ON THE REPRESENTATION OF RECURSIVELY ENUMERABLE SETS IN WEAK ARITHMETICS[1]

S. N. Manukian

Algebras $\Theta_1, \Theta_2, \Theta_3$ on two-dimensional recursively enumerable sets of natural numbers are introduced; arithmetical systems A_1 and A_2 complete in the signatures, correspondingly, $\{0,', =, <\}$ and $\{0,', =\}$ are considered. It is proved that a two-dimensional arithmetical set is inductively representable in Θ_1 (correspondingly, Θ_2) if and only if it is expressible by a formula in A_1 (correspondingly, A_2). It is proved that a two-dimensional arithmetical set is inductively representable in Θ_3 if and only if it is expressible by a formula of a special kind in A_2. Algebras $\Omega_1, \Omega_2, \Omega_3$ of two-dimensional recursively enumerable fuzzy sets are introduced; theorems are proved establishing relations between these algebras and arithmetical systems A_1 and A_2.

§1. INTRODUCTION. In this paper some relations between weak arithmetics and algebras on recursively enumerable sets are considered. The term „weak arithmetic" is interpreted here in its direct sense as „subsystem of Peano's arithmetic" ([4], [5], [16], [17]). The contents of this paper may be considered as a continuation of the investigations described in [11] and [12]. Let us recall that in [11] and [12] the algebras Θ and Θ° on two-dimensional recursively enumerable sets of natural numbers are introduced; it is proved in [12] that all two-dimensional recursively enumerable sets of natural numbers are inductively representable in Θ; it is proved in [12] also that a two-dimensional recursively enumerable set of natural numbers is inductively representable in Θ° if and only if it is expressible by a formula in M.Presburger's arithmetical system ([1], [4], [5], [15]) (which is complete in the signature $\{0,', +, =\}$). Below some subalgebras $\Theta_1, \Theta_2, \Theta_3$ of Θ° are introduced and the arithmetical systems A_1 and A_2 complete in the signatures, correspondingly, $\{0,', =, <\}$ and $\{0,', =\}$ are considered; it is proved (theorem 2.1) that the relations between Θ_1 (correspondingly, Θ_2) and A_1 (correspondingly, A_2) are the

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same as relations between the algebra Θ° and M.Presburger's system; namely, every two-dimensional recursively enumerable set of natural numbers is inductively representable in Θ_1 (correspondingly, Θ_2) if and only if it is expressible by a formula in A_1 (correspondingly, A_2). It is proved (theorem 2.2) that a two-dimensional arithmetical set of natural numbers is inductively representable in Θ_3 if and only if it is expressible by a formula of a special kind (i.e. so-called „positive formula") in A_2. It is proved also that the sets of elements inductively representable in the algebras $\Theta, \Theta^{\circ}, \Theta_1, \Theta_2, \Theta_3$ are different (theorem 2.3).

Below some results concerning recursively enumerable fuzzy sets are given which may be considered also as a continuation of investigations described in [11] and [12]. Let us recall that in [11] and [12] the algebras Ω and Ω° on two-dimensional recursively enumerable fuzzy sets are introduced; it is proved in [12] that all two-dimensional recursively enumerable fuzzy sets are inductively representable in Ω (up to the equivalence); a statement is proved (theorem 5.1 in [12]) establishing that two-dimensional recursively enumerable fuzzy sets inductively representable in Ω° may be considered as fuzzy analogues of sets expressible by formulas in M.Presburger's system. Below some subalgebras $\Omega_1, \Omega_2, \Omega_3$ of Ω° are introduced; it is proved (theorems 4.1-4.2) that the relations between the algebras Ω_1, Ω_2 and arithmetical systems A_1, A_2 are similar to the relations between the algebra Ω° and M.Presburger's system. Some properties of the system Ω_3 and its relations with the system A_2 are established in the theorems 4.3-4.5. It is proved that the sets of elements inductively representable in the algebras $\Omega, \Omega^{\circ}, \Omega_1, \Omega_2, \Omega_3$ are different (theorem 4.6).

The theorems proved below show algebraic properties of weak arithmetical systems as well as their relations with the corresponding systems of fuzzy logic. These theorems give possibilities for investigations of algebraic transformations of the sets representable in the mentioned systems.

The formulations of the theorems 2.1 and 4.1 were published in [13].

§2. Let us give (cf. [12]) some definitions used below. The *n-dimensional arithmetical set* is defined as a set of n-tuples $(x_1, x_2, ..., x_n)$, where all x_i are natural numbers $0, 1, 2, \ldots$. The notion of *algebra* is interpreted as „universal algebra ([2], [6]) with a fixed set of basic elements"; so, every algebra is defined by a *main set* M, by a set of *operations* $f_1, f_2, ...$ on M and by a set of *basic elements* $a_1, a_2, ...$ in M. In every algebra considered below the set of operations and the set of basic elements are finite. We say that an element $a \in M$ is *inductively representable* in a given algebra $(M, f_1, f_2, ..., a_1, a_2...)$ if it can be obtained by the operations $f_1, f_2...$ from the basic elements $a_1, a_2....$ We say that an n-dimensional function f on a set M is *expressible* by some functions $f'_1, ..., f'_k$ on a set M' such that $M \subseteq M'$, if there exists a term ([1], [4], [5]) containing no other functional symbols, except $f'_1, ..., f'_k$, no other variables, except $x_1, x_2, ..., x_n$, and such that it expresses the value $f(x_1, x_2, ..., x_n)$ for every $x_1, x_2, ..., x_n \in M$. We say that an algebra $W = (M, f_1, f_2, ..., a_1, a_2, ...)$ is a *subalgebra* of an algebra $W' = (M', f'_1, f'_2, ..., a'_1, a'_2, ...)$ if the following conditions hold: $M \subseteq M'$; every function f_i is expressible by a finite subset of the set $\{f'_1, f'_2, ...\}$; every element $a \in M$ inductively representable in W is inductively representable in W'. We say that W is a proper subalgebra of W' if W is a *subalgebra* of W' and there exists an element $a \in M$ which is inductively representable in W' but not in W (cf.[6]).

In all the algebras considered below in the sections 2 and 3 the main set M is the set of all two-dimensional recursively enumerable sets of natural numbers 0,1,2, (TRES) ([1], [5]). In all the algebras considered below in the sections 4 and 5 the main set M is the set of all two-dimensional recursively enumerable fuzzy sets (TREFS) ([7]-[13]).

We shall use the following operations on the set of all TRESes (cf. [11], [12]).

The *set-theoretical sum* $A \cup B$ and *set-theoretical intersection* $A \cap B$ of TRESes A and B are defined in the usual way.

The *composition* $A \circ B$ of TRESes A and B is defined by the following generating rule (g.r.): if $(x, y) \in A$ and $(y, z) \in B$ then $(x, z) \in A \circ B$.

