg\neg^\circ(\theta((A_1 \& A_2 \& ... \& A_n) \supset \neg\neg^\circ\sigma\theta)[B^+]_\neg));$$

$$\vdash_{HLU} \neg\neg^\circ(\theta((A_1 \& A_2 \& ... \& A_n) \supset \neg\neg^\circ\tau\theta)[B^-]_\neg));$$

or, using lemma 3.9 and the points (c) and (d) in lemma 3.4,

$$\theta(A_1), \theta(A_2), ..., \theta(A_n) \vdash_{HLU} \sigma\theta([B^+]_\neg);$$

$$\theta(A_1), \theta(A_2), ..., \theta(A_n) \vdash_{HLU} \tau\theta([B^-]_\neg).$$

Now let us note that the formulas $[B^+]_\neg$ and $[B^-]_\neg$ are obtained from B^+ and B^- by such a substitution when every elementary formula $p^+(s_1, s_2, ..., s_k)$ is replaced by $p(s_1, s_2, ..., s_k)$, and $p^-(s_1, s_2, ..., s_k)$ by $\neg p(s_1, s_2, ..., s_k)$. From other side, $[B^+]_{\neg\neg}$ and $[B^-]_{\neg\neg}$ are obtained from B^+ and B^- by such a substitution when every elementary formula $p^+(s_1, s_2, ..., s_k)$ is replaced by $\neg\neg^\circ p(s_1, s_2, ..., s_k)$, and $p^-(s_1, s_2, ..., s_k)$ by $\neg\neg^\circ\neg p(s_1, s_2, ..., s_k)$. However by the points (k) and (l) in lemma 3.4 we obtain

$$Exm(p) \vdash_{HLU} (p(s_1, s_2, ..., s_k) \sim \neg\neg^\circ p(s_1, s_2, ..., s_k));$$

$$Exm(p) \vdash_{HLU} (\neg p(s_1, s_2, ..., s_k) \sim \neg\neg^\circ\neg p(s_1, s_2, ..., s_k));$$

hence, if $p_1, p_2, ..., p_m$ is the list of all predicate symbols contained in B, then

$$\theta(A_1), \theta(A_2), ..., \theta(A_n), Exm(p_1), Exm(p_2), ..., Exm(p_m) \vdash_{HLU} \sigma\theta([B^+]_{\neg\neg});$$

$$\theta(A_1), \theta(A_2), ..., \theta(A_n), Exm(p_1), Exm(p_2), ..., Exm(p_m) \vdash_{HLU} \tau\theta([B^-]_{\neg\neg}),$$

hence

$$\text{Luk}(\Omega) \vdash_{HLU} \sigma\theta([B^+]_{\neg\neg});$$

$$\text{Luk}(\Omega) \vdash_{HLU} \tau\theta([B^-]_{\neg\neg});$$

or, using lemma 3.8,

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \sigma\neg\neg^{\circ}B;$$
$$\mathrm{Luk}(\Omega) \vdash_{HLU} \tau\neg\neg^{\circ}\neg B;$$

Here three cases are possible: (1) σ is empty, τ is $\neg$; (2) σ is $\neg$, τ is empty; (3) both σ and τ are $\neg$ (let us recall that σ and τ cannot be together empty). In the case (1) we have

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg\neg^{\circ}B \vdash_{HLU} B.$$

In the case (2) we have

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg\neg^{\circ}\neg B \vdash_{HLU} \neg B.$$

Finally, in the case (3) we have

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg^{\circ}B;$$
$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg^{\circ}\neg B;$$

that is

$$\mathrm{Luk}(\Omega) \vdash_{HLU} B \supset \neg B;$$
$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg B \supset \neg\neg B;$$

hence

$$\mathrm{Luk}(\Omega) \vdash_{HLU} (B \supset \neg B)\&(\neg B \supset \neg\neg B).$$

So, it is proved that if Ω is complete in the classical sense, then $\mathrm{Luk}(\Omega)$ is Luk-complete.

Now let us suppose that $\mathrm{Luk}(\Omega)$ is Luk-complete (hence it is Luk-consistent). Let B be any closed formula belonging to $\mathrm{L}^*(\mathrm{Z})$. We shall prove that either $\Omega \vdash_{HK} B$, or $\Omega \vdash_{HK} \neg B$.

Let us consider the formula $\theta(B)$; this formula is closed (corollary of lemma 3.1). We have, using the Luk-completeness of $\mathrm{Luk}(\Omega)$ that either

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \theta(B),$$

or

$$\mathrm{Luk}(\Omega) \vdash_{HLU} \neg\theta(B),$$

or

$$\mathrm{Luk}(\Omega) \vdash_{HLU} (\theta(B) \supset \neg\theta(B))\&(\neg\theta(B) \supset \theta(B)).$$

So, for some n and m

$$\theta(A_1), \theta(A_2), ..., \theta(A_n), Exm(p_1), Exm(p_2), ..., Exm(p_m) \vdash_{HLU} G,$$

where G is either $\theta(B)$, or $\neg\theta(B)$, or $(\theta(B) \supset \neg\theta(B))\&(\neg\theta(B) \supset \theta(B))$. Using the form (1) of deduction theorem for HLU mentioned above we obtain

$$\vdash_{HLU} (\theta(A_1), \theta(A_2), ..., \theta(A_n), Exm(p_1), Exm(p_2), ..., Exm(p_m)) \supseteq G.$$

