INFINITE GAMES AND FINITE MACHINES: FROM LARGE CARDINALS TO COMPUTERS[1]

Dedicated to Denis Richard in his roaring sixties

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We give a new proof of finite state determinacy, inspired by Martin's proof of analytic determinacy using the large cardinal axiom of sharps. The proof suggests a new direction in order to study the feasible aspects of finite state determinacy. It is on the borderline between Logic and Computer Science, but our presentation aims at a good understanding no matter which side of the line you prefer.

§1. INTRODUCTION

Let A be a set of binary „infinite words" : $A \subset 2^\omega$; the associated *game* $G(A)$ lets player I and player II choose $z \in 2^\omega$. This *play* z is won by player I iff z belongs to A, and by player II otherwise. (Thus A is player I 's *winning set* ; while the winning set of player II is the complement $2^\omega - A$). The choice of z is made in an infinite succession of turns : at turn n player I chooses $z(2n) \in 2$ and player II replies with $z(2n+1) \in 2$.

$G(A)$ is *determined* iff one of the player has a *winning strategy (=: w.s.)* That is, a map $\sigma : 2^{<\omega} \longrightarrow 2$ such that the player is guaranteed to win every play z in which he played $z(n) = \sigma(z|n)$ whenever it was his turn to play.

An *analytic* set A is one accepted by some non deterministic Turing machine with oracle.

Martin [M] proved „*analytic determinacy*" : determinacy of $G(A)$ in case the winning set of one of the players is analytic. But Martin's proof uses „*sharps*" : a large cardinal axiom, much stronger and daring than ordinary Set Theory, although the strength of the latter is already way beyond the current needs of mathematics (as of today). And

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Harrington [H] (building on ideas of Harvey Friedman) proved a converse, to the effect that sharps are *equivalent* to analytic determinacy. This underlies the fact that the winning strategy proved to exist is in the general case of the most extremely *non* effective kind. In fact, set theory's topic of large cardinals is the most infinitary part of mathematics; and analytic (more generally projective) determinacy is the most spectacular outcome of this infinitary riot.

Back to earth : the study of FS (:= *finite state*) machines is the very first chapter of Computer Science. Can we combine fruitfully these two opposite extremes, and use ideas coming from large cardinals to produce results about finite state machines ? Surprisingly we here give a modest but definitely positive answer : we shall supplement Martin's proof so that it yields *FS determinacy*: the existence of some FS computable winning strategy in G(A), whenever the winning set A itself is FS accepted. This nice theorem is one of the few in existence to be entirely mathematical and at the same time entirely about Computers. It had three proofs due to Rabin, Büchi-Landweber, Gurevich-Harrington (the latter proof has been optimized by Thomas, see [*T*]). The new and highly paradoxical proof of FS determinacy we present suggests quite a new perspective explained in the conclusion, in order to adress some main unsolved problems connecting effective determinacy with Computer Science : the P-time realization of FS strategies, and the P-time decision of the winner of a „parity game".

Section 4 below exposes most of Martin's proof of analytic determinacy. This is not abusive : if we simply gave a reference for it, we could not reasonably explain our own proof. In addition Martin's proof is short and most commendable reading. And we provide it with a new and accessible form S of sharps, since the original form of this axiom looks hopelessly away from Computer Science.

§2. PREVIEW OF THE PROOF

Recall that A is a *finite state accepted* subset of 2^ω if there is an FS machine such that A consists of the words $z \in 2^\omega$ which are accepted by this machine. We call it the *acceptor* A, we denote Q its *set of states* (so Q is finite) and for every $s \in 2^{<\omega}$ we denote $q(s)$ the state which the acceptor A enters upon reading the word s. Thus when acceptor A reads an infinite word $z \in 2^\omega$, there is a subset of Q

$Inf\ z := \{q;\ q = q(z|n)$ for infinitely many $n < \omega\}$.
which forms a loop in the transition graph of the acceptor A. And the acceptor has a fixed family $\mathcal{F}$ of *accepting* loops such that z is accepted iff $Inf\ z$ belongs to $\mathcal{F}$.

Nota-Bene : except in last section we assume that the acceptor A is deterministic (:= $q(s)$ is uniquely defined ; this does not change the class of FS accepted set).

Let us call *rejecting* (for player I) all other loops in Q. In order for player I to win the game $G(A)$, he must avoid a rejecting loop to repeat itself eventually during the play z. Here is a way (*) to ensure this :

*) player I plays ordinals in addition to his moves, in such a way that whenever a rejecting loop L starts being repeated by the acceptor A (reading the play z during its performance), the ordinals chosen by player I when each repetition occurs start building a strictly decreasing sequence of ordinals.

Remark 1. All such sequences are finite, hence if player I manages for the whole play z to ensure (*), then every sequence of repetitions of a rejecting loop L is finite and play $z \in A$ and is won by player I .

This idea can be made precise in different ways, which are not equivalent. Indeed, the whole sequel depends on a careful selection of the precise version of (*) that is used.

For any ordinal γ we denote $G(\gamma)$ the version of the game $G(A)$ that follows :

- in addition to his moves $z(2n)$ for G(A), player I must produce ordinals $< \gamma$ in the above way (*)
- if he does so for the whole infinite play he wins at the end
- and if he cannot at some finite stage, player II *immediately* wins.

Thus we turned $G(A)$ to a game $G(\gamma)$ that is *open* for player player II := he wins a play iff he already wins it at a finite stage. Gale and Stewart proved *open determinacy :* determinacy of every game which is open for at least one player. Thus there is a strategy σ_γ that is winning for one of the players (which one depends on A, γ). From the family of strategies σ_γ we now deduce a w.s. in the original game $G(A)$.

Case 1 : there is some γ such that the strategy σ_γ is winning for player I . Then by R.0, σ_γ is winning for player I also in $G(A)$ (just omitting to exhibit the ordinals provided by σ_γ, since they are not required in the original game).

Case 2 : not Case 1, hence σ_γ is winning for player $I\!I$. In contrast with the first case, this strategy in $G(\gamma)$ is no longer a strategy at all in $G(A)$: for it depends on ordinals that player I duly provides in $G(\gamma)$ but not in $G(A)$. Here enter the axiom „sharps" and Martin's analytic determinacy proof : as long as A is analytic, sharps enable Martin to define a *mean value* $\sigma = \int_\gamma \sigma_\gamma$ of these strategies for all *countable* γ's. This mean value no longer depends on the ordinals provided by player I during the play of $G(\gamma)$ because instead it uses their „average". For that reason σ is a strategy for player $I\!I$ in the original game $G(A)$; and using the fact that the strategies σ_γ which σ comes from were winning ones, Martin's proof shows that σ is winning for player $I\!I$.

Nota-Bene : actually Martin's idea only applies to a modification of the above $\dot{G}(\gamma)$ and σ_γ. But the modification still uses the same ideas : Gale-Stewart and a version of (*)+R.0.

The disjunction of the two cases produces a winning strategy for $G(A)$ in any case, hence proves its determinacy. But it does so in a way that seems infinitely far away from FS determinacy. For the set of countable ordinals and the strategies σ_γ -let alone their „mean value" which depends on a large cardinal- are everything but effective -let alone FS !

Nevertheless there are many possible variations in the definition of the open game $G(\gamma)$ and its w.s. σ_γ. And we found one of these variations which in Case 2 yields an average strategy σ that *is* FS. This shows FS determinacy in case player $I\!I$ has a winning strategy. And interchanging the roles of the two players puts an end to the proof : it allows to show the other case.

Let us finally preview how the outrageously non effective average strategy $\int_\gamma \sigma_\gamma$ is to become an FS one. We recall Gurevich-Harrington's contribution to FS determinacy. Let A be a FS accepted subset of 2^ω.