The *transitive closure* $* A$ of a TRES A is defined by the following g.r.: if $(x, y) \in A$ then $(x, y) \in * A$; if $(x, y) \in * A$ and $(y, z) \in * A$ then $(x, z) \in * A$.

The *arithmetical sum* $A \diamond B$ of TRESes A and B is defined by the following g.r.: if $(x, y) \in A$ and $(y, z) \in B$ then $(x, y + z) \in A \diamond B$.

The *inversion* A^{-1} of a TRES A is defined by the following g.r.: if $(x, y) \in A$ then $(y, x) \in A^{-1}$.

We shall consider as basic elements the following TRESes (cf. [11]-[13]): $\bar{R} = \{(x, y) | \ y = x + 1\}$; $\bar{Q} = \{(x, y) | \ x < y\}$; $\bar{Z}_0 = \{(x, y) | \ x = 0\}$; $\bar{S} = \{(x, y) | \ x \neq y\}$.

Now let us define some algebras (cf. [11], [12]) on the set of all TRESes. The algebra Θ is defined by the list of operations $(\cup, \cap, \circ, *, \circ, ^{-1})$ and by the list of basic elements containing only one element $\bar{R}$. The algebra Θ° is defined by the list of operations $(\cup, \cap, \circ, \circ, ^{-1})$ and by the list $(\bar{R}, \bar{Q})$ of basic elements. The algebras $\Theta_1, \Theta_2, \Theta_3$ are defined by the same list of operations $(\cup, \cap, \circ, ^{-1})$ and by the following lists of basic elements: $(\bar{R}, \bar{Q}, \bar{Z}_0)$ for Θ_1, $(\bar{R}, \dot{\bar{S}}, \bar{Z}_0)$ for Θ_2, $(\bar{R}, \bar{Z}_0)$ for Θ_3.

The algebras Θ and Θ° are introduced in [11] and [12]. It is proved in [12] that every TRES is inductively representable in Θ; a TRES is inductively representable in Θ° if and only if it is expressible by a formula in M.Presburger's system.

Now let us consider some arithmetical systems. We use the language of first order predicate calculus ([1], [4], [5]) containing the logical symbols &(conjunction), $\vee$(disjunction), $\supset$(implication), $\neg$(negation), $\sim$(equivalence), $\forall$(generality), $\exists$(existence). By x' we denote the successor function $x' = x + 1$. By $x^{(k)}$ we denote the term $x''^{\cdots\prime}$, where the symbol $'$ is repeated k times. By $Subst(F; x_1, x_2, ..., x_k; t_1, t_2, ..., t_k)$, where F is a formula, $x_1, x_2, ..., x_k$ are variables, $t_1, t_2, ..., t_k$ are terms, we denote the formula obtained by the admissible substitution of the terms $t_1, t_2, ..., t_k$ for the variables $x_1, x_2, ..., x_k$ in the formula F. The system A_1 in the signature $\{0, ', =, <\}$ is defined by the following axioms:

$\forall x(x = x)$;

$\forall x \forall y \forall z((x = y) \supset (x = z \supset y = z))$;

$\forall x \forall y(x = y \supset x' = y')$;

$\forall x \forall y \forall z((x = y) \supset (y < z \supset x < z))$;

$\forall x \forall y \forall z((x < y) \supset (y = z \supset x < z))$;

$\forall(\neg(x = 0) \supset \exists y(x = y')$;

$\forall x(x < x')$; $\forall x \neg(x < 0)$;

$\forall x \forall y(x < y \vee x = y \vee y < x)$;

$\forall x \forall y((x < y) \supset \neg(y < x'))$;

$\forall x \forall y \forall z((x < y) \supset (y < z \supset x < z))$.

It is easily seen that the system A_1 is equivalent to the system A_L considered in [1] and to the systems (A) and (B) considered in [4]. It is proved in [1] and [4] that the corresponding systems are complete in the mentioned signature; hence it is true also for A_1.

The system A_2 in the signature $\{0, ', =\}$ is defined by the following axioms:

$\forall x(x = x)$;

$\forall x \forall y \forall z((x = y) \supset (x = z \supset y = z))$;

$\forall x \forall y(x = y \supset x' = y')$;

$\forall x \neg(x' = 0)$;

$\forall x \forall y(x' = y' \supset x = y)$;

and by the scheme of induction

$\forall u \forall v ... \forall w((Subst(F; x; 0) \& \forall x(F \supset Subst(F; x; x'))) \supset \forall x(F))$;

this gives an axiom for every formula F in the mentioned signature and for every list of variables $x, u, v, ..., w$. It is easily seen that the system A_2 is equivalent to the system A_S considered in [1]; it is proved in [1] that it is complete in the mentioned signature; hence it is true also for A_2.

We say that an k -dimensional arithmetical set A is *expressed* (or is *expressible*) in the system L by a formula F containing no other free variables, except $x_1, x_2, ..., x_k$, if for every given constant terms $t_1, t_2, ..., t_k$ the following condition holds: the formula $Subst(F; x_1, x_2, ..., x_k; t_1, t_2, ..., t_k)$ is deducible in the system L if and only if the k-tuple $(n_1, n_2, ..., n_k)$ of the values of the terms $t_1, t_2, ..., t_k$ belongs to A.

A formula F in the signature $\{0, ', =\}$ is said to be *positive* if it contains no other logical symbols, except $\exists, \&, \vee, \neg$, and all the negation symbols contained in it relate only to elementary subformulas $(t = s)$, where the terms t and s together contain no more than one variable.

Theorem 2.1. A two-dimensional arithmetical set is inductively representable in Θ_1 (correspondingly, Θ_2) if and only if it is expressible by a formula in A_1 (correspondingly, A_2).

Theorem 2.2. A two-dimensional arithmetical set is inductively representable in Θ_3 if and only if it is expressible by a positive formula in A_2.

Theorem 2.3. Every next algebra in the sequence $\Theta, \Theta^\circ, \Theta_1, \Theta_2, \Theta_3$ is a proper subalgebra of the preceding one.

§3. In this section we shall give the proofs of the theorems 2.1. - 2.3.

We say that the set Φ of formulas in the logical system L *admits the elimination of quantifiers* (cf. [1], [4]) if for every formula $F \in \Phi$ there exists a formula $F_1 \in \Phi$ which is quantifier-free and equivalent to F in L.

We say that the logical system L *admits the elimination of quantifiers* if the set of all formulas in L admits the elimination of quantifiers.

Lemma 3.1. The systems A_1 and A_2 admit the elimination of quantifiers.

The proof is actually given in [1].

Lemma 3.2. The set of positive formulas in the system A_2 admits the elimination of quantifiers.

The proof is actually obtained from the considerations in [1]. Indeed, it is easily seen that the algorithm of the elimination of quantifiers in A_S (which is equivalent to A_2) described in [1] gives for every positive formula in A_S its quantifier-free positive equivalent.