Hence, similarly to the proof of lemma 3.10 (considering G instead of $\neg(D_0 \supset D_0)$), we can conclude that

$$A_1, A_2, ..., A_n \vdash_{HK} [G^+]_\neg.$$

Now, if G is $\theta(B)$, then, using lemma 3.7, we obtain

$$A_1, A_2, ..., A_n \vdash_{HK} B.$$

If G is $\neg\theta(B)$, then, using the same lemma, we obtain

$$A_1, A_2, ..., A_n \vdash_{HK} \neg B.$$

Finally, if G is $(\theta(B) \supset \neg\theta(B))\&(\neg\theta(B) \supset \theta(B))$, then $[G^+]_\neg$ is

$$([\theta(B)^+]_\neg \supset [\theta(B)^-]_\neg)\&([\theta(B)^+]_\neg \supset [\theta(B)^-]_\neg)\&$$
$$\&([\theta(B)^-]_\neg \supset [\theta(B)^+]_\neg)\&([\theta(B)^-]_\neg \supset [\theta(B)^+]_\neg).$$

This formula is equivalent in HK to

$$([\theta(B)^+]_\neg \supset [\theta(B)^-]_\neg)\&([\theta(B)^-]_\neg \supset [\theta(B)^+]_\neg)\&$$

or, using lemma 3.7, to

$$(B \supset \neg B)\&(\neg B \supset B).$$

So, we obtain

$$A_1, A_2, ..., A_n \vdash_{HK} B \supset \neg B$$
$$A_1, A_2, ..., A_n \vdash_{HK} \neg B \supset B$$

or

$$A_1, A_2, ..., A_n \vdash_{HK} \neg B$$
$$A_1, A_2, ..., A_n \vdash_{HK} B$$

and the system Ω in this case would be classically inconsistent, but it is not so. Hence the considered case, when G is $(\theta(B) \supset \neg\theta(B))\&(\neg\theta(B) \supset \theta(B))$, is impossible. This completes the proof of theorem 2.1.

§4. In this section we shall give the proof of theorem 2.2. The proof will be based on lemmas 4.1- 4.6 given below. We shall use also lemmas 3.1 - 3.10 proved in the preceding section.

Lemma 4.1 (cf. lemma 9.6 in [7], pp.130-131). If A is any formula, $A \in$LP, and $p_1, p_2, ..., p_m$ are all predicate symbols contained in A, then

$$Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} \neg(A^+\&A^-).$$

The proof is similar to that of lemma 3.2.

Lemma 4.2. If A and B are any formulas in LP, and $p_1, p_2, ..., p_m$ are all predicate symbols contained in A and B, then

$$Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} (A \supseteq B)^+ \sim (A^+ \supset B^+).$$

The proof is similar to that of lemma 3.3.

Lemma 4.3 (cf. lemma 9.15 in [7], pp.149-150). For every predicate symbols $p_1, p_2, ..., p_m$

$$\vdash_{HLU} [\theta(Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m))]_{\neg\neg}.$$

Proof: It is sufficient to prove that

$$\vdash_{HLU} [\theta(Dis(p_i))]_{\neg\neg}$$

for $1 \leq i \leq m$; after this the required statement is easily obtained using lemma 3.9, the point (2) of lemma 3.4 and the definition of the operation $[]_{\neg\neg}$. However, for any n-dimensional predicate symbol p the formula $[\theta(Dis(p))]_{\neg\neg}$ is

$$\neg\neg^{\circ}\forall x_1 \neg\neg^{\circ}\forall x_2 ... \neg\neg^{\circ}\forall x_n \neg\neg^{\circ}\neg(\neg\neg^{\circ}(\neg\neg^{\circ}p(x_1, x_2, ..., x_n) \& \neg\neg^{\circ}\neg p(x_1, x_2, ..., x_n))).$$

We shall prove that this formula is deducible in HLU. Using the point (d) of lemma 3.4 as well as deductive properties of HLU we conclude that it is sufficient to prove

$$\vdash_{HLU} \neg^{\circ}(\neg\neg^{\circ}p(x_1, x_2, ..., x_n) \& \neg\neg^{\circ}\neg p(x_1, x_2, ..., x_n))$$

or

$$\neg\neg^{\circ}p(x_1, x_2, ..., x_n) \& \neg\neg^{\circ}\neg p(x_1, x_2, ..., x_n) \vdash_{HLU} \neg(\neg\neg^{\circ}p(x_1, x_2, ..., x_n) \& \neg\neg^{\circ}\neg p(x_1, x_2, ..., x_n))$$

But we can deduce in HLU both $p(x_1, x_2, ..., x_n)$ and $\neg p(x_1, x_2, ..., x_n)$ from the formula $\neg\neg^{\circ}p(x_1, x_2, ..., x_n) \& \neg\neg^{\circ}\neg p(x_1, x_2, ..., x_n)$; so, this formula gives a contradiction in HLU. This completes the proof.