Proposition : oblivious determinacy - In $G(A)$, one of the players has a winning strategy σ such that $\sigma(z|n)$ only depends on $LAR(z|n)$; where for $s \in 2^{<\omega}$

- $LAR^0(s) :=$ the sequence $i_0 < \ldots < i_k = lh(s)$ which enumerates all stages i_j at which some state $q^j \in Q$ is entered *for the last time* by machine A reading s (thus $k < N := |Q|$, $q^j = q(s|i_j)$, $q^k = q(s)$ and $LAR^0(s) \in [lh(s)+1]^{\leq N}$)

∘ $LAR(s) := (q^0, ..., q^k) \in Q^{\leq N}$ (with $i \neq j \longrightarrow q^i \neq q^j$).

Such a strategy σ is said to be *oblivious* because it can forget most of s, retaining only the information $LAR(s)$ which is bounded independently of $lh(s)$. We shall later set

$$LAR^2(s) := (LAR(s|i_0), ..., LAR(s|i_k))$$

where $(i_0, ..., i_k) = LAR^0(s)$ (thus in particular $LAR(s)$ is the last element of $LAR^2(s)$). The dependence of $\sigma(s)$ on $LAR^2(s)$ (instead of $LAR(s)$) is a weaker oblivion property for a strategy σ that we call *2*-oblivious.

Let us come back to the games $G(\gamma)$, the winning strategies σ_γ and the average $\int_\gamma \sigma_\gamma$: their precise, modified definition will ensure that this family of games and strategies is oblivious in some sense, so that the average will become 2-oblivious. By the Remark (a) below, a 2-oblivious strategy is an FS one. And this ends our preview of the proof of FS determinacy.

Remark

(a) Let LAR denote the (finite) set of possible values of the function $LAR(s)$; the reader can easily devise an FS machine L such that LAR is the set of states of L, and L enters the state $LAR(s)$ upon reading s. Likewise there is an FS machine L^2 with set of states LAR^2 which enters the state $LAR^2(s)$ upon reading s.

    Strategy σ is *2-oblivious* $:= \sigma\ (s) = w(LAR^2(s))$ for some function w. Note that w has finite domain and image : it is the writing function of an „FS transducer" (L^2, w) which computes σ.

(b) It looks as if our new FS determinacy proof is not as good as the Gurevich-Harrington one : the latter obtains an oblivious w.s., which is better than 2-oblivious. But this is *not* a weakness of our proof in the perspective we shall present for future research : see R.19 below.

There are quite a few ways to use the mathematical theory of infinite games in Computer Science: see [T]. We end this section by recalling one of these ways, which has great practical content. Present days industry raises a large number of problems of the form : design a (non terminating and FS) processor P working in real time interaction with its environment, and that satisfies for a certain specification. FS determinacy is the theoretical background of a successful modelization of this problem.

(a) One imagines an infinite game in which player player I is the environment: its possible moves are all stimuli which the environment might send at once to the processor ; player $I\!I$ is the processor: its possible moves are all the reactions which the processor might have to make at once. The plays are all infinite sequences of alternate moves of player I and player $I\!I$ - coded so as to coincide with all elements of 2^ω.

(b) The specification is then represented by the set A of all plays $z \in 2^\omega$ such that the moves of player $I\!I$ are a satisfactory response to the moves of player I , according to the specification; thus our problem becomes: find a transducer σ which is a w.s. for player $I\!I$ in the game $G(A)$.

(c) If A happens to be FS accepted, then by FS determinacy one of the players has a w.s. σ which is an FS transducer. If this player is player $I\!I$, then σ is the abstract form of the desired processor P; and if it is player I , then no processor of any kind will ever satisfy the specification. Moreover using Büchi's lemma which says that an FS acceptor accepts a non empty set of infinite words iff it accepts an ultimately periodic one, we can *effectively* determine which player has the w.s. - and find out the transducer σ which realizes it.

The infinite length of $G(A)$ is an imaginary feature and makes us expect that the above „model of real world processors" is a crude and falsely idealized one. But if a w.s. σ for $G(A)$ is performed by an FS processor P then P has done everything σ has to do every time a loop is completed in the transition graph of P. (For otherwise player I could induce infinite repetition of the unsatisfactory loop, to win the play and defeat σ). Now suppose P is a real world processor, with about 10^6 states say. Today, its speed is counted in gigaherz : P completes a loop every fraction of a second !

Thus although $G(A)$ seems to allow σ and P unlimited hence unreasonable amounts of time to complete their task, in practice P is very quickly effective. So that the above model of processor design has some (heuristic) value for a very large class of applications - for instance, in the design of processors used in modern planes. Indeed, the expanding mathematical theory of this model is giving guidemarks and ideas of algorithms for the design of such processors. Which is an extraordinary fate for a theoretical research about *infinite* games...

But FS determinacy is only the beginning of this remarkable story. Today there are unsolved theoretical problems which are as beautiful, and have much more precise and strong bearing in applied CS ... only they are much harder. Namely :

1. P-time realization of FS strategies
2. P-time decision of the winner of a parity game (= a special case of FS games).

While FS determinacy contents itself with the mere existence of a FS winning strategy, these problems ask to quickly compute it and to quickly decide which of the two players has it. The first, obvious step in the study of this „P-time version of FS determinacy" is to examine the now 4 proofs of FS determinacy and see whether we might extract from them additional information (on the moves of the w.s. , or on the winner). The answer looks rather negative to us for the first three proofs. Whereas our new one opens a track; but this will be discussed in last section.

§3. OBLIVION

We start the full account of the proof just previewed : we work towards a modified definition of the games $G(\gamma)$ which via Lemma 7 and section 6 leads to „oblivious", hence FS winning strategies. This will use not only the results below up to Lemma 6 but also the particular proofs we are providing for them.

We write $s \subset s'$ if sequence s' is a proper extension of s. For $s \subset s' \in 2^{<\omega}$ we denote $q(s, s')$ the set of states which machine A reading s' visits after s is read through : $q(s,s') := \{q(s'|i); lh(s) < i \leq lh(s')\}$.

Lemma 1: *„normal form"*

If $s \subset s' \in 2^{<\omega}$ we set : $s'Rs \longleftrightarrow$ there is $s^- \subset s$ such that

1) $q(s,s')$ is a rejecting loop and $q(s,s') = q(s^-, s)$.

Then for each $z \in 2^\omega$,

$z \in A$ iff the relation R_z is well founded

- where $R_z := R \restriction \{z|n; n < \omega\}$.

Proof L.1

We prove (L.1,$\longrightarrow$) in contrapositive form. So we assume that R_z is ill founded : for some infinite sequence (s_p) we have $s_{p+1} R s_p \subset z$. Thus $q(s_p, s_{p+1})$ is rejecting; we show that it becomes constant for all sufficiently large p

There is $n < \omega$ such that $q(s_{n-1}, s_n)$ is not *strictly* contained in $q(s_{n+p}, s_{n+p+1})$ for any p.

Claim : $q(s_{n-1}, s_{n+p}) \subseteq q(s_{n-1}, s_n)$ for all $p < \omega$.

This being true for $p = 0$, it suffices to show a contradiction if one assumes it true for some p but false for $p+1$. Indeed, let q be an element of $q(s_{n-1}, s_{n+p+1})$ but not in $q(s_{n-1}, s_n)$. Then $s_{n+p+1} R s_{n+p}$ implies :

$$+)\ q \in q(s^-_{n+p}, s_{n+p+1}).$$

Remark 2.

The only property of R needed to ensure $(+)$ is "$s'Rs \longrightarrow (1)$", and not the converse.