We shall consider the following TRESes (cf. [12]): the set $\bar{V}$ of all pairs (x,y) of natural numbers; the set $\bar{E} = \{(x,y)|x = y\}$; the sets $\bar{W}_{k,l} = \{(x,y)|(x = k)\&(y = l)\}$ for all natural numbers k,l; the sets $\bar{Z}_k = \{(x,y)|x = k\}$ for all natural numbers k; the empty set $\bar{O}$.

Lemma 3.3. All the sets $\bar{V}, \bar{E}, \bar{W}_{kl}, \bar{Z}_k, \bar{O}$ are inductively representable in Θ_3.

The proof is given by the following equalities: $V = \bar{Z}_0^{-1} \circ \bar{Z}_0; \bar{E} = \bar{R} \circ (\bar{R}^{-1}); \bar{W}_{00} = \bar{Z}_0 \circ (\bar{Z}_0^{-1}); \bar{W}_{k,l+1} = \bar{W}_{k,l} \circ R; \bar{W}_{k+1,l} = \bar{R}^{-1} \circ \bar{W}_{k,l}; \bar{Z}_{k+l} = \bar{R}^{-1} \circ \bar{Z}_k; \bar{O} = \bar{R}^{-1} \cap \bar{R}$.

Note. It follows from the given equalities that $\bar{E}$ and $\bar{O}$ are inductively representable also in $\Theta, \Theta^\circ, \Theta_1, \Theta_2$.

Lemma 3.4. Every next algebra in the sequence $\Theta, \Theta^\circ, \Theta_1, \Theta_2, \Theta_3$ is a subalgebra of the preceding one.

Proof: It is sufficient to consider conditions concerning basic elements. Θ° is a subalgebra of Θ (cf. [12]) because $\bar{Q} = *\bar{R}$; Θ_1 is a subalgebra of Θ° because $\bar{Z}_0 = ((\bar{E} \diamond \bar{E}) \cap (\bar{E} \diamond \bar{E})^{-1}) \circ (\bar{Q} \cup \bar{E})$; Θ_2 is a subalgebra of Θ_1 because $\bar{S} = \bar{Q} \cup (\bar{Q}^{-1})$; Θ_3 is obviously a subalgebra of Θ_2.

Corollary. All the sets mentioned in the lemma 3.3 are inductively representable also in $\Theta, \Theta^\circ, \Theta_1, \Theta_2$.

Lemma 3.5. Every two-dimensional arithmetical set inductively representable in Θ_1 is expressible by a formula in A_1.

Proof: (cf. [12], lemma 3.3) The basic sets $\bar{R}, \bar{Q}, \bar{Z}_0$ in Θ_1 are expressible in A_1 by the formulas, correspondingly, $y = x', x < y, x = 0\&y = y$. If sets A and B are expressible by the formulas, correspondingly, $F(x,y)$ and $G(x,y)$ with free variables x and y, then the sets $A \cup B, A \cap B, A \circ B, A^{-1}$ are expressible by the formulas, correspondingly,

$F(x,y) \vee G(x,y), F(x,y) \& G(x,y), \exists z(F(x,z) \& G(z,y)), F(y,x)$ (where z is not contained in $F(x,y)$ and $G(x,y)$). This completes the proof.

Lemma 3.6. Every two-dimensional arithmetical set expressible by a formula in A_1 is inductively representable in Θ_1.

Proof. Let an arithmetical set be expressible by a formula $F(x,y)$ in A_1 with free variables x and y. Without loss of generality we may suppose (see lemma 3.1) that $F(x,y)$ is quantifier-free and contains only the logical symbols $\&, \vee, \neg$. We can reduce $F(x,y)$ to a disjunctive normal form and eliminate in it all the negations (indeed, the formula $\neg(t < s)$, where t and s are any terms, may be replaced by $(s < t) \vee (s = t)$; the formula $\neg(s = t)$ may be replaced by $(s < t) \vee (t < s)$). So we may suppose that $F(x,y)$ is obtained by $\&$ and $\vee$ from elementary formulas. Clearly, every elementary formula in $F(x,y)$ may be reduced to one of the forms $(x = y^{(k)}), (x < y^{(k)}), (x^{(k)} < y), (0 = y^{(k)}), (x = 0^{(k)}), (0 < y^{(k)}), (x < 0^{(k)}), (0^{(k)} < y), (x^{(k)} < 0), (0 = 0^{(k)}), (0 < 0^{(k)}), (0^{(k)} < 0)$ (or to a form obtained from the mentioned ones by replacing y by x and x by y). The sets expressed by these formulas are obtained in Θ_1 as follows (where it is supposed that the members $\bar{R}$ in the expressions $(\bar{R} \circ \bar{R} \circ \dots \circ R)$ are repeated k times). The set $\{(x,y)|x = y^{(k)}\}$ is $\bar{E}$ when $k = 0$ and $(\bar{R} \circ R \circ \dots \circ \bar{R})^{-1}$ when $k > 0$. The set $\{(x,y)|x < y^{(k)}\}$ is $\bar{Q}$ when $k = 0$ and $\bar{Q} \circ (\bar{R} \circ \bar{R} \circ \dots \circ \bar{R})^{-1}$ when $k > 0$. The set $\{(x,y)|x^{(k)} < y)\}$ is $\bar{Q}$ when $k = 0$ and $(\bar{R} \circ \bar{R} \circ \dots \circ \bar{R}) \circ \bar{Q}$ when $k > 0$. The set $\{(x,y)|0 = y^{(k)}\}$ is $\bar{Z}_0^{-1}$ when $k = 0$ and $\bar{O}$ when $k > 0$. The set $\{(x,y)|x = 0^{(k)}\}$ is $\bar{Z}_k$. The set $\{(x,y)|0 < y^{(k)}\}$ is $\bar{V} \circ \bar{R}$ when $k = 0$ and $\bar{V}$ when $k > 0$. The set $\{(x,y)|x < 0^{(k)}\}$ is $\bar{O}$ when $k = 0$ and $\bar{Z}_0 \cup Z_2 \cup \dots \cup \bar{Z}_{k-1}$ when $k > 0$. The set $\{(x,y)|0^{(k)} < y\}$ is $\bar{V} \circ \bar{R}$ when $k = 0$ and $(\bar{V} \circ \bar{R}) \circ (\bar{R} \circ \bar{R} \circ \dots \circ \bar{R})$ when $k > 0$. The set $\{(x,y)|x^{(k)} < 0\}$ is $\bar{O}$. The set expressible by a formula containing no variables is either $\bar{V}$ or $\bar{O}$. The transformation of a formula, when x is replaced by y, and y by x, corresponds to the operation $^{-1}$ applied to the set expressed by this formula. The logical operations $\&$ and $\vee$ on the formulas correspond to the operations $\cap$ and $\cup$ on the sets expressed by these formulas. This completes the proof.

Lemma 3.7. Every two-dimensional arithmetical set inductively representable in Θ_2 is expressible by a formula in A_2.