Lemma 4.4. Let Ω be an axiomatic system in a signature Z considered in the framework of J.Lukasiewicz's logic. Then for any formula $A \in L^*(Z^+)$ there exists a formula $B \in L^*(Z)$ containing the same free variables as A and such that

$$\Omega^+ \vdash_{HK} (B^+ \sim A).$$

Proof: We use the induction on the process of constructing for the formula A. If A is an elementary formula having the form $p^+(s_1, s_2, ..., s_n)$ or $p^-(s_1, s_2, ..., s_n)$, then we can define the formula B as, correspondingly, $p(s_1, s_2, ..., s_n)$ or $\neg p(s_1, s_2, ..., s_n)$; obviously in such a cases B^+ coincides with A. (Let us recall that every predicate symbol in the signature Z^+ has the form p^+ or p^-, where $p \in Z$).

Let A be a formula having the form $(A_1 \& A_2)$. By induction we can conclude that there exist formulas B_1 and B_2 in $L^*(Z)$ such that

$$\Omega^+ \vdash_{HK} (B_1^+ \sim A_1);$$
$$\Omega^+ \vdash_{HK} (B_2^+ \sim A_2).$$

Then the required formula B for the formula A can be defined as $(B_1 \& B_2)$. Indeed, $(B_1 \& B_2)^+$ is $B_1^+ \& B_2^+$ and the deducibility

$$\Omega^+ \vdash_{HK} (B_1^+ \& B_2^+) \sim (A_1 \& A_2);$$

is easily obtained from the deducibilities given above for A_1 and A_2.

Let A be a formula having the form $(A_1 \supset A_2)$. By induction we can conclude that there exist formulas B_1 and B_2 in L(Z) such that

$$\Omega^+ \vdash_{HK} (B_1^+ \sim A_1);$$
$$\Omega^+ \vdash_{HK} (B_2^+ \sim A_2).$$

Then the required formula B for the formula A can be defined as $(B_1 \supseteq B_2)$. Indeed, using lemma 4.2, we have

$$\Omega^+ \vdash_{HK} (B_1 \supseteq B_2)^+ \sim (B_1^+ \supset B_2^+);$$

(let us recall that the system Ω^+ contains the axioms $Dis(p)$ for all $p \in$(Z). The deducibility

$$\Omega^+ \vdash_{HK} (B_1 \supseteq B_2)^+ \sim (A_1^+ \supset A_2^+);$$

is easily obtained from the deducibilities mentioned above.

Let A be a formula having the form $\neg A_1$. By induction we can conclude that there exists a formula B_1 in L(Z) such that

$$\Omega^+ \vdash_{HK} (B_1^+ \sim A_1);$$

Then the required formula B for the formula A can be defined as $\neg^\circ B_1$, i.e. $(B_1 \supset \neg B_1)$. Indeed, B^+ is the formula $(B_1 \supset \neg B_1)^+$, that is

$$(B_1^+ \supset \neg B_1^-) \& (B_1^+ \supset \neg B_1^-)$$

or, which is equivalent in HK

$$(B_1^+ \supset \neg B_1^-).$$

However, we have

$$\Omega^+ \vdash_{HK} (B_1^+ \supset B_1^-) \sim \neg B_1^+.$$

Indeed, for the establishing of this deducibility it is sufficient to prove that

$$\Omega^+, B_1^+ \supset B_1^- \vdash_{HK} \neg B_1^+,$$
$$\Omega^+, \neg B_1^+ \vdash_{HK} B_1^+ \supset B_1^-,$$

The first of these statements is obtained using lemma 4.1, because the formulas $Dis(p)$ are axioms of Ω^+ for all $p \in Z$, hence a contradiction in HK is deducible from $\Omega^+, B_1^+ \supset B_1^-, B_1^+$ the second of them is obvious. So, B^+ and $\neg B_1^+$ are equivalent in Ω^+, that is

$$\Omega^+ \vdash_{HK} (B^+ \sim \neg B_1^+),$$

hence

$$\Omega^+ \vdash_{HK} (B^+ \sim \neg A_1).$$

Finally, if A is a formula having the form $(A_1 \vee A_2), \forall x(A_1)$ or $\exists x(A_1)$, and there exist formulas B_1 and B_2 in $L^*(Z)$ such that

$$\Omega^+ \vdash_{HK} (B_1^+ \sim A_1),$$
$$\Omega^+ \vdash_{HK} (B_2^+ \sim A_2),$$

then the required formula B for the formula A can be defined as, correspondingly, $(B_1 \vee B_2), \forall x(B_1)$ or $\exists x(B_1)$. Indeed, the formulas $(B_1 \vee B_2)^+, (\forall x(B_1))^+, (\exists x(B_1))^+$ are, correspondingly, $B_1^+ \vee B_2^+, \forall x(B_1^+), \exists x(B_1^+)$, and the required statements are obtained similarly to the considered case when A is $(A_1 \& A_2)$.F This completes the proof.

Lemma 4.5. If Ω is an axiomatic system considered in the framework of J.Lukasiewicz's logic, then Ω is Luk-consistent if and only if Ω^+ is consistent in the classical sense.