$(+)$ implies : $s^-_{n+p} \subset s^-_{n-1}$ hence $q(s_{n-1}, s_n)$ included in $q(s^-_{n+p}, s_{n+p+1}) = q(s_{n+p}, s_{n+p+1})$. The inclusion is strict due to the element q. This contradiction with the choice of n proves the Claim.

Thus $q(s_{n+p}, s_{n+p+1})$ is constant or decreasing. The first case ends our proof of (L.1,$\longrightarrow$); and the latter case happens at most $|Q|$ times. That is: $k < |Q|$ if $n = n(0) < n(1) < ... < n(k)$ are such that i) each $n(i)$ has the same maximality property as n; and ii) $q(s_{n(i)}, s_{n(i+1)})$ goes decreasing each time. Thus eventually $q(s_{n(k)+p}, s_{n(k)+p+1})$ gets constant and equal for all p to $Inf\ z$ which is thus rejecting : $z \in A$ is false and (L.1,$\longrightarrow$) is proved (in contrapositive form).

Now we show (L.1,$\longleftarrow$) in contrapositive form. Assume that $z \notin A$: $Inf\ z$ is rejecting. Set

- $m_0 :=$ smallest $m < \omega$ such that $q(z|m, z) = Inf\ z$
- $m_{i+1} :=$ smallest $m > m_i$ such that $Inf\ z = q(z|m_i, z|m)$.

Since $q(z|m_{i-1}, z|m_i) = q(z|m_i, z|m_{i+1}) = Inf$ z is rejecting, $z|m_{i+1} R z|m_i$ for each $i > 0$. Thus R is not well founded.

L.1 proved

We next want to linearize R : extend it to an order $<$ on $2^{<\omega}$ so that R is well founded on $\{z|n; n < \omega\}$ iff $<$ is well ordered on it. For any $s \in 2^{<\omega}$ and $z \in 2^\omega$ we shall set $R_s := R \restriction \{s|n; n \leq lh(s)\}$ - hence $R_z = \cup_{n<\omega} R_{z|n}$.

Lemma 3. : „*linearization*"

Let R' denote the transitive closure of R and define for $S = s \in 2^{<\omega}$ and $S = z \in 2^\omega$ a linear order $<_S$ extending R_S by the clauses :

- $s \subset s'$ and $lh(s') = lh(s) + 1$ implies : $<_s \subset <_{s'}$ and in the order $<_{s'}$, s' is immediate predecessor of the set $\{s|k; k \leq lh(s), s'R's|k\}$ if the set is non empty.
- s' is the greatest element of $<_{s'}$ otherwise.
- $<_z := \cup_{n<\omega} <_{z|n}$ for $z \in 2^\omega$

Then whenever $z \in 2^\omega$ we have : R_z is well founded iff $<_z$ is well ordered.

Proof L.3 The „if" part of L.3 is clear since $R_z \subset <_z$. We prove the reciprocal in contrapositive form : from an infinite descending sequence (s_n) for $<_z$, we deduce a sequence (s^n) descending for R_z. Note that for this purpose our definition of $<_s$ is most natural : it is descending only when this is *forced* by R_s. However in order to prove the reciprocal we need the alternative definition of $<_s$ provided by the Fact below. The initial definition by induction on s remains useful in order to prove properties of $<_s$ by the same induction. In particular it makes inductively clear that $s \subset s' \longrightarrow <_s \subset <'_s$ and $<_s$ extends R, R' to a linear order of the restrictions of s.

Fact 4. For $t \in 2^\omega, k < \omega$ recursively set

- $t^0 := min_\subset\{t';\ t = t' \text{ or } tR't'\}$
- $t^{k+1} := min_\subset\{t' \supset t^k;\ t = t' \text{ or } tR't'\}$ (t^{k+1} *undefined* if the latter set is empty)

Assume $s \subset t$ and $s^i = t^i$ for each $i < k$, then

(a) $t^k \subset s^k$ implies $t <_t s$

(b) $s^k \subset t^k$ implies $s <_t t$

(c) $t^k = s^k$ implies

- i) if $s^k = s$, then $t <_t s$
- ii) if $s^k \subset s$, then t^{k+1} and s^{k+1} both are defined.

Remarks a) This decides the relation „$t <_t s$" except in case (c.ii). But in this case the comparison for $\subset$ between t^{k+1} and s^{k+1} offers a new chance to decide „$t <_t s$".

b) Using this chance at most $lh(s)$ times eventually decides „$t <_t s$", since s^k defined implies $k \leq lh(s)$ (because $sR's^{k+1}R's^k$).

c) Note that the definitions imply : $(t^k)^i = t^i$ $(i \leq k)$; $tR't'$ and $t^i = t'^i$, $i < j \longrightarrow$ $(t^j \subsetneq t'^j$ or $t' = t^{j-1})$.

Proof F.4 (c.i) is clear : $tR't^k = s^k = s$ implies $t <_t s$. (c.ii) clear too : if necessary t^{k+1}, s^{k+1} are provided by t, s.
We adress (a+b); it clearly holds when $k = 0, t$ of length 1. We inductively assume that (a+b) holds :

- for any shorter sequence in place of t
- and for each smaller or equal integer in place of k.

(a) - $(t^k)^k = t^k \subset s^k$ implies : $t^k <_t s$ by induction hypothesis (b) with s, t^k, k in place of t, s, k. Thus $t <_t t^k <_t s$.

(b) - Let X denote $\{t';\ tR't'\}$; if X is empty then $s <_t t$ by definition of $<_t$. Otherwise our inductive hypothesis applies to each $t' \in X$; and by the above R.c there is j such that $t^i = t'^i$ for each $i < j$ and either $j = k + 1$ or $j \leq k$ and $(t^j \subset t'^j$ or $t' = t^{j-1})$.
Assume $s \subset t'$; if $j = k + 1$ then $s <_{t'} t'$ by induction hypothesis (b) with t', s, k in place of t, s, k. If $j \leq k$ then $t' = t^{j-1}$ is excluded (by $s \subset t'$), so $s^j \subseteq t^j \subset t'^j$ implies

$s <_{t'} t'$ by induction hypothesis (b) with t', s, j in place of t, s, k. Now assume $t' \subset s$; if $j = k+1$ then $s <_s t'$ by induction hypothesis (a) with s, t', k in place of t, s, k. If $j \leq k$ and $t' = t^{j-1}$ then $s <_s s^{j-1} = t^{j-1} = t'$. If $j \leq k$ and $t^j \subset t'^j$ then $s^j = t^j \subset t'^j$ implies $s <_s t'$ by induction hypothesis (a) with s, t', j in place of t, s, k.

In any case we got $s <_t t'$, hence $s <_t X$. And $s <_t t$, since t is *immediate* predecessor for $<_t$ of X.

F.4 proved

Proof L.3 Assume that $<_z$ is not well ordered : $s_{n+1} <_z s_n$ for some sequence (s_n) such that $\cup_{n<\omega} s_n = z$. By extracting a suitable subsequence we can assume $\forall n\ s_{n+1} \subset s_n$. Then by F.4.b applied with $t = s_{n+1}, s = s_n$ we conclude that $s^0_{n+1} \subseteq s^0_n$ for each n. By extracting a suitable subsequence we can assume : s^0_n is a constant s^0. Inductively assume that (by extracting subsequences) we have : $\forall i \leq k\ \forall n < \omega$ (s^i_n is a constant s^i). Then $n > 0 \longrightarrow s^{k+1}_n$ exists, by F.4.c.ii. Then by F.4.b $s^{k+1}_{n+1} \subseteq s^{k+1}_n$; hence by extracting a subsequence we can assume that s^{k+1}_n is a constant s^{k+1}. Note that for any s, $s^{k+1} R' s^k$ whenever it is defined. Hence by induction on $k < \omega$ we are constructing an infinite sequence $(s^k)_{k<\omega}$ such that $\cup_k s^k = z$ and $s^{k+1} R' s^k$ for all k.