The proof is the same as that of lemma 3.5 with the following change: instead of $\bar{Q}$ in Θ_1 the basic set $\bar{S}$ in Θ_2 is considered; it is expressed by the formula $\neg(x = y)$ in A_2.

Lemma 3.8. Every two-dimensional arithmetical set expressible by a formula in A_2 is inductively representable in Θ_2.

Proof: Let an arithmetical set be expressed by a formula $F(x, y)$ in A_2 containing only free variables x and y. As in the proof of lemma 3.6 we may suppose that $F(x, y)$ is quantifier-free, contains only the logical operations $\&, \vee, \neg$, and is reduced to a disjunctive normal form. So $F(x, y)$ can be obtained by $\&$ and $\vee$ from elementary formulas and their negations. Every formula of the mentioned kind can be reduced to one of the forms: $(x = y^{(k)}), \neg(x = y^{(k)}), (0 = y^{(k)}), \neg(0 = y^{(k)}), (x = 0^{(k)}), \neg(x = 0^{(k)}), (0 = 0^{(k)}), \neg(0 = 0^{(k)}))$ (or to forms obtained from the mentioned ones by replacing y by x and x by y). The sets $\{(x, y)|x = y^{(k)}\}, \{(x, y)|0 = y^{(k)}\}, \{(x, y)|x = 0^{(k)}\}$ are obtained in Θ_2 in the same way as it given for Θ_1 in the proof of lemma 3.6. The set $\{(x, y)|\neg(x = y^{(k)})\}$ is $\tilde{S}$ when $k = 0$ and $\tilde{S} \circ ((\tilde{R} \circ \tilde{R} \circ ...\tilde{R})^{-1})$ when $k > 0$ (here and below it is supposed that R is repeated k times in every such expression). The set $\{(x, y)|\neg(0 = y^{(k)})\}$ is $\tilde{V} \circ \tilde{R}$ when $k = 0$ and $\tilde{V}$ when $k > 0$. The set $\{(x, y)|\neg(x = 0^{(k)})\}$ is $\tilde{R}^{-1} \circ \tilde{V}$ when $k = 0$ and $\tilde{Z}_0 \cup \tilde{Z}_2 \cup ... \cup \tilde{Z}_{k-1} \circ ((\tilde{R} \circ \tilde{R} \circ ... \circ \tilde{R})^{-1} \circ (\tilde{R}^{-1} \circ \tilde{V}))$ when $k > 0$.The remaining part of the proof is the same as in the proof of lemma 3.6.

Lemma 3.9. Every two-dimensional arithmetical set inductively representable in Θ_3 is expressible by a positive formula in A_2.

The proof is the same as that of lemma 3.5 and lemma 3.7; it is easily seen that the proofs of lemmas 3.5 and 3.7 give a positive formula for every set belonging to Θ_3.

Lemma 3.10. Every two-dimensional arithmetical set expressible by a positive formula in A_2 is inductively representable in Θ_3.

The proof is actually given in the proofs of lemmas 3.6 and 3.8; it is easily seen that the proofs of lemmas 3.6 and 3.8 give a set inductively representable in Θ_3 for every set expressible by a positive formula in A_2.

The proofs of the theorems 2.1 and 2.2 are now obtained immediately using the lemmas 3.5 - 3.10.

Now let us give the proof of the theorem 2.3. We shall consider some auxiliary notions and establish some properties of them.

An one-dimensional (correspondingly, two-dimensional) set of natural numbers is said to be cofinite when its complement to the set of all natural numbers (correspondingly, to the set of all pairs of natural numbers) is finite.

Let A be any two-dimensional arithmetical set. By $\sigma_n(A)$ we denote the quantity of points (x, y) such that $x \leq n$, $y \leq n$, $(x, y) \in A$ (obviously, $\sigma_n(A) \leq (n+1)^2$ for every two-dimensional arithmetical set A). We say that A is 0-oriented, if

$$\lim_{n \to +\infty} \frac{\sigma_n(A)}{(n+1)^2} = 0,$$

and 1-oriented, if

$$\lim_{n \to +\infty} \frac{\sigma_n(A)}{(n+1)^2} = 1.$$

Obviously, if a two-dimensional arithmetical set is 0-oriented (correspondingly, 1-oriented), then its complement is 1-oriented (correspondingly, 0-oriented).

We say that A is *strongly 1-oriented*, if there exist such natural numbers k, l that $(x, y) \in A$ for every point (x, y) satisfying the conditions $x \geq k, y \geq l$. Clearly, every strongly 1-oriented set is 1-oriented. We say that A is *extremal* (correspondingly, *strongly extremal*), if A is either 1-oriented or 0-oriented (correspondingly, either strongly 1-oriented or 0-oriented).

The statement established in the following lemma is actually formulated in [1] (without proof).

Lemma 3.11. Every one-dimensional set expressible by a formula in A_1 or A_2 is either finite or cofinite.

Indeed, if an one-dimensional set is expressible by a formula in A_1 or A_2 by a formula $F(x)$ containing at most one free variable x, then we may suppose without loss of generality that $F(x)$ is quantifier-free and is obtained by & and $\vee$ from the elementary formulas and their negations. It is easily seen that every elementary formula or its negation in A_1 or A_2 containing no more than one variable expresses either finite or cofinite set. Hence it is so also for $F(x)$, because the following statement holds: if a set A is either finite or cofinite, and a set B possesses the same property, then $A \cup B$ and $A \cap B$ are also either finite or cofinite.

Lemma 3.12. Every two-dimensional arithmetical set expressible by a formula in A_2 is extremal.

Proof: Let a two-dimensional set is expressible by a formula $F(x, y)$ in A_2 containing only free variables x and y. Similarly to the preceding lemmas we may suppose that $F(x, y)$ is quantifier-free and is obtained by & and $\vee$ from the elementary formulas and their negations. It is easily seen that every elementary formula having the form $t = s$, where t and s are different terms in A_2, expresses in A_2 an 0-oriented set, and every negation of an elementary formula having the mentioned form expresses in A_2 an 1-oriented set, hence they are extremal. (Clearly, the formulas $t = t$ and $\neg(t = t)$ express in A_2 the sets, correspondingly, $\tilde{V}$ and $\tilde{O}$, so, they are extremal). Hence the set expressible by $F(x, y)$ is extremal, because the following statement holds: if A and B are extremal then $A \cup B$ and $A \cap B$ are also extremal. This completes the proof.

The statement established by the following lemma is similar to a statement formulated in [1] (without proof).

Lemma 3.13. The set $\tilde{Q}$ is not inductively representable in Θ_2.

Proof: If $\tilde{Q}$ would be inductively representable in Θ_2 then it would be expressible by a formula in A_2, but it is not so because

$$\lim_{n\to+\infty} \frac{\sigma_n(A)}{(n+1)^2} = \frac{1}{2},$$

hence $\tilde{Q}$ is not extremal.