Proof: We shall prove that Ω is Luk-inconsistent if and only if Ω^+ is inconsistent in the classical sense. Let us suppose that $\Omega = (A_1, A_2, ...)$ in the signature Z is Luk-inconsistent. Then for some n

$$A_1, A_2, ..., A_n \vdash_{HLU} \neg(D_0 \supset D_0) ,$$

where D_0 is some fixed closed formula in L P. Using the first form of the deduction theorem for HLU we have, that the formula

$$(A_1 \& A_2 \& ... \& A_n) \supseteq (D_0 \supseteq D_0)$$

is deducible in HLU. Let us denote this formula by Q. So, $\vdash_{HLU} Q$. Using lemma 3.6 we obtain

$$\vdash_{HK} O_0(Q),$$

or

$$\vdash_{HK} (\rho(0, Q) \supset ((A_1 \& A_2 \& ... A_n) \supseteq \neg(D_0 \supset D_0))^+.$$

Let $p_1, p_2, ..., p_t$ be a list of predicate symbols in Z containing all predicate symbols taking part in Q. Then we have by lemma 4.1

$$Dis(p_1), Dis(p_2), ..., Dis(p_t) \vdash_{HK} (\rho(0, Q),$$

and by lemma 4.2,

$$Dis(p_1), Dis(p_2), ..., Dis(p_t) \vdash_{HK} (A_1 \& A_2 \& ... A_n)^+ \supset (\neg(D_0 \supset D_0))^+$$

or

$$Dis(p_1), Dis(p_2), ...; Dis(p_t), A_1^+, A_2^+, ..., A_n^+ \vdash_{HK} (D_0^+ \& D_0^-),$$

hence

$$\Omega^+ \vdash_{HK} D_0^+ \& D_0^-.$$

But, using lemma 4.1, we have

$$\Omega^+ \vdash_{HK} \neg(D_0^+ \& D_0^-).$$

Hence Ω^+ is inconsistent.

Now let us suppose that Ω^+ is inconsistent in the classical sense. Then for some n and m

$$A_1^+, A_2^+, ..., A_n^+, Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} \neg(D_0 \supset D_0),$$

where D_0 is some fixed closed formula in L*P. We conclude that

$$\vdash_{HK} (A_1^+ \& A_2^+ \& ... \& A_n^+ \& Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m)) \supset \neg(D_0 \supset D_0),$$

or, using lemma 3.5,

$$\vdash_{HLU} \theta((A_1^+ \& A_2^+ \& ... \& A_n^+ \& Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m)) \supset \neg(D_0 \supset D_0)).$$

By a substitution for predicate symbols we obtain

$$\vdash_{HLU} [\theta((A_1^+ \& A_2^+ \& ... \& A_n^+ \& Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m)) \supset \neg(D_0 \supset D_0))]_{\neg\neg}.$$

Using the definition of the operation θ, we can conclude that for any formulas A, B, C the formula $[\theta((A \& B) \supset C)]_{\neg\neg}$ is

$$\neg\neg^\circ(\neg\neg^\circ([\theta(A)]_{\neg\neg} \& [\theta(B)]_{\neg\neg}) \supset [\theta(C)]_{\neg\neg}),$$

and, using the points (c) and (d) of lemma 3.4, we obtain

$$\vdash_{HLU} \neg\neg^\circ([\theta((A_1 \& A_2 \& ... \& A_n)^+)]_{\neg\neg} \& [\theta(Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m))]_{\neg\neg}) \supset [\theta(\neg(D_0 \supset D_0))]_{\neg\neg}$$

or

$$[\theta((A_1 \& A_2 \& ... \& A_n)^+)]_{\neg\neg}, [\theta(Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m))]_{\neg\neg} \vdash_{HLU} [\theta(\neg(D_0 \supset D_0))]_{\neg\neg}.$$

Now, using lemmas 3.8 and 4.3, we have

$$\neg\neg^\circ(A_1 \& A_2 \& ... \& A_n)^+) \vdash_{HLU} [\theta(\neg(D_0 \supset D_0))]_{\neg\neg}$$

or

$$A_1 \& A_2 \& ... \& A_n)^+ \vdash_{HLU} [\theta(\neg(D_0 \supset D_0))]_{\neg\neg}.$$

Similarly to the proof of theorem 2.1 we conclude that $\theta(\neg(D_0 \supset D_0))$ is equivalent in HLU to

$$\neg(\theta(D_0) \supset \theta(D_0)).$$

and we obtain

$$A_1 \& A_2 \& ... \& A_n)^+ \vdash_{HLU} \neg([\theta(D_0)]_{--} \supset [\theta(D_0)]_{--})$$

from other side we have obviously

$$A_1 \& A_2 \& ... \& A_n)^+ \vdash_{HLU} ([\theta(D_0)]_{--} \supset [\theta(D_0)]_{--})$$

So, Ω is Luk-inconsistent. This completes the proof.

Lemma 4.6. Let $\Omega = (A_1, A_2, ...)$ be an axiomatic system in a signature Z considered in the framework of J.Lukasiewicz's logic. Let B be any formula in the language L(Z). Then B is a Luk-corollary of Q if and only if B^+ is a corollary Ω^+ in the classical sense.