L.3 proved

Remark 5 : the preceding proof and lemma actually work for *any* relation R on $2^{<\omega}$ such that tRs implies $s \subset t$

When R comes from the „first normal form" (L.1) of A and $<$ denotes the linearization of R by the above lemma, we immediately obtain :

Lemma 6. : *„linear normal form"*

$z \in A$ iff $<_z$ is well ordered - whenever $z \in 2^\omega$.

There remains to bring oblivion to these normal forms. The concatenation of two sequences s, t is denoted $s^\frown t$.

Lemma 7. : *„oblivious normal forms"*

In L.1's definition of $s'Rs$, to the condition

1) $q(s,s')$ is a rejecting loop and $q(s,s') = q(s^-,s)$.

let us add

$$2)\ q(s^-,s') \neq q(s'',s') \text{ whenever } s \subset s'' \subset s'.$$

Then

(a) R and $<$ keep satisfying : $z \in A$ iff R_z is well founded iff $<_z$ is well ordered - whenever $z \in 2^\omega$.

(b) R becomes in addition „*oblivious*" := for $s \in 2^{<\omega}$

- i) the sets $\{t \in 2^{<\omega}; s^\frown t R s\}$ and $\{(t,t');\ t,t' \in 2^{<\omega}, s^\frown t' R s^\frown t\}$ only depend on $LAR(s)$
- ii) the set $\{i \leq lh(s); s^\frown t R s|i\}$ is contained in $LAR^0(s)$.

(c) $<_s$ becomes in addition „*2-oblivious*" := for $s \in 2^{<\omega}$, and $(i_j)_{j\leq k} = LAR^0(s)$

- i) the set $\{(t,t'); t,t' \in 2^{<\omega}, s^\frown t' <_{s^\frown t'} s^\frown t\}$ only depends on $LAR^2(s)$; the set $\{(t,j);\ t \in 2^{<\omega}, s^\frown t <_{s^\frown t} s|i_j\}$ only depends on $LAR^2(s)$
- ii) min $\{i \leq lh(s);\ s^\frown t <_{s^\frown t} s|i\}$ and max $\{i \leq lh(s);\ s^\frown t >_{s^\frown t} s|i\}$ belong to $LAR^0(s)$.

Proof L.7

(a) The modified R still satisfies : $tRs \longrightarrow s \subset t$; by R.5 this was the only condition for L.3 to apply. Thus here we only need to prove the part of (a) relative to R. Below (a) only denotes this part.

(a,$\longleftarrow$). The modified R still satisfies : $s'Rs \longrightarrow$ (1). By R.2 this implies (a,$\longleftarrow$).

(a,$\longrightarrow$) in contrapositive form. We assume that $z \notin A$: $Inf\ z$ is rejecting. And we set

- $m_0 :=$ smallest $m < \omega$ such that $q(z|m, z) = Inf\ z$

- $m_{i+1} :=$ smallest $m > m_i$ such that $Inf\ z = q(z|m_i, z|m)$.

Thus $q(z|m_{i-1}, z|m_i)=q(z|m_i, z|m_{i+1}) = Inf\ z$ is rejecting, and (1) is satisfied for each $i < \omega$ by $s := z|m_i, s' := z|m_{i+1}$. But (2) is not guaranteed, so $s'Rs$ cannot be concluded. However we shall define n_i such that $m_i \leq n_i < m_{i+1}$ and both (1+2) are satisfied if $z|n_i$ replaces $z|m_i$:

- 1') $q(z|n_{i-1}, z|n_i) = q(z|n_i, z|n_{i+1})$ is rejecting
- 2') $q(z|n_{i-1}, z|n_{i+1}) \neq q(s'', z|n_{i+1})$ whenever $z|n_i \subset s'' \subset z|n_{i+1}$.

This will ensure $z|n_{i+1}Rz|n_i$.

Inductively assume that we got $n_i < m_{i+1}$ for $i \leq p$; we shall obtain n'_i with the same property but up to $i = p+1$ and with $n_i \leq n'_i$ for $i \leq p$. If we succeed this can be repeated ω times, and the changing value of n_i during this process is non decreasing bounded by m_{i+1}. Hence it stabilizes : at some stage p(i), n'_i remains equal to n_i forever. This provides a final sequence of values $n_i, i < \omega$ satisfying $z|n_{i+1}Rz|n_i$ when R involves (1+2).

It will show ill foundedness of R_z, proving (a,$\longrightarrow$).

Obtaintion of n'_i, $i \leq p+1$:

$n'_{p+1} := m_{p+1}$; $n'_p := \max k : q(z|k, z|n'_{p+1}) = Inf\ z$; $n'_{p-1} := \max k : q(z|k, z|n'_p) = Inf\ z$.

Note that $n'_{p-1} < m_p$ as required, because $n'_p < m_{p+1}$ implies $q(z|m_p, z|n'_p) \neq Inf\ z$ by definition of m_{p+1}. Repeating this leads to n'_i successively from $i = p-2$ down to $i = 1$.

Thus L.7.a is proved.

L.7.b : R is oblivious. Set $LAR^0(s) = (i_0, ...i_k)$. If $s^\frown tRs|i$ and $i_j < i \leq i_{j+1}$, then by (2) $q(s|i_j, s) \neq q(s|i, s)$. By definition of $LAR^0(s)$ this implies $i = i_{j+1}$; thus $\{i : s^\frown tRs|i\} \subseteq LAR^0(s)$. We checked (b.ii); further we have $s^\frown tRs$ iff there is $s^- \subset s$ with $q(s^-, s) = q(s, s^\frown t)$ rejecting. Clearly this only depends on $LAR(s)$ and $q(s, s^\frown t)$, hence on $(LAR(s), t)$. We checked a part of (b.i).

Next we show for $t, t' \in 2^{<\omega}$ that both conditions (1) and (2) in the definition of $s^\frown t'Rs^\frown t$ have their truth which only depends on $(LAR(s), t, t')$. The rejecting character of $q(s^\frown t, s^\frown t')$ (which is part of (1)) only depends on $q(s), t, t'$ - and $q(s)$ is the last element of $LAR(s)$. Thus this part of (1) only depends on $(LAR(s), t, t')$; and the same clearly applies to (2). There remains the other part of (1): existence of $s' \subset s^\frown t$ such that $q(s', s^\frown t) = q(s^\frown t, s^\frown t')$. It has two cases.

- $s \subsetneq s'$: existence of s' then only depends on $(q(s), t, t')$.
- $s' \subset s$: since $q(s', s^\frown t) = q(s', s) \cup q(s, s^\frown t)$ and the possible values of $q(s', s)$ only depend on $LAR(s)$, existence of s' then only depends on $(LAR(s), t, t')$.

Thus we checked the remaining part of (b.i).

Next we show L.7.c : $<$ is 2-oblivious. $<_s$ satisfies (ii) of 2-oblivious because by definition $s^\frown t$ is immediate predecessor for $<_s$ of $\{(s^\frown t)|i; s^\frown tR'(s^\frown t)|i\}$ and since R is oblivious, the latter set intersects $\{s|i; i \leq lh(s)\}$ only where $i \in LAR^0(s)$. There remains (i) of 2-oblivious.

- $s^\frown t <_{s^\frown t} s$ iff $s^\frown tR's|i \leq_{s^\frown t} s$ for some $i \in LAR^0(s)$. By (i) of obliviousness of R (applied to $s|i$ in place of s) the latter only depends on $LAR(s|i), t$, hence on $LAR^2(s)$ and t.
- $s^\frown t' <_{s^\frown t'} s^\frown t$ iff there is $j \geq lh(s^\frown t)$ such that : $s^\frown t'R'(s^\frown t')|j$ *and* $\not\exists k < j$ $s^\frown tR'(s^\frown t)|k$. Truth of the first conjunct for any given j only depends on $t', LAR(s)$ and truth of the secund for any given j, k only depends on $t, LAR^2(s)$. The whole only depends on $t, t', LAR^2(s)$.