Lemma 3.14. Every two-dimensional arithmetical set expressible by a positive formula in A_2 is strongly extremal.

Proof: Similarly to the preceding lemmas we may suppose that the considered set is expressed by a positive quantifier-free formula $F(x, y)$ in A_2 with free variables x and y which can be obtained by & and $\vee$ from its subformulas having the form $t = s$ or $\neg(t = s)$; in the last case t and s together may contain no more than one variable. But it is easily seen that the set expressed in A_2 by a formula $t = s$, where t and s are different terms, is 0-oriented, the set expressed by a formula $t = t$ is $\tilde{V}$, and the set expressed by a positive formula $\neg(t = s)$ is either $\tilde{O}$ (when t and s coincide) or strongly 1-oriented. So all the sets expressible by the mentioned formulas are strongly extremal. Hence the set expressible by $F(x, y)$ in A2 is strongly extremal, because the following statement holds: if A and B are strongly extremal then $A \cup B$ and $A \cap B$ are also strongly extremal. This completes the proof.

Lemma 3.15. The set $\tilde{S}$ is not inductively representable in Θ_3.

Proof: If $\tilde{S}$ would be inductively representable in Θ_3, then in would be expressible by a positive formula in A_2 but it is not so because $\tilde{S}$ is extremal but not strongly extremal.

Proof of the theorem 2.3: Using lemma 3.4 we may conclude that it is sufficient to consider only conditions concerning elements inductively representable in the given algebras. Clearly, the algebra Θ° is a proper subalgebra of Θ because, as it is proved in [12], every set inductively representable in Θ° is recursive but it is not so for Θ. The algebra Θ_1 is a proper subalgebra of Θ° becouse, for example, the set $\tilde{V} \circ (\tilde{E} \circ \tilde{E})$ is inductively representable in Θ° but not in Θ_1. Indeed, this set is the set of pairs (x, y) such that x is any natural number and y is any even number; if this set would be inductively representable in Θ_1 then it would be expressible by a formula $F(x, y)$ in A_1, hence the formula $\exists x F(x, y)$ would express in A_1 a set of even numbers, but it is not so, because this set is neither finite nor cofinite. The algebra Θ_2 is a proper subalgebra of Θ_1 because the set $\tilde{Q}$ is inductively representable in Θ_1 but not in Θ_2. Finally, the algebra Θ_3 is a proper subalgebra of Θ_2, because the set $\tilde{S}$ is inductively representable in Θ_2 but not in Θ_3. This completes the proof.

Corollary. In the following sequence of operations on TRESes: $^{-1}, \cap, \cup, \circ, \diamond, *$ no next operation is expressible by the preceding ones.

Indeed, this statement concerning $*$ and $\diamond$ follows immediately from the theorem 2.3. In order to prove this statement for $^{-1}, \cap, \cup, \circ$ let us consider the algebras $\rho_1, \rho_2, \rho_3, \rho_4$ defined by the list of operations, correspondingly, $(^{-1}), (^{-1}, \cap), (^{-1}, \cap, \cup), (^{-1}, \cap, \cup, \circ)$ and by the list containing only one basic element $\tilde{R}$. Let us denote the sets of elements inductively representable in $\rho_1, \rho_2, \rho_3, \rho_4$, correspondingly, by J_1, J_2, J_3, J_4. It is easily seen that J_4 is infinite, and J_1, J_2, J_3 are finite and contain, correspondingly, $2, 3, 4$ elements. This completes the proof.

Note. The complement of a set inductively representable in the algebra Θ is in general not inductively representable in it. Of contrary, the complement of any set inductively representable in $\Theta^{\circ}, \Theta_1, \Theta_2$ is always inductively representable in the corresponding algebra. However Θ_3 returns us to the mentioned property of Θ, because the complement of a set inductively representable in $\Theta3$ is in general not inductively representable in it.

§4. Let us recall some definitions concerning fuzzy sets ([3], [14], [18]) and recursively enumerable fuzzy sets. The *n-dimensional recursively enumerable fuzzy set* is defined ([7]-[13]) as a recursively enumerable set of $(n+1)$-tuples having the form $(x_1, x_2, ..., x_n, \varepsilon)$, where all x_i are natural numbers, and ε is a binary rational number $\frac{k}{2^m}$ such that $0 \leq \frac{k}{2^m} \leq 1$. The connections between this notion and the general concept of fuzzy set ([3], [14],

[18]) are described in [12] (see [12], pp.86-87). We consider below only two-dimensional recursively enumerable fuzzy sets (TREFS). We say that a TREFS W *covers* the TREFS V if for every $(x, y, \varepsilon) \in V$, where $\varepsilon > 0$, there exists such $\delta = \varepsilon$ that $(x, y, \delta) \in W$. We say that TREFSes W and V are *equivalent* if W covers V, and V covers W. (This notion of equivalence is used in [7]-[8] and [11]-[13]; it is different from the notion of equivalence considered in [9]). If A is a TRES then the fuzzy image of A is defined as the TREFS W consisting of all the triples having the form $(x, y, 0)$, and of the triples $(x, y, 1)$ such that $(x, y) \in A$. A TREFS W is said to be *standard* if all the triples having the form $(x, y, 0)$ belong to W. A TREFS W is said to be *n-discrete* for some natural number n if in every triple $(x, y, \varepsilon) \in W$ the third component ε has the form $\frac{k}{2^m}$, where $0 \leq k \leq 2^n$. A TREFS is said to be *discrete* if it is n-discrete for some n. For every TREFS W we define its ε_0*-level* where $0 \leq \varepsilon_0 \leq 1$, as the set of pairs (x, y) such that $(x, y, \varepsilon_0) \in W$. Clearly, every ε_0 -level of a TREFS is a TRES. The ε_0 -level of a TREFS W will be denoted by $W[\varepsilon_0]$; we shall say that ε_0 is the *index* of the set $W[\varepsilon_0]$ in W (cf. [11], [12]).

We consider the following operations on TREFSes (cf. [11], [12]).

The *sum* $W + V$ of TREFSes W and V is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ and $(x, y, \delta) \in V$ then $(x, y, min(1, \varepsilon + \delta)) \in W + V$.

The *product* $W \cdot V$ of TREFSes W and V is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ and $(x, y, \delta) \in V$ then $(x, y, \varepsilon \cdot \delta) \in W \cdot V$.

The *composition* $W \circ V$ of TREFSes W and V is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ and $(y, z, \delta) \in V$ then $(x, z, \varepsilon \cdot \delta) \in W \circ V$.

The *additive-transitive closure* $\oplus W$ of a TREFS W is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ then $(x, y, \varepsilon) \in \oplus W$; if $(x, y, \varepsilon) \in \oplus W$ and $(y, z, \delta) \in \oplus W$ then $(x, z, min(1, \varepsilon + \delta)) \in \oplus W$.