Proof: Let B be a Luk-corollary of Ω. Then for some n

$$A_1, A_2, ..., A_n \vdash_{HLU} B,$$

or, by the form (1) of the deduction theorem for HLU, the formula

$$(A_1 \& A_2 \& ... \& A_n) \supseteq B$$

is deducible in HLU. Let us denote this formula by Q. So, $\vdash_{HLU} Q$. Using lemma 3.6 we have for any l

$$\vdash_{HK} O_0(Q),$$

or

$$\vdash_{HK} \rho(0, Q) \supset Q^+.$$

Using lemma 4.1 we obtain

$$\Omega^+ \vdash_{HK} \rho(0, Q).$$

hence

$$\Omega^+ \vdash_{HK} Q^+.$$

that is

$$\Omega^+ \vdash_{HK} ((A_1 \& A_2 \& ... \& A_n) \supseteq B)^+,$$

hence, by lemma 4.2,

$$\Omega^+ \vdash_{HK} (A_1 \& A_2 \& ... \& A_n)^+ \supset B^+,$$

or

$$\Omega^+, A_1^+, A_2^+, ..., A_n^+ \vdash_{HK} B^+.$$

But $A_1^+, A_2^+, ..., A_n^+$ are axioms belonging to Ω^+, hence

$$\Omega^+ \vdash_{HK} B^+.$$

which means that B^+ is the corollary of Ω^+ in the classical sense.

Now let us suppose that

$$\Omega^+ \vdash_{HK} B^+.$$

Then for some n and m

$$A_1^+, A_2^+, ..., A_n^+, Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} B^+,$$

hence

$$\vdash_{HK} (A_1^+ \& A_2^+ \& ... \& A_n^+ \& Dis(p_1) \& Dis(p_2) \& ... \& Dis(p_m)) \supset B^+.$$

Similarly to the proof of lemma 4.5 (considering B^+ instead of $\neg(D_0 \supset D_0)$), we conclude that

$$A_1, A_2, ..., A_n \vdash_{HLU} [\theta(B^+)]_{\neg},$$

or, using lemma 3.8 and the point (c) of lemma 3.4,

$$A_1, A_2, ..., A_n \vdash_{HLU} B.$$

Hence B is Luk-corollary of Ω. This completes the proof.

Note. Lemma 4.6 is similar to lemma 6.3 in [13] (see [13], p.49). But the proof of lemma 6.3 in [13] is given on the set-theoretical level, using Luk-models of axiomatic systems. The proof of lemma 4.6 is given on the constructive level; it uses only constructive properties of the calculi HLU and HK.

Proof of theorem 2.2: Let $\Omega = (A_1, A_2, ...)$ be an axiomatic system in a signature Z considered in the framework of J.Lukasiewicz's logic. Let us suppose that Ω is Luk-complete (hence it is Luk-consistent). Let A be any closed formula in the language $L^*(Z^+)$. We shall prove, that either $\Omega^+ \vdash_{HK} A$ or $\Omega^+ \vdash_{HK} \neg A$ (which means that Ω^+ is complete in the classical sense). By lemma 4.4 there exists a closed formula B in $L^*(Z)$ such that

$$\Omega^+ \vdash_{HK} (B^+ \sim A.$$

The formula B belongs also to L(Z), hence, by the Luk-completeness of Ω we conclude that one of the following cases takes place:

$$case(1) \qquad \Omega \vdash_{HLU} B;$$
$$case(2) \qquad \Omega \vdash_{HLU} \neg B;$$
$$case(3) \qquad \Omega \vdash_{HLU} (B \supset \neg B) \& (\neg B \supset B).$$

In the case 1, using lemma 4.6, we have

$$\Omega^+ \vdash_{HK} B^+,$$

hence

$$\Omega^+ \vdash_{HK} A.$$

In the case 2, using the same lemma, we have

$$\Omega^+ \vdash_{HK} (\neg B)^+,$$

hence

$$\Omega^+ \vdash_{HK} B^-,$$

Using lemma 4.1 we conclude, that a contradiction can be deduced in HK from Ω^+ and B^+ ; hence

$$\Omega^+ \vdash_{HK} \neg B^+,$$

and

$$\Omega^+ \vdash_{HK} \neg A,$$

Finally, in the case 3, using lemma 4.6, we have

$$\Omega^+ \vdash_{HK} ((B \supset \neg B)\&(\neg B \supset B),$$

that is

$$\Omega^+ \vdash_{HK} (B^+ \supset B^-)\&(B^+ \supset B^-)\&(B^- \supset B^+)\&(B^- \supset B^+),$$

Similarly to the case 2, using lemma 4.1 we conclude that a contradiction can be deduced in HK from Ω^+ and B^+ ; hence

$$\Omega^+ \vdash_{HK} \neg B^+,$$

and

$$\Omega^+ \vdash_{HK} \neg A,$$

Now let us suppose that the system Ω^+ is complete in the classical sense (hence it is consistent in the classical sense). Let B be any closed formula in the language L(Z). We shall prove that either Q. $\Omega \vdash_{HLU} B$, or $\Omega \vdash_{HLU} \neg B$, or $\Omega \vdash_{HLU} (B \supset \neg B)\&(\neg B \supset B)$.