L.7 proved.

§4. FROM INDISCERNIBLES TO DETERMINACY : MARTIN'S PROOF

We consider $G(A)$ in a much more general case than in section 3 : namely we only assume that A is co-analytic. This is *equivalent* to say that A has normal forms as the ones of L.1+6 (only they are not oblivious as in the FS case). Thus to every $s \in 2^{<\omega}$ is associated a linear order $<_s$ on $\{s|k; k \leq lh(s)\}$ so that for $z \in 2^\omega$ we have: $z \in A$ iff $<_z$ is well ordered, where $<_z := \cup_{k<\omega} <_{z|k}$. A trivial modification (done for notational convenience) allows to assume that actually $<_s$ only orders the initial segments of s of even length: namely, set $s|2i <_s s|2j$ in the modified order $<_s$ iff $s|i <_s s|j$ in the original one .

Then for each ordinal γ define an auxiliary game G_γ : in addition to $z(2n)$, player I chooses an ordinal $\alpha_n < \gamma$. So the *positions* in G_γ are of the form $((s, \alpha_i)_{i<n})$ with $s \in 2^*$ and $lh(s) = 2n$ or $2n+1$; they are *legal* (for player I) iff the map: $s|2i \longmapsto \alpha_i$ is order preserving: $<_s \longrightarrow \gamma$. And play $(z, (\alpha_i)_{i<\omega})$ is *won by* player I iff all its positions $(z \restriction 2n, (\alpha_i)_{i<n})$ are legal. Observe that in such a case the map $:s|2i \longmapsto \alpha_i$ is order preserving from $<_z$ to γ, hence $<_z$ is a well order and (because of the normal form of A) play z in $G(A)$ is also won by player I . Thus a w.s. τ' for player I in G_γ provides a w.s. τ in the original game $G(A)$. This shows the first point of the Remark below :

Remark 8.

(a) $\exists\gamma$ player I has a w.s. in $G(\gamma) \longrightarrow$ player I has a w.s. in $G(A)$.

(b) The converse is true, in fact with a countable value of γ.

(We shall not use this (b), so its proof is in appendix)

(c) Each game G_γ is *open* for player $I\!I$. Hence by the Gale Stewart theorem it is determined.

(d) For every $s \in 2^{<\omega}$ the property for any ordinals $\bar{\alpha} < \gamma$ that $(s, \bar{\alpha})$ is legal in G_γ only depends on the order type of $\bar{\alpha}$ (w.r.to the order of the ordinals). Indeed let p_s denote the order type of $\{s|2m \, ; \, 2m \leq lh(s)\}$ for $<_s$; then $(s, \bar{\alpha})$ is legal iff p_s is the order type of $\bar{\alpha}$.

Notation : we let $p_s(\bar{\alpha}, \gamma)$ mean that $\bar{\alpha}$ is a sequence of ordinals $< \gamma$ with order type p_s.

Martin's idea is to deduce the determinacy of $G(A)$ from that of G_γ for all $\gamma < \omega_1$; there are two cases.

Case 1 : there is γ such that it is player I which has a w.s. in $G_\gamma(A)$. By the above Remark (a), this is also a w.s. in $G(A)$. Thus $G(A)$ is determined in case 1 without need for a large cardinal.

Case 2 : not case 1. Then by open determinacy (R.8.c) it is player $I\!I$ which has a w.s. σ_γ in $G_\gamma(A)$. In particular this is true for each countable γ, that is for each $\gamma < \omega_1$. (ω_1 denotes the first uncountable ordinal and cardinal)

Now we assume the (large cardinal) „axiom of sharps": it is then easy to choose the family σ_γ, $\gamma < \omega_1$ and to provide an uncountable subset C of ω_1 so that for each $s \in 2^{<\omega}$ the statement

$$\sigma_\gamma(s, \bar{\alpha}) = 0$$

takes *constant value* for all $\bar{\alpha}, \gamma$ in C such that $p_s(\bar{\alpha}, \gamma)$.

One then says that C is *indiscernible* w.r.to the strategies σ_γ, $\gamma < \omega_1$. Or w.r.to the statement $\sigma_y(s, \bar{x}) = 0$, for each $s \in 2^{<\omega}$.

Remark. We did not state the axiom of sharps; and its above consequence is „easy" only for the reader familiar with the axiom. In the opposite case, the reader can admit this step - temporarily since section 7 will provide for the detail that is missing here.

Such a set C yields a w.s. σ for player $I\!I$ in $G_\gamma(A)$: suppose $z \restriction 2n+1$ is a position in this game; let $\bar{\alpha}, \gamma$ be any sequence from C such that $p_s(\bar{\alpha}, \gamma)$. Set $\sigma(z \restriction 2n+1) = \sigma_\gamma(z \restriction 2n+1, \bar{\alpha})$. The indiscernibility of C ensures that this definition does not depend on the choice in C of $\bar{\alpha}, \gamma$. The latter fact ensures that strategy σ is indeed winning for player $I\!I$: assume on the contrary that z is a play of G in which player $I\!I$ applied σ , yet z is won by player I . Thus $<_z$ is a wellordering; let : $z|2i \longmapsto \alpha_i$ be its isomorphism onto its order type γ. And let $(\xi_{\alpha_i})_{i<\omega}$ be a sequence from C with same order type as $(\alpha_i)_{i<\omega}$. The indiscernibility of C implies:

$$\forall n < \omega \; z(2n+1) = \sigma_{\xi_\gamma}(z \restriction 2n+1, (\xi_{\alpha_i})_{i \le n}).$$

Thus $(z, (\xi_{\alpha_i})_{i<\omega})$ is a play of G_{ξ_γ} in which player $I\!I$ applied his w.s. σ_{ξ_γ}. Yet by construction we ensured that the play remained legal for player I hence is won by him, which is a contradiction.

We proved that $G(A)$ is determined in the two cases that exist, assuming the axiom of sharps in the secund one. The proof applies to all analytic games $G(A)$, by interchanging the role of the two players if it is A that is analytic in place of its complement.

§5. ADDENDUM TO MARTIN'S PROOF

Let us redo Case 2 of the proof in the preceding section. So we assume this case; hence for every $\gamma < \omega_1$ player $I\!I$ has a w.s. σ_γ in $G_\gamma(A)$. All what Martin used about σ_γ to get $\dot{\sigma}$ was the existence of a set C of ω_1 indiscernibles w.r. to this family (σ_γ). Here we are going to provide in addition a specific construction of σ_γ : in next section this addendum yields finite state determinacy and in section 7 it yields an equivalent S of the axiom of sharps, which is easier to state.

Let G be any game which is open for player $I\!I$; if player $I\!I$ has a w.s. for G then he has a *canonical* one, based on backtrack analysis from the set of positions where player $I\!I$ *already won*. Below we define this canonical strategy but not in the above said, usual way. When $G = G_\gamma$ this will define σ_γ.

We need the notion of *ordinal height* of a binary relation ψ. We recall it with notations that suit various instances used later. We add on top of the class of ordinals an element denoted ∞. Let X be any subset of dom ψ; when ψ is indicated by the context, $||X||$ denotes the ordinal height of the relation $\psi \restriction X$:

Notation

- $||X|| := 0$ if X is empty and $||X|| := \infty$ if $\psi \restriction X$ is not well founded
- otherwise, $\infty > ||X||$ which is the smallest ordinal γ with an application $h : X \longrightarrow \gamma$ such that for $x \neq y$ in X, $x\psi y \longrightarrow h(x) < h(y)$

Remark. For $x \in X$, set $X_x = \{y \in X; y\psi x \text{ and } y \neq x\}$

(a) $||X|| = sup_{x \in X} ||X_x|| + 1$

(b) $||X_x||$ equals $h(x)$ for the map h of the above notation : it is the „height of x" (for the relation $\psi \restriction X$). In case X equals the full domain of ψ we denote it also $||x||$.