The *multiplicative-transitive closure* $\otimes W$ of a TREFS W is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ then $(x, y, \varepsilon) \in \otimes W$; if $(x, y, \varepsilon) \in \otimes W$ and $(y, z, \delta) \in \otimes W$ then $(x, z, \varepsilon \cdot \delta) \in \otimes W$.

The *arithmetical sum* $W \diamond V$ of TREFSes W and V is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ and $(x, z, \delta) \in V$ then $(x, y + z, \varepsilon \cdot \delta) \in W \diamond V$.

The *inversion* W^{-1} of a TREFS W is defined by the following g.r.: if $(x, y, \varepsilon) \in W$ then $(y, x, \varepsilon) \in W^{-1}$.

We shall consider as basic elements the following TREFSes: the TREFS H consisting of all the triples $(x, y, 0)$ and $(x, y, \frac{1}{2})$; the fuzzy images R, Q, Z_0, S of the TRESes, correspondingly, $\tilde{R}, \tilde{Q}, \tilde{Z}_0, \tilde{S}$.

Let us define some algebras on the set of all TREFSes (cf. [11], [12]). The algebra Ω is defined by the list of operations $(+, \cdot, \circ, \oplus, \otimes, \diamond, ^{-1})$ and the list of basic elements (R, H). The algebra Ω° is defined by the list of operations $(+, \cdot, \circ, \diamond, ^{-1})$ and the list of basic elements (R, Q, H). The algebras $\Omega_1, \Omega_2, \Omega_3$ are defined by the same list of operations $(+, \cdot, \circ, -1)$ and by the following lists of basic elements: (R, Q, Z_0, H) for Ω_1; (R, S, Z_0, H) for Ω_2; (R, Z_0, H) for Ω_3.

The algebras Ω and Ω° are introduced in [11] and [12]; it is proved in [12] that every TREFS is inductively representable in Ω (up to the equivalence); a TREFS is inductively representable in Ω° if and only if (up to the equivalence) it is discrete and all its ε_0 -levels are expressible by the formulas in M.Presburger's system.

Theorem 4.1. A TREFS is inductively representable in Ω_1 (correspondingly, Ω_2) (up to the equivalence) if and only if it is discrete and all its ε_0 -levels are expressible by the formulas in the system A_1 (correspondingly, A_2).

More precisely: every TREFS inductively representable in Ω_1 (correspondingly, Ω_2) is discrete and all its ε_0 -levels are expressible by the formulas in the system A_1 (correspondingly,A_2); if a TREFS W is discrete and all its ε_0 -levels are expressible by the formulas in A_1 (correspondingly, A_2) then a set V can be constructed which is equivalent to W and inductively representable in Ω_1 (correspondingly, Ω_2).

The formulation of the following theorem was suggested to the author by Yu.V.Matiyasevich.

Theorem 4.2. (Yu.V. Matiyasevich) A TREFS W is inductively representable in Ω_1 (correspondingly, Ω_2) (up to the equivalence) if and only if W is n-discrete for some natural number n and there exists a formula F in A_1 (correspondingly, A_2) such that F contains only free variables z, x, y, and for every k, where $0 \le k \le 2^n$, the level $W[\frac{k}{2^n}]$ of W is expressible by the formula $Subst(F; z; 0^{(k)})$ in A_1 (correspondingly, A_2).

Theorem 4.3. If a TREFS is inductively representable in Ω_3 then it is discrete and all its ε_0 -levels are representable by positive formulas in A_2.

The statement similar to that given by Yu.V.Matiyasevich holds also in the case of

Ω_3.

Theorem 4.4. If a TREFS W is inductively representable in Ω_3 then W is n-discrete for some natural number n and there exists a positive formula F in A_2 such that F contains only free variables z, x, y, and for every k, where $0 \leq k \leq 2^n$, the level $W[\frac{k}{2^n}]$ of W is expressible by the formula $Subst(F; z; 0^{(k)})$ in A_2.

The question, whether the statements reverse to the theorems 4.3 and 4.4 are true or not, is open. We shall prove only the following statement.

Theorem 4.5. If all ε_0 -levels of a discrete TREFS W are 0-oriented and expressible by positive formulas in A_2 then a TREFS V can be constructed which is equivalent to W and inductively representable in Ω_3.

Theorem 4.6. Every next algebra in the sequence $\Omega, \Omega^\circ, \Omega_1, \Omega_2, \Omega_3$ is a proper subalgebra of the preceding one.

§5. In this section we shall give the proofs of the theorems 4.1 - 4.6.

Lemma 5.1. The fuzzy image of every TRES inductively representable in Θ_1 (correspondingly, in Θ_2 or Θ_3) is inductively representable in Ω_1 (correspondingly, in Ω_2 or Ω_3).

Proof: The proof is similar to that of lemma 4.2 in [12]. Indeed, if some TRES A is inductively representable in Θ_1 (correspondingly, in Θ_2 or Θ_3) then there exists a process of obtaining the TRES A in Θ_1 (correspondingly, in Θ_2 or Θ_3); considering fuzzy images of all sets taking part in this process we obtain the constructing process of the fuzzy image of A in Ω_1 (correspondingly, in Ω_2 or Ω_3), because the fuzzy images of basic elements in Θ_1 (correspondingly, in Θ_2 or Θ_3) are basic elements in Ω_1 (correspondingly, in Ω_2 or Ω_3), and the operations $\cup, \cap, *, ^{-1}$ on TRESes correspond to the operations $+, \cdot, *, ^{-1}$ on their fuzzy images.

Corollary. The fuzzy images V, E, W_{kl}, Z_k, O of the TRESes $\bar{V}, \bar{E}, \bar{W}_{kl}, \bar{Z}_k, \bar{O}$ are inductively representable in Ω_1, Ω_2 and Ω_3.

Note. The set $\bar{O}$ is empty, but its fuzzy image O is not empty; it contains the triples $(x, y, 0)$ for all natural numbers x and y.

Lemma 5.2. If a TREFS W is inductively representable in Ω_1 (correspondingly, in Ω_2), then it is discrete, and all its ε_0-levels are expressible by formulas in A_1 (correspondingly, A_2).