Let us consider the formulas B^+ and B^-. They are closed (lemma 3.1). Hence, by the completeness of Ω^+ we have

$$\Omega^+ \vdash_{HK} \sigma B^+,$$

$$\Omega^+ \vdash_{HK} \tau B^-,$$

where crand rare either empty, or negation symbols. They cannot be together empty, otherwise Ω^+ would be inconsistent in the classical sense (lemma 4.1), but it is not so. We have for some n and m

$$A_1^+, A_2^+, ..., A_n^+, Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} \sigma B^+;$$

$$A_1^+, A_2^+, ..., A_n^+, Dis(p_1), Dis(p_2), ..., Dis(p_m) \vdash_{HK} \tau B^-;$$

Similarly to the proof of lemma 4.5 (considering σB^+ and τB^- instead of $\neg(D_0 \supset D_0)$)

we obtain

$$A_1, A_2, ..., A_n \vdash_{HLU} [\theta(\sigma B^+)]_{\neg\neg};$$
$$A_1, A_2, ..., A_n \vdash_{HLU} [\theta(\tau B^-)]_{\neg\neg};$$

or, using the definition of θ and lemma 3.8,

$$A_1, A_2, ..., A_n \vdash_{HLU} \neg\neg^\circ\sigma\neg\neg^\circ B;$$
$$A_1, A_2, ..., A_n \vdash_{HLU} \neg\neg^\circ\tau\neg\neg^\circ\neg B;$$

hence, using the point (c) of lemma 3.4,

$$A_1, A_2, ..., A_n \vdash_{HLU} \sigma\neg\neg^\circ B;$$
$$A_1, A_2, ..., A_n \vdash_{HLU} \tau\neg\neg^\circ\neg B;$$

Similarly to the proof of theorem 2.1 we conclude: if σ is empty, τ is $\neg$, then

$$A_1, A_2, ..., A_n \vdash_{HLU} \neg\neg^\circ B \vdash_{HLU} B;$$

if σ is $\neg$, τ is empty, then

$$A_1, A_2, ..., A_n \vdash_{HLU} \neg\neg^\circ\neg B \vdash_{HLU} \neg B;$$

if σ and τ are $\neg$, then

$$A_1, A_2, ..., A_n \vdash_{HLU} \neg^\circ B;$$
$$A_1, A_2, ..., A_n \vdash_{HLU} \neg^\circ\neg B;$$

hence

$$A_1, A_2, ..., A_n \vdash_{HLU} (B \supset \neg B)\&(\neg B \supset B).$$

The case, when cr and t are empty is impossible, as it is noted above. This completes the proof of theorem 2.2.

§5. In this section we shall introduce and investigate arithmetical systems HLUS and HLUL in the framework of J.Lukasiewicz's logic.

The system HLUS in the signature $Z_1 = \{0, ', =\}$ is defined by the following axioms:

($HLUS_1$) $\forall x(x = x)$;

($HLUS_2$) $\forall x\forall y\forall z((x = y) \supset (x = z \supset y = z))$;

($HLUS_3$) $\forall x\forall y(x = y \supset x' = y')$;

($HLUS_4$) $\forall x\neg(x' = 0)$;

($HLUS_5$) $\forall x\forall y(x' = y' \supset x = y)$;

and by the scheme of induction

($HLUS_B$) $(\forall\forall)(Subst(B, x, 0) \supseteq (\forall x(B \supseteq Subst(B, x, x')) \supseteq \forall x(B)))$,

where x is any variable, B is any formula in $L(Z_1)$.

The system HLUL in the signature $Z_2 = \{0,', <, =\}$ is defined by the following axioms:

(HLUL$_1$) $\forall x(x = x)$;

(HLUL$_2$) $\forall x\forall y\forall z((x = y) \supset (x = z \supset y = z))$;

(HLUL$_3$) $\forall x\forall y(x = y \supset x' = y')$;

(HLUL$_4$) $\forall x\forall y\forall z((x = y) \supset (y < z \supset y < z))$;

(HLUL$_5$) $\forall x\forall y\forall z((x < y) \supset (y = z \supset x < z))$;

(HLUL$_6$) $\forall x\forall y((x = y) \vee \neg(x = y))$;

(HLUL$_7$) $\forall x(\neg(x = 0) \subset \exists y(x = y'))$;

(HLUL$_8$) $\forall x(x < x')$;

(HLUL$_9$) $\forall x\forall y(x < y \vee x = y \vee y < x)$;

(HLUL$_{10}$) $\forall x\forall y\forall z((x < y) \subset (y < z \subset x < z))$;

(HLUL$_{11}$) $\forall x\forall y((x < y) \sim \neg(y < x))$;

(HLUL$_{12}$) $\forall x\neg(x' < 0)$;

(HLUL$_{13}$) $\forall x\forall y\neg^{\circ}(x < y \& y < x')$.

Theorem 5.1. The system HLUS is Luk-complete.