Notation

(a) We say that a position t in the game G is *legal* (for player I) if it is not already won by player $I\!I$ (in the sense: player $I\!I$ wins any full play extending t. If $G = G(\gamma)$ this is the former notion of legality)

(b) Then for $p \leq \omega$ we denote $G_{t,p}$ the game G played from position t for p moves, and won by player I iff he remained in legal position. $G_{t,\omega}$ is also denoted G_t.

(c) Let X^t denote $\cup_{p<\omega} \{\tau;\ \tau$ is a w.s. for player I in $G_{t,p}$ $\}$. We use the preceding notation $||X||$

- when X is X^t for some t
- and w.r.to the relation : $x\psi y$ iff $y \subset x$.

Remark. Thus $||X^t|| = \infty$ iff there is an infinite chain $\tau_1 \subset \tau_2 ...$ of w.s. τ_p for player I in $G_{t,p}$

For convenience we assume that in G player $I\!I$ is only allowed to play 0 or 1 - as in $G(\gamma)$. And that t is of odd length (so it is player $I\!I$ which plays the next move).

Proposition 9.

(a) $||X^t|| = \infty$ iff player I has a w.s. in G_t

(b) $||X^t|| = \min(||X^{t\frown 0}||, ||X^{t\frown 1}||) + 1$

(c) let $\sigma\ (t) = 0$ iff $||X^{t\frown 0}|| \leq ||X^{t\frown 1}||)$, $\sigma\ (t) = 1$ otherwise. σ is a w.s. for player $I\!I$ from every position t such that $||X^t|| < \infty$.

Proof P.9

(a) results from the above Remark, since the union of an infinite chain of w.s. τ_p for player I in $G_{t,p}$ becomes a w.s. in G_t.

(b) Observe that $\tau \in X^t \longrightarrow \tau = \tau_0 \cup \tau_1$ where $\tau_i \in X^{s\frown i}$; then it is easy to prove by induction on $||X^t_\tau||$ that

$||X^t_\tau|| = \min(||X^{t\frown 0}_{\tau_0}||, ||X^{t\frown 1}_{\tau_1}||) + 1$. When $\tau =$ <> this is (b).

(c) Set $o(s) := ||X^s||$; from (b) follows that

- (oa1) s of odd length $\longrightarrow$ $o(s) := \min(o(s\frown 0), o(s\frown 1)) + 1$. And s of even length $\longrightarrow$ $o(s) := \sup_a o(s\frown a) + 1$, where $s\frown a$ ranges over all legal one move extensions of s.

∘ (oa2) The strategy σ satisfies $o(t^\frown(z|n+1)) < o(t^\frown(z|n))$ for every n and every play $t^\frown z$ in which player $I\!I$ uses σ from position t.

Then there is a finite stage n such that $o(t^\frown(z|n)) = 0$, hence player $I\!I$ already won the play at this stage. Hence (c)..

P.9 proved

Remark.

Condition (oa1) defines the well-known canonical assignment of ordinals to the winning positions of a player with an open winning set. And (oa2) is then the usual definition of the canonical w.s. of this player : the one by bactrack analysis. It is equivalent to the definition via $||X^t||$ which we took; but only the latter suits the sequel.

Next we fix G to be $G_\gamma(A)$ and we let σ_γ denote σ in this case. A position in the game is then of the form $t = (s, \bar{\alpha})$ where $p_s(\bar{\alpha}, \gamma)$ holds. And we denote $A^s(\gamma)_{\bar{\alpha}}$ the corresponding set X^t :

$A^s(\gamma)_{\bar{\alpha}} = \{\tau : \exists p < \omega\ \tau$ is a w.s. for player I in $G(\gamma)$ played only p times from position $(s, \bar{\alpha})\}$. From now on $\bar{\alpha}, \gamma$ denote ordinals such that $p_s(\bar{\alpha}, \gamma)$.

Remark 10.

Thus $\sigma_\gamma(s, \bar{\alpha}) = 0$ iff

$$+)\ ||A^{s^\frown 0}(\gamma)_{\bar{\alpha}}|| \leq ||A^{s^\frown 1}(\gamma)_{\bar{\alpha}}||.$$

And for a subset C of ω_1, to be indiscernible w.r.to the σ_γ 's is to give constant value for each s to the above statement.

§6. PROOF OF FINITE STATE DETERMINACY

We continue the preceding section returning to the assumption that A is FS accepted and that the normal form $<$ used to define $(G(\gamma),\ \sigma_\gamma)$ is the 2-oblivious linear one of L.7.

Lemma 11. Whenever s is of odd length and $i \in \{0, 1\}$, then $A^{s^\frown i}(\gamma)_{\bar{\alpha}}$ only depends on $\bar{\alpha} \restriction LAR^0(s)$ and on $LAR^2(s)$. More precisely assume that $LAR^2(s) = LAR^2(s')$, that $p_{s'}(\bar{\beta}, \gamma)$ holds and $\bar{\alpha} \restriction LAR^0(s) = \bar{\beta} \restriction LAR^0(s')$. Then

(a) for $t \in 2^{<\omega}, \bar{\delta} < \gamma$, the legality of $(s^\frown t, \bar{\alpha}^\frown \bar{\delta})$ and that of $(s'^\frown t, \bar{\beta}^\frown \bar{\delta})$ is same question

(b) $A^{s^\frown i}(\gamma)_{\bar{\alpha}} = A^{s'^\frown i}(\gamma)_{\bar{\beta}}$.

Proof L.11 (a) results of the 2-obliviousness of $<_s$. (b) is easy consequence of (a).

L.11 proved

Recall that the w.s. σ for player $I\!I$ is defined by : $\sigma(s) = 0$ iff

$$+) \; ||A^{s^\frown 0}(\gamma)_{\bar{\alpha}}|| \leq ||A^{s^\frown 1}(\gamma)_{\bar{\alpha}}||$$

whenever $\bar{\alpha}, \gamma$ belong to C.

Claim 12. σ is 2-oblivious : if $LAR^2(s) = LAR^2(s')$ then $\sigma\,(s) = \sigma\,(s')$.

Indeed, the assumption implies that $<_s \restriction LAR^0(s) = <_{s'} \restriction LAR^0(s')$; then since C is infinite we can choose inside C a sequence of elements $\bar{\alpha}, \bar{\beta}, \gamma$ as in the preceding lemma : $p_s(\bar{\alpha}, \gamma)$ and $p_s(\bar{\beta}, \gamma)$ both hold. And by preceding lemma, the value of (+) is unchanged if $(s', \bar{\beta})$ replaces $(s, \bar{\alpha})$.

Thus we showed that if player I has no w.s. in $G(A)$ then player $I\!I$ has a 2-oblivious one. Now since A is FS accepted, so is the complement of A. Hence we can interchange the role of player I and player $I\!I$: if it is player $I\!I$ which has no w.s. in $G(A)$ then by the same token player I has a 2-oblivious w.s. . This implies 2-oblivious determinacy : one of the players has a w.s. of the form $\sigma : s \longmapsto w(LAR^2(s))$ for some function w. As remarked in the preview, it implies FS determinacy.