Proof: The proof is similar to that of lemma 5.2 in [12]. Indeed, it is sufficient to prove that every TREFS W inductively representable in Ω_1 (correspondingly, Ω_2)

is discrete, and all its ε_0 -levels are inductively representable in Θ_1 (correspondingly, Θ_2). The basic elements in Ω_1 and Ω_2 are discrete, because H is 1-discrete, and every fuzzy image of a TRES is 0-discrete. Clearly, if W is m-discrete, and V is n-discrete, then W^{-1} is m-discrete, $W+V$ is $max(m,n)$-discrete, $W \circ V$ and $W \cdot V$ are $(m+n)$ -discrete. Obviously, all the ε_0 -levels in the basic elements in Ω_1 (correspondingly, Ω_2) are inductively representable in Θ_1 (correspondingly, Θ_2). Now, if a TREFS U is $W \circ V$, where W and V are inductively representable in Ω_1 or Ω_2, and all their ε_0-levels are inductively representable in Θ_1 or Θ_2, then every non-empty ε_0 -level of U can be represented in the form $(W[\delta_1] \circ V[\eta_1]) \cup (W[\delta_2] \circ V[\eta_2]) \cup ... \cup (W[\delta_t] \circ V[\eta_t])$, where $\delta_1, \delta_2, ..., \delta_t, \eta_1, \eta_2, ..., \eta_t$, are indexes of all levels in W and V such that $\delta_i \cdot \eta_i = \varepsilon_0, 1 \leq i \leq t$. Hence $U[\varepsilon_0]$ is inductively representable in the corresponding algebra (Θ_1 or Θ_2). Similarly, if U is $W+V$, where W and V satisfy the mentioned conditions, then every non-empty level $U[\varepsilon_0]$ can be represented in the form $(W[\delta_1] \cap V[\eta_1]) \cup (W[\delta_2] \cap V[\eta_2]) \cup ... \cup (W[\delta_t] \cap V[\eta_t])$, where $\delta_1, \delta_2, ..., \delta_t, \eta_1, \eta_2, ..., \eta_t$, are indexes of all levels in W and V satisfying for all $i, 1 \leq i \leq t$, the following condition: $\delta_i + \eta_i \geq 1$ when $\varepsilon_0 = 1$, and $\delta_i + \eta_i = \varepsilon_0$ when $\varepsilon_0 < 1$. If U is $W \cdot V$, then every non-empty level $U[\varepsilon_0]$ can be represented in the form given above, where $\delta_i \cdot \eta_i = \varepsilon_0$ for $1 \leq i \leq t$. Finally, if U is W^{-1}, then every level $U[\varepsilon_0]$ is $(W[\varepsilon_0])^{-1}$. This completes the proof.

Proof of the theorem 4.3: It is easily seen that all ε_0 -levels in the basic elements of Ω_3 are expressible in A_2 by the following positive formulas: $(x = x)\&(y = y), y = x', (x = 0)\&(y = y), (0 = 0')\&(x = x)\&(y = y)$. Using the expressions given in the proofs of lemmas 5.2 and 3.5 it is easily seen that if all ε_0 -levels of TREFSes U and V are expressible by positive formulas in A_2 then the sets $U+V, U \cdot V, U \circ V, U^{-1}$ possess the same property. This completes the proof.

Proof of the theorem 4.4: Let the TREFS W is inductively representable in Ω_3. Using lemma 5.2 and theorem 4.3 we may conclude that W is n-discrete for some n and all the levels $W[\varepsilon_0]$ are expressible by positive formulas in A_2. Now let us construct the formula F which is the conjunction of formulas $\neg(z = 0^{(k)}) \vee F_k$ for all k such that $0 \leq k \leq 2^n$, where every F_k is the positive formula in A_2 expressing the level $W[\frac{k}{2^n}]$ of W. It is easily seen that the formula F satisfies all the conditions of the theorem 4.4. This completes the proof.

Lemma 5.3. If a TREFS W is discrete and all its ε_0-levels are expressible by formulas in A_1 (correspondingly, A_2), then there exists a set V which is equivalent to W and inductively representable in Ω_1 (correspondingly, in Ω_2).

Proof: The proof is similar to that of lemma 5.3 in [12]. Let W be an n-discrete TREFS such that all its ε_0-levels are expressible by formulas in the system A_1 (correspondingly, A_2). Obviously, if some sets A and B are expressible by formulas in A_1 or A_2, then it is so also for the set $A - B$. Let us construct the TREFS W^* such that

$W^*(1) = W(1)$, $W^*[\frac{k}{2^n}] = W[\frac{k}{2^n}] - (W[\frac{k+1}{2^n}] \cup W[\frac{k+2}{2^n}] \cup ... \cup W[1])$

for all k such that $0 < k \leq 2^n - 1, W^*[0] = \tilde{V}$. It is easily seen that the TREFS W^* is equivalent to W, all its ε_0-levels are expressible by formulas in A_1 (correspondingly, A_2), and all the sets $W^*[\frac{k}{2^n}] \cap W^*[\frac{l}{2^n}]$ are empty when $k > 0, l < 0, k \neq l$. Now let us construct the TREFSes U_n for all $n > 1$ such that $U_1 = H, U_{n+1} = U_n \cdot H$; clearly, U_n contains only triples $(x, y, 0)$ and $(x, y, \frac{1}{2^n}$ for all natural numbers x, y. Let us construct also the TREFSes $Y_{n,k}$ for all $n \geq 1$ and $k \geq 0$ such that $Y_{n,0}$ contains only the triples $(x, y, 0)$ for all x, y, and $Y_{n,k+1} = Y_{n,k} + U_n$; clearly, $Y_{n,k}$ contains the triples $(x, y, \frac{q}{2^n}$ for all natural numbers x, y and for all q such that $0 \leq q \leq min(2^n, k)$. Finally, let us construct the set V as the sum of all sets having the form $V_K^* \cdot Y_{n,k}$ for $0 \leq k \leq 2^n$, where V_k^* is the fuzzy image of $W^*[\frac{k}{2^n}]$. It is easily seen that the TREFS V satisfies all the conditions of lemma. This completes the proof.

The proof of the theorem 4.1 is now obtained immediately using lemmas 5.2 and 5.3.

Proof of the theorem 4.2: Let us suppose that a TREFS W is inductively representable in Ω_1 (correspondingly, Ω_2). Using theorem 4.1 we may conclude that there exists a TREFS V which is equivalent to W, n-discrete for some n, and all the ε_0 -levels of V are expressible by formulas in A_1 (correspondingly, A_2). A formula F satisfying the conditions of theorem 4.2 is now constructed in the same way as it is described in the proof of the theorem 4.4.

Now let us suppose that a TREFS W is n-discrete for some n and there exists a formula F in A_1 (correspondingly, A_2) with only free variables z, x, y, such that for every k, where $0 \leq k \leq 2^n$, the level $W[\frac{k}{2^n}$ of W is expressible by the formula $Subst(F; z; 0^{(k)})$ in A_1 (correspondingly, A_2). So, every ε_0-level $W[\varepsilon_0]$, where $0 \leq \varepsilon_0 \leq 1$, either is empty, or is expressible by some formula $Subst(F; z; 0^{(k)})$ in A_1 (correspondingly, A_2). Hence all the ε_0 -levels of W are expressible by formulas in A_1 (correspondingly, A_2). Using the theorem 4.1 we may conclude that a set V can be constructed which is equivalent to W and is inductively representable in Ω_1 (correspondingly, Ω_2). This completes the proof.

Now we shall prove some lemmas in order to prove theorem 4.5.

Lemma 5.4. Every two-dimensional finite or cofinite arithmetical set is inductively representable in Θ_3.