Theorem 5.2. The system HLUL is Luk-complete.

Let us define an auxiliary notion. We shall say, that a three-valued predicate is a *quasi-classical* one if its logical value in any point is either „true" or „false". We shall say, that a Luk-model of some axiomatic system is a *quasi-classical* one if the interpretations of all predicate symbols in this Luk-model are quasi-classical predicates. We shall prove below, that the system HLUS can have only quasi-classical Luk-models; of contrary, the system HLUL has no quasi-classical Luk-model.

Let us note that if a Luk-complete axiomatic system Ω has only quasi-classical models, then, however, we cannot say that it is complete in the classical sense. Indeed, if a formula B is the logical constant U („undefined"), then neither B nor $\neg B$ is a Luk-corollary of Ω.

Proof of theorem 5.1: The system HLUS is Luk-consistent because it has a Luk-model, where the main set is the set of all natural numbers, the symbols 0 and $'$ have usual interpretations, and the interpretation of $=$ is a three-valued quasi-classical predicate, such that its value in any point is equal to the value of the classical predicate $=$.

Taking in the scheme of induction the formula $(x = 0) \vee \neg(x = 0)$ as B, x as x, and the empty group of quantifiers as $(\forall\forall)$, we obtain

$$HLUS \vdash_{HLU} \forall x((x = 0) \vee \neg(x = 0)).$$

Further, taking in the scheme of induction the formula $((x' = y) \vee \neg(x' = y))$ as B, y as x, the empty group of quantifiers as $(\forall\forall)$, we obtain

$$HLUS, \forall y((x = y) \vee \neg(x = y)) \vdash_{HLU} \forall y((x' = y) \vee \neg(x' = y)).$$

hence

$$HLUS \vdash_{HLU} \forall x(\forall y((x = y) \vee \neg(x = y)) \supseteq \forall y((x' = y) \vee \neg(x' = y))).$$

Finally, taking in the scheme of induction the formula $\forall y((x = y) \vee \neg(x = y))$ as B, x as x, the empty group of quantifiers as $(\forall\forall)$, we obtain, using the preceding deducibilities:

$$HLUS \vdash_{HLU} \forall x \forall y((x = y) \vee \neg(x = y)).$$

(cf.[3], point *158, p.193; [7], lemma 4.2, p.66). This deducibility means that the single predicate symbol $=$ in Z_1 can have only quasi-classical Luk-interpretations in models of HSUS, hence, the system HSUS can have only quasi-classical models. Now let us construct the axiomatic system HLUS$^+$ in the signature $Z_1^+ = \{0, ', =^+, =^-\}$. Using the axiom

$$\forall x \forall y \neg((x =^+ y) \& (x =^- y))$$

in HLUS$^+$ and the formula

$$\forall x \forall y((x = y) \vee \neg(x = y))^+,$$

that is

$$\forall x \forall y \neg((x =^+ y) \vee (x =^- y))$$

(which is a classical corollary of HLUS$^+$ by lemma 4.6), we obtain that

$$\text{HLUS}^+ \vdash_{HK} \forall x \forall y((x =^- y) \sim \neg(x =^+ y));$$

so, we can eliminate the predicate symbol $=^-$ from Z_1^+ and construct a system equivalent (in an obvious sense) to HLUS$^+$ in the signature $\{0, ', =^+\}$. It is easily seen, that all the axioms of the system A_s, considered in [1] (where we replace $=$ by $=^+$) are deducible in HK from HLUS$^+$. But it is proved in [1], that A_s is complete in the corresponding signature. Hence HLUS$^+$ is complete in the classical sense, and by theorem 2.2 HLUS is Luk-complete.

Proof of theorem 5.2: The system HLUL is Luk-consistent because it has a Luk-model, where the main set is the set of all natural numbers, the symbols $0, ', =$ have usual interpretations, and the symbol $<$ is interpreted as a three-valued predicate having the value „undefined" on every pair of equal natural numbers, having the value „true" on every pair (x, y), where x is less than y, and having the value „false" on every pair (x, y), where y is less than x.

Let us consider the system $HLUL^+$ in the signature $Z_2^+ = \{0,', <^+, <^-, =^+, =^-\}$. Similarly to the proof of theorem 5.1, using the axiom $HLUL_6$, we obtain

$$HLUL^+ \vdash_{HK} \forall x \forall y((x =^- y) \sim \neg(x =^+ y)).$$

Besides, using the axiom $HLUL_{11}$, we obtain

$$HLUL^+ \vdash_{HK} \forall x \forall y((x <^- y) \sim \neg(x <^+ y)).$$

Hence, replacing $x =^- y$ by $\neg(x =^+ y)$, and $x <^- y$ by $\neg(y <^+ x)$, we can eliminate $=^-$ and $<^-$ from Z_2^+ and construct a system equivalent (in an obvious sense) to $HLUL^+$ in the signature $\{0,', <^+, =^+\}$. It is easily seen that all the axioms of the system A_L considered in [1] (where we replace $=$ by $=^+$, and $<$ by $<^+$) are deducible in HK from $HLUL^+$. But it is proved in [1], that A_L is complete. Hence $HLUL^+$ is complete in the classical sense. By theorem 2.2 HLUL is Luk-complete, and the theorem 5.2 is proved.