Remarks

(a) The proof is easily extended to the case of a (deterministic) Turing acceptor A ; it yields a strategy σ that is 2-oblivious in the sense : $\sigma\,(s)$ only depends on $LAR^2(s)$ *and* on the full configuration of A upon reading s (Nota-Bene : A may have erased s while reading it, so we are not always in the trivial case where s belongs to that configuration). This no longer implies effectiveness of σ except in much more special cases. But it yields the maximum extension of „oblivion" : extension to all games $G(A)$ such that A is boolean combination of rank two Borel sets. Indeed :

- oblivion (in the sense of keeping only a bounded memory of the succession of states used to accept) becomes false beyond the above sets ;
- and these are exactly the Turing accepted ones if you allow the Turing machine to have an arbitrary infinite word written on a tape and used as an oracle.

(b) We point out how the use of sharps is eliminated from our proof. Remember that we used the family of strategies σ_{γ} „only" for $\gamma < \omega_1$. But here we first blow γ up to the much larger ordinal and cardinal : $\lambda := \beth(\omega)$. The Erdos Rado partition theorem „$\lambda \longrightarrow (\omega_1)^N_\omega$ for every $N < \omega$" implies the existence of an uncountable subset C of λ such that the relation

$$||A^{s^\frown 0}(\gamma)_{\bar{a}}|| \leq ||A^{s^\frown 1}(\gamma)_{\bar{a}}||$$

takes constant value on C, for each s in a given *finite* set E. Fix E so that $LAR^2(s)$ takes all possible values on it. Then since the σ_γ 's are 2-oblivious, it follows that the above property holds not only for $s \in E$ but for all $s \in 2^\omega$. That is : C is indiscernible w.rto the σ_γ's. The rest of the proof simply uses this set $C \subset \lambda$ instead of $C \subset \omega_1$.

§7. A SIMPLE EQUIVALENT OF THE AXIOM OF SHARPS

We return to the general case of an analytic game as in section 4, and give complements to understand the nature of Martin's averaging procedure, and the nature of „sharps".

(Believe it or not, this is a useful step to understand the perspective opened by the new proof of FS determinacy !)

Definition Subset C of ω_1 is *closed* if the sup of any countable sequence of ordinals from C still belongs to C. It is *cofinal* if it is so in ω_1.

Let us say that a property on ω_1 is true *in average* iff it holds on some closed cofinal set C. It is easy to prove that every countable intersection of closed cofinal sets is again closed cofinal : in other words being true „in average" is closed under countable conjunctions - a very good property. But is the average value in this sense *defined ?* Let us discuss and illustrate this question for a particular property.

We define the average strategy $\sigma = \int \sigma_\gamma$ by $\sigma(s) :=$ *average value of* $\sigma_\gamma(s, \bar{\alpha})$ (when $\bar{\alpha}, \gamma$ ranges over ω_1 and satisfies $p_s(\bar{\alpha}, \gamma)$). We consider this in case where $lh(s) = 1$, so $lh(\bar{\alpha}) = 0$:

- $\sigma(s) = 1$ if there is a closed cofinal C on which $\sigma_\gamma(s) = 1$,
- $\sigma(s) = 0$ if there is a closed cofinal C on which $\sigma_\gamma(s) = 0$.

This is well defined iff : *) at least one of the two cases happens, and **) at most one.

Remarks

(a) The truth of (**) is an immediate consequence of closure of closed cofinal sets under intersection. But (*) is not obvious :

(b) using the axiom of choice one can obtain a set $X \subset \omega_1$ such that neither the property $x \in X$, nor its negation are true in average

(c) due to the canonical construction in section 5 of the σ_γ's, the statement „$\sigma_\gamma(s) = 0$" is not ad hoc and pathological as the set X of (b). Yet even in this favorable case, the existence of the average is non obvious.

(d) In fact, to guarantee that all the statements of the form „$\sigma_\gamma(s, \bar{\alpha}) = 0$" have an average value, the theory ZFC is not enough : we need the additional axiom S defined below.

Notation 13.

- For any set $X \subset \omega^{<\omega}$ and any ordinal γ, the *stretching of X along γ* is the set denoted $X(\gamma)$ of all sequences $\bar{\alpha} \in \gamma^{<\omega}$ which in the order of γ have same order type as some $\bar{m} \in X$. And $||X(\gamma)||$ denotes the height of this set relative to the relation $x\psi y := y \subset x$.

  (Thus $||X(\gamma)|| < \infty$ means that $X(\gamma)$ is well founded in the sense of having no „infinite branch" $s_1 \subset s_2 \subset \ldots$ And $||X(\gamma)_{\bar{\alpha}}||$ denotes the height of $\bar{\alpha}$ for the relation ψ - see N.4)

∘ We let S be the statement :

for any two countable subsets X^0, X^1 of $\omega^{<\omega}$ and for any $\bar{m} \in X^0 \cap X^1$ there is a closed cofinal subset C of ω_1 such that whenever $\bar{\alpha} < \gamma$ are ordinals of C with $\bar{\alpha}$ of same order type as $\bar{m}$ – hence $\bar{\alpha} \in X^0(\gamma) \cap X^1(\gamma)$ – the statement

$$||X^0(\gamma)_{\bar{\alpha}}|| \leq ||X^1(\gamma)_{\bar{\alpha}}||$$

has constant value.

Remark 14.

If $||X(\gamma)|| < \infty$ for each γ then we say that X has *well founded stretchings*. Any instance of S where at least one of X^0, X^1 has an ill founded stretching is superfluous as an axiom added to ZFC : for it is easy to prove this instance from ZFC and with a set C simply of the form $\{\gamma; \delta < \gamma < \omega_1\}$ for some $\delta < \omega_1$.

The map : $\gamma \longmapsto ||X(\gamma)||$ from ordinals to ordinals (plus ∞) is called the stretching map of X. When X varies over countable subsets of $\omega^{<\omega}$ and especially the relevant ones which have well founded stretchings, it is a natural way to define maps over ω_1 that are as effective as possible (from X taken as a parameter or „oracle"; and in spite of the non effective nature of ω_1). Thus case $\bar{m} =<>$ of S says that any two such maps are comparable modulo some closed cofinal set ; and the general case of S is a similar property.

We are going to see that this partition property S of ω_1 implies analytic determinacy.

Fact 15

(a) For every $t \in 2^{<\omega}$ there is $X \subset \omega^{<\omega}$ such that $A^t(\gamma)_{\bar{\alpha}} = X(\gamma)_{\bar{\alpha}}$ for all $\bar{\alpha} < \gamma$.

(b) In fact, X is recursive in t and in the map : $s \longmapsto <_s$ (of the normal form of A).

Proof F.15 Let θ be a recursive enumeration of $2^{<\omega}$ such that all sequences of length $< p$ are enumerated before the ones with length p. We represent any strategy $\tau \in A^t(\gamma)_{\bar{\alpha}}$ by the sequence $\bar{\alpha}^\frown(\mu_i, \nu_i, \tau(\theta(i))_{i<2^p})$ where p is the number of moves of player $I\!I$ which τ must answer, where $\tau(\theta(i))$ denotes the answer of τ after player $I\!I$ played the sequence $\theta(i)$ and where $\mu_i < \nu_i < \gamma$ if the bit answered by τ after player $I\!I$ played $\theta(i)$ is 0 – while $\gamma > \mu_i > \nu_i$ in the opposite case. Let X be the image of $A^t(\omega)$ under this representation. R.8.d readily implies that (a+b) are satisfied.

F.15 is proved.

Theorem 16. S implies analytic determinacy.

Proof T.16

Remember from last section that the only use of sharps was to prove in „Case 2" the existence of a cofinal set $C \subset \omega_1$ that is indiscernible w.r.to the winning strategies $\sigma\ \gamma$ of player $I\!I$.