Proof: If a two-dimensional arithmetical set A is finite then it can be represented as a sum of TRESes $\bar{W}_{kl}$ for some k and l. If A is cofinite then it is expressible in A_2 by a conjunction of formulas having the form $\neg(x = 0^{(k)} \vee \neg(y = 0^{(l)})$ for some k and l; hence it is expressible by a positive formula in A_2. This completes the proof.

Lemma 5.5. If a formula F containing at most two free variables has the form $(t = s)\&(t_1 = s_1)$, where t and s are different terms in A_2, t_1 and s_1 are any terms in A_2, then either F is equivalent to $t = s$ in A_2, or it expresses in A_2 a finite set.

Proof: Let F be a formula in A_2 having the form $(t = s)\&(t_1 = s_1)$ containing no other free variable, except x and y, and let t and s are different. If t_1 and s_1 are equal, or $t_1 = s_1$ is equivalent to $t = s$, then F is equivalent to $t = s$. Let us suppose now that F is not equivalent to $t = s$. Hence t_1 is different from s_1. Clearly, in this case every formula $t = s$ and $t_1 = s_1$ may be reduced to one of the forms: $(x = y^{(k)}), (x^{(k)} = y), (0 = y^{(k)}), (0^{(k)} = y), (x = 0^{(k)}), (x^{(k)} = 0), (x = x^{(k)}), (y = y^{(k)}), (0 = 0^{(k)})$. It can be easily seen now that the set expressed by every conjunction of two non-equivalent formulas which have one of these forms and satisfy the mentioned conditions, is either empty or contains only one point (x, y). This completes the proof.

Lemma 5.6. If A and B are two-dimensional arithmetical sets such that A is inductively representable in Θ_3 and 0-oriented, B is inductively representable in Θ_2, then the set $A - B$ is inductively representable in Θ_3.

Proof: If the sets A and B satisfy the mentioned conditions then they are expressible by the formulas F and G in A_2 containing no other free variables, except x and y. We may suppose that F is positive, F and G are quantifier-free and F is reduced to a disjunctive normal form. The set $A - B$ is expressed by the formula $F\&\neg G$; we denote $\neg G$ by H. So $A - B$ is expressed by the formula $F\&H$; we may suppose that F and H are reduced to a disjunctive normal forms, hence the formula $F\&H$ may be transformed to a disjunction of formulas having the form

$$F_1 \& F_2 \& ... \& F_k \& H_1 \& H_2 \& ... \& H_l \tag{5.1}$$

where all F_i and H_j are elementary formulas in A_2 or negations of elementary formulas, and the set expressed by $F_1 \& F_2 \& ... \& F_n$ is 0-oriented. It is sufficient to prove that the formula (5.1) is equivalent to some positive formula in A_2. Clearly, it is so if some F_i or H_j has the form $\neg(t = t)$ (in this case the set expressed by (5.1) is empty); if some F_i or H_j has the form $(t = t)$ then this subformula may be dropped from (5.1). So, we may suppose that every F_i and every H_j in (5.1) have the form $(t = s)$ or $\neg(t = s)$, where t and s are different. But in this case at least one F_i has the form $(t = s)$, because the set expressed by $F_1 \& F_2 \& ... \& F_k$ is 0-oriented. We may suppose that F_1 has the form $(t = s)$. The formula (5.1) is equivalent to the following one

$$F_1 \& (F_1 \& F_2) \& (F_1 \& F_3) \& ... \& (F_1 \& F_k) \& (F_1 \& H_1) \& (F_1 \& H_2) \& ... \& (F_1 \& H_l) \tag{5.2}$$

so, it is sufficient to prove that every formula $F_1, (F_1 \& F_i), (F_1 \& H_j)$ is equivalent to a positive formula in A_2. Obviously, F_1 is a positive formula. Every formula $(F_1 \& F_i)$ and $(F_1 \& H_j)$ has one of the form $(t = s) \& (t_1 = s_1)$ or $(t = s) \& \neg(t_1 = s_1)$ (where t_1 and s_1 are different). But $(t = s) \& (t_1 = s_1)$ is a positive formula. The formula $(t = s) \& \neg(t_1 = s_1)$ is equivalent to $(t = s) \& \neg((t = s) \& (t_1 = s_1))$. Using lemma 5.5 we may conclude that in the considered case the formula $(t = s) \& (t_1 = s_1)$ is equivalent to $t = s$ or expresses in A_2 a finite set. In the first case the formula $(t = s) \& \neg(t_1 = s_1)$ expresses in A_2 an empty set. In the second case the formula $\neg((t = s) \& (t_1 = s_1))$ expresses in A_2 a cofinite set, hence, using lemma 5.4, we may conclude that the formula $\neg((t = s) \& (t_1 = s_1))$ is equivalent in A_2 to some positive formula. This completes the proof.

Proof of the theorem 4.5: If a TREFS is n-discrete for some n, and all its ε_0-levels are 0-oriented and inductively representable in Θ_3, then using lemma 5.6 we may conclude that the construction described in the proof of lemma 5.3 gives a representation in Ω_3 of some TREFS V equivalent to W. This completes the proof.

Note. The construction described in the proof of lemma 5.3 may be applied to the TREFSes whose ε_0-levels are 0-oriented, but in the general case such a proof does not hold, because the difference $A - B$ of sets inductively representable in Θ_3 is in general not inductively representable in Θ_3. For example, it is so, when $A = \tilde{V}$, $B = \tilde{E}$.

Proof of the theorem 4.6: Using theorem 2.3 we may conclude that there exist TRE-

Ses C_1, C_2, C_3, C_4 which are inductively representable, correspondingly, in $\Theta, \Theta^\circ, \Theta_1, \Theta_2$, but not in $\Theta^\circ, \Theta_1, \Theta_2, \Theta_3$. Let us consider the fuzzy images W_1, W_2, W_3, W_4 of the sets, correspondingly, C_1, C_2, C_3, C_4. Using theorems 2.1, 2.2, 4.1 and 4.3 proved above, as well as theorems 3.1, 4.1 and 5.1 proved in [12] we may conclude that the TREFSes W_1, W_2, W_3, W_4 are inductively representable, correspondingly, in $\Omega, \Omega^\circ, \Omega_1, \Omega_2$ but not in $\Omega^\circ, \Omega_1, \Omega_2, \Omega_3$. This completes the proof.

The theorems proved above show that the fuzzy sets inductively representable in Ω_1 and Ω_2 may be considered as fuzzy analogues of two-dimensional sets expressible in the systems A_1 and A_2. The relations of sets inductively representable in Ω_3 and the logical means of the system A_2 are now in general not so clear. The problem concerning an exact logical description of sets inductively representable in Ω_3 remains open.

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Institute for Informatics and Automation Problems of the
National Academy of Sciences of Armenia, Yerevan, Armenia
E-mail: zaslav@ipia.sci.am