Note 1. The system HLUL cannot have quasi-classical Luk-models. Indeed, using the axiom $HLUL_{11}$ we obtain

$$HLUL \vdash_{HK} \forall x \forall y((x < x) \sim \neg(x < x)),$$

And so, in every Luk-model of HLUL the one-dimensional predicate $x < x$ is equivalent to its negation, which is impossible in quasi-classical Luk-models.

Note 2. If we replace in the axiom $HLUL_{13}$ the weak negation $\neg^\circ$ by the usual negation $\neg$, then the system obtained by such a transformation, will be Luk-inconsistent. Indeed, we can deduce from the transformed axiom $HLUL_{13}$ the formula

$$\forall x \forall y(\neg(x < y) \vee \neg(y < x'))$$

and the formula

$$\neg(0 < 0) \vee \neg(0 < 0').$$

But we have, using axiom $HLUL_8$,

$$HLUL, \neg(0 < 0) \vee \neg(0 < 0') \vdash_{HLU} \neg(0 < 0),$$

and, using the deducibility

$$HLUL \vdash_{HLU} \forall x \forall y((x < x) \sim \neg(x < x)),$$

we obtain in the transformed system both $0 < 0$ and $\neg(0 < 0)$ which means that this system is Luk-inconsistent.

A Luk-inconsistent system will be obtained also if, for example, we replace x' by x in the axiom $HLUL_{12}$.

Note 3. Three-valued analogue of M.Presburger's system considered in [13] has only quasi-classical Luk-models. It can be proved similarly to the proof of theorem 5.1.

ACKNOWLEDGEMENT

The author is grateful to G.E. Mints for valuable discussions and suggestions.

REFERENCES

1. H.B.Enderton. A Mathematical Introduction to Logic. 2nd edition, San Diego, Harcourt, Academic Press (2001).

2. D. Hilbert and P. Bernays. Grundlagen der Mathematik, Band 1. Zweite Auflage, Berlin - Heidelberg - New York, Springer-Verlag (1968). French translation: D. Hilbert et P. Bernays. Fondements des mathématiques, tome 1, L'Harmattan (2001). Russian translation: Д. Гильберт, П. Бернайс. Основания математики (логические исчисления и формализация арифметики). Перевод Н.М. Нагорного под ред. С.И. Адяна, М., „Наука"(1979).

3. S.C.Kleene. Introduction to Metamathematics, D.Van Nostrand Comp., Inc., New York - Toronto (1952).

4. J.Lukasiewicz. O logice trojwartosciowej. Ruch filozoficzny (Lwow), 5, 169-171 (1920).

5. E.Mendelson. Introduction to Mathematical Logic. D. van Nostrand Comp., Inc., Princeton, New Jersey - Toronto - New York - London (1964).

6. I.D.Zaslavsky. On the predicate and arithmetic calculi of the symmetric constructive logic. Soviet Math. Dokl., vol.14, pp.751-754 (1973).

7. I.D.Zaslavsky. Symmetric Constructive Logic. Publ. House of Acad. Sci. Arm. SSR (1978) (in Russian).

8. I.D.Zaslavsky. On a logically but not functionally complete calculus in three-valued logic. In.: The Tbilisi Symposium on Logic, Language and Computation, selected papers, CSLI publications, Stanford, California, pp.309-313 (1998).

9. I.D.Zaslavsky. On three-valued arithmetic. In.: Proc. Intern. Conf. on Computer Science and Information Technologies, CSIT'01, Yerevan, pp.133-136 (2001).

10. I.D.Zaslavsky. Dual realizability in symmetric logic. Annals of Pure and Applied Logic, vol.113, pp.389-397 (2002).

11. I.D.Zaslavsky. On three-valued logical models and their applications to arithmetical theories. Submitted to Denis Richard 60th Birthday Conference, Clermont-Ferrand, France (2002).

12. I.D.Zaslavsky. On the Completeness Properties of the Systems of Symmetric Constructive and Three-valued Logic. In: International Conference „21st Days of Weak Arithmetics", St.Petersburg, Russia, June 7-9, 2002, Abstracts, St.Petersburg, pp.17-18 (2002). URL http://at.yorku.ca/cgi-bin/amca/cail-01.

13. I.D.Zaslavsky. Formal axiomatic theories on the base of three-valued logic. „Zapiski nauchnykh seminarov POMI" (Proceedings of Scientific Seminars in St.Petersburg Department of Steklov Institute of Mathematics, Russian Acad. Sci.), vol.304, „Theory of Complexity of Computations VIII", St.Petersburg, pp.19-74 (2003) (in Russian).

14. Igor Zaslavsky. On Luk-completeness of axiomatic systems based on three-valued logic. In: International Conference „23rd Days of Weak Arithmetics", Yerevan, Armenia, June 2-5, 2004, Abstracts, Yerevan, pp.14-15 (2004).

16 September 2005

Institute for Informatics and Automation Problems of the
National Academy of Sciences of Armenia, Yerevan, Armenia
E-mail: zaslav@ipia.sci.am