Where $\sigma\ \gamma((s, \bar{a})) = 0$ iff

$$E)\ ||A^{s\frown 0}(\gamma)_{\bar{a}}|| \leq ||A^{s\frown 1}(\gamma)_{\bar{a}}||.$$

So here we only needs to construct C by use of S. With R.14 and F.15, S implies for each $s \in 2^{\omega}$ the existence of a closed cofinal set C_s over which the above statement (E) has constant value. By intersecting the countably many sets C_s we obtain a closed cofinal set C over which all these statements have constant value. Thus set C is indiscernibles w.r.to the $\sigma\ \gamma$'s.

T.16 proved

We now recall the original definition of „sharps", but the next result will dispense the reader unfamiliar with it to worry : he can safely consider that this axiom coincides with S. Only if he wants to understand the motivation (the „intuitive reason of truth") for this axiom S, and why it has a bold effect on the whole universe of sets although it looks as a plausible statement concerning only ω_1 and countable objects X^0, X^1 - only then must the reader learn about the constructible universe of sets L and about the original form of the axiom of sharps.

Definition 17

- For every $\rho \in \omega^{\omega}$, $L(\rho)$ denotes the smallest transitive submodel of the „universe of all sets" V that contains all ordinals, contains ρ and satisfies ZFC.
- The axiom of sharps says that for every $\rho \in \omega^{\omega}$ there is an uncountable subset of ω_1 which is indiscernible w.r.to every first order formula with parameter ρ, interpreted in this universe $L(\rho)$ of „all sets constructible from ρ"
- „$\rho^{\#}$ exists" is the above axiom for a fixed ρ.

We recall that a subset B of 2^ω is $\Sigma_1^1(\rho)$ iff it is the projection of some closed subset of $2^\omega x \omega^\omega$ that himself is recursive in ρ. This is also equivalent to say that the complement of B has a normal form with a map $: s \longmapsto <_s$ that is recursive in ρ. And a set is analytic iff it is $\Sigma_1^1(\rho)$ for some ρ.

Corollary 18.

- S is equivalent to the axiom of sharps
- In fact let $S(\rho)$ denote S restricted to the sets X, X' which are recursive(ρ); then $S(\rho)$ is equivalent to the axiom: „$\rho^\#$ exists", as well as to $\Sigma_1^1(\rho)$ determinacy.

Proof C. 18 That the axiom of sharps implies S is a routine argument .... for people familiar with this axiom. Now S implies analytic determinacy which (by the hard theorem of Harrington which gives a converse to Martin's analytic determinacy result) implies the axiom of sharps. This is easily refined to $S(\rho)$ and $\rho^\#$: use F.15.b in addition to 15.a.

C.18 proved

§8. WHAT'S NEXT ?

1. Final form of our proof of FS determinacy. In case A is co-analytic last section provided a w.s. σ for player *II* defined by : $\sigma\ (s) = 0$ iff

+) there is a closed cofinal subset of ω_1 on which $||A^{s\frown 0}(\gamma)_{\bar{a}}|| \leq ||A^{s\frown 1}(\gamma)_{\bar{a}}||$.

When A is in addition FS accepted and in (+) we use its oblivious normal form, then the proof in section 5 applies unchanged to show that this strategy σ is 2-oblivious hence FS.

In this form our proof remains highly non effective, but all its steps have become rather canonical - including finally the averaging step.

2. The next challenge The Gurevich Harrington theorem gives an upper bound on the number of states needed by an FS processor which computes a w.s. σ in an FS accepted game $G(A)$: namely $|LAR|$, the total number of values of $LAR(s)$. It has been proved that this bound is an optimal one. Thus σ may need 100! states if acceptor A has 100 states. It shows that in general the number of states of σ is too large to be permanently stored in a real world processor.

But if acceptor A has 100 states, each one can be coded by 8 bits, hence 800 bits suffice to store any value of $LAR(s)$: this does not exceed the real world computers capacity ! Hence while reading z they can store the current value $LAR(z|n)$ on a tape and (quasi instantaneously) compute $LAR(z|n+1)$ from $LAR(z|n)$. Then since the Gurevich Harrington proof provides a w.s. σ of the form $w(LAR(z|n))$ for some finite function w, a real world processor P could perhaps exist, that registers the current value of $LAR(z|n)$ and computes from it $w(LAR(z|n))$ which for even n is the desired reaction for P. Such a processor which permanently (re-)computes an automaton σ without storing it, we call a *virtual* automaton. This leads to the problem of „P-time realizability of FS strategies"

:

is there a polynomial p(x) such that for every FS acceptor A with N states we can find a w.s. σ in $G(A)$ and a function L such that

- σ (s) is computed from $L(s)$ by a Turing machine bounded in time and in size by p(N); and
- $L(s^\frown 0), L(s^\frown 1)$ are computed from $L(s)$ in the same way.

Nota-Bene : here the machine A is allowed to be non deterministic ; but the accepting family is of the form $F \subset Q$ instead of $\mathcal{F} \subset P(Q)$ and the accepting condition is changed to : $Inf z \cap F$ is non empty. The change does not alter the class of FS accepted sets but allows (the binary code of) the acceptor A to be of size N^2.

This problem is the sharp and purely mathematical form of a frequently encountered problem in Computer Science : in case an automaton depends on too many states to be stored entirely, can it nevertheless exist as a virtual automaton; and how to compute the latter ?

Remark 19. The Gurevich Harrington proof and its improvement by W Thomas have an obvious superiority over our proof: namely „oblivious" is better than „2-oblivious". But in the above perspective of computing virtual automata, this superiority becomes a deadly drawback : to optimize the number of states needed by an FS w.s. is very impressive but *of no use* if this number is „unfeasible". And it is practically certain that in order to reach this useless optimum, the proof has to sacrifice other goals and become unfit for the study of virtual automata. Whereas our construction of σ depending on $LAR^2(s)$ provided by the above (+) offers a track that is not harmed in advance, since it allows to use more than the minimum $LAR(s)$. In fact,

- Girard (unpublished) elaborated a proof of analytic determinacy where Martin's averaging procedure is replaced by a subtle variant called ,,equalization".
- Berardi [B] has studied equalization in a case where it becomes effective.
- We conjecture that in the FS case an oblivious version of Girard's proof can be made, much as we did for Martin's proof. And that it leads to a variant of the definition (+) of σ that remains canonical but that in addition becomes effective and sufficiently efficient to have a bearing on the computation of virtual automata ...

In conclusion, the present work comes with a series of ideas, methods and results, as a base of future research on the P-time version of FS determinacy and on other challenges of effective determinacy.

APPENDIX

We promised a proof of R.8.b: if player I has a w.s. τ for $G(A)$ then there is a countable ordinal γ such that player I has a w.s. for $G_\gamma(A)$.

Denote T the set of all legal positions in $G_\omega(A)$ which can be reached when player I is applying τ. On T define a tree relation (:= partial order with initial sections linear and finite): $(s,(\alpha_i)_{i<n})R(s',(\alpha'_i)_{i<n'})$ iff the second position extends the first one and $\alpha_{n-1} > \alpha'_{n'-1}$. An infinite branch of this tree T would give a play $z \in 2^\omega$ won by player I in $G(A)$ since player I applies τ, *and* would give an infinite descending sequence $(\alpha_{n_p})_{p<\omega}$ for the order $<_z$. This is a contradiction since $z \in A \longleftrightarrow <_z$ is well ordered; so T has no infinite branch. In other words $||T|| < \infty$ where $||T||$ is the ordinal height of T with respect to the relation $\psi := R$. Let γ be this ordinal; here is a w.s. τ^* for player I in $G_\gamma(A)$: use τ to choose $z(2n)$; and let α_n be the height in T of the position reached before move $2n$. That is : $\alpha_n = ||(z \restriction 2n, (\alpha_i)_{i<n})||$.

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