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Quantum Processes and Possibility of Their Control

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Abstract

The dissipation and decoherence (for example, the effects of noise in quantum computations), interaction with thermostat or in general with physical vacuum, measurement and many other complicated problems of open quantum systems are a consequence of interaction of quantum systems with the environment. These problems are described mathematically in terms of complex probabilistic process (CPP). Particularly, treating the environment as a Markovian process we derive an Langevin-Schrödinger type stochastic differential equation (SDE) for describing the quantum system interacting with environment. For the $1D$ randomly quantum harmonic oscillator (QHO) L-Sh equation has a solution in the form of orthogonal CPP. On the basis of orthogonal CPP the stochastic density matrix (SDM) method is developed and in its framework relaxation processes in the uncountable dimension closed system of "QHO+environment" is investigated. With the help of SDM method the thermodynamical potentials, like nonequilibrium entropy and the energy of ground state are exactly constructed. The dispersion for different operators is calculated. In particular, the expression for uncertain relations depending on parameter of interaction between QHO and environment is obtained. The Weyl transformation for stochastic operators is specified. Ground state Winger function is developed in detail.

Introduction

Recently a great number of papers [1] concerning the "quantum chaos", i.e. with the quantum analogues of classical systems possessing the dynamic chaos features, have been published. The investigations are conducted along different directions, such as analysis of distribution of energy levels; definition and calculation of quantities, which are responsible for the presence of chaos (corresponding to the classical Lyapunov exponents and KS-entropy)in the quantum systems; study of localization and delocalization of wave functions around the classical orbits; etc. Although in most cases mentioned above one is faced with the necessity to describe a quantum system statistically, so far there was not paid much attention to a stochastic behavior of the wave function itself.

Many problems of great importance in the field of the non-relativistic quantum mechanics, such as description of Lamb shift, spontaneous transitions in atoms, etc., remain unsolved due to the fact that the concept of environment (which as a rule is random) has not been considered within the framework of the standard quantum mechanics. It is obvious that a quantum object immersed into the thermostat (or more fundamentally the physical vacuum)

is an open system. Various approaches [2] to the description of such systems exist, mainly in application to the problem of continuous measurements. One of them is based on the consideration of the wave function as a random process, for which a stochastic differential equation (SDE) is derived. But the equation is obtained by the method which is extremely difficult for application even in case of comparatively simple type of interaction between the system and the environment, so that some new ideas are needed [3]-[5]. Moreover sometimes it becomes necessary to consider the wave function as a random complex process even in closed systems (for example, when a classical analogue of the quantum system has the features of the dynamical chaos) [6]-[8].

To describe the cases mentioned above, recently a radically new mathematical scheme (theory of stochastic quantum processes (TSQP was proposed). This scheme is used to describe the non-relativistic quantum system, in general case strongly interacting with the randomly environment [9]. Note, that the wave function of closed system "quantum object + thermostat" at that time is described by L-Sch type SDE. On the base of TSQP stochastic density matrix method was developed permitting construction of all the thermodynamic potentials of quantum subsystem.

In the present paper relaxation processes in a closed uncountable dimension system "quantum oscillator and thermostat" are investigated within the framework of non-perturbational method. In particular, it enables to describe the evolution process inside thermostat. Exact representations are found for both widening and shift (analogous to the Lamb shift) of the ground state energy level of the immersed into the thermostat (physical vacuum) QHO as well as the entropy of an ground state are calculated.

1 Formulation of the problem

We shall consider the closed system "quantum object + thermostat" within the framework of Langevin-Schrödinger type SDE

$$i\partial_t \Psi_{stc} = \hat{H}\Psi_{stc}, \tag{1.1}$$

where 1D evolution operator $\hat{H}$ is assumed to be quadratic over the space variable:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2(t)x^2. \tag{1.2}$$

In expressions (1.1)-(1.2) the frequency $\Omega(t)$ is stochastic function of time. Let them have the form:

$$\Omega^2(t) = \Omega_0^2 + \sqrt{2\epsilon}f(t), \tag{1.3}$$

where $\Omega_0 = const$ and $f(t)$ is independent Gaussian stochastic process with the zero mean and $\delta-$ shaped correlation function:

$$< f(t)f(t') >= \delta(t-t'), \qquad < f(t) >= 0. \tag{1.4}$$

Constant ϵ is characterized the power of stochastic force $f(t)$. The equation (1.1) have asymptotic solution $\Psi_{as}(n|x,t)$ in the limit of $t \to -\infty$:

$$\Psi_{as}(n|x,t) = e^{-i(n+1/2)\Omega_0 t}\phi(n|x),$$
$$\phi(n|x) = \left(\frac{1}{2^n n!}\sqrt{\frac{\Omega_0}{\pi}}\right)^{1/2} e^{-\Omega_0 x^2/2} H_n\left(\sqrt{\Omega_0}x\right), \tag{1.5}$$

where $\phi(n|x)$ is the wave function of a stationary oscillator and $H_n(x)$ is the Hermitian polynomial. The formal solution of the problem (1.1)-(1.5) may be written down explicitly for arbitrary $\Omega(t)$. It has the following form:

$$\Psi_{stc}(x,t|\{\xi\}) = \frac{1}{\sqrt{r}} exp\Big\{\frac{i}{2}\dot{r}r^{-1}x^2\Big\}\chi\Big(\frac{x}{r},\tau\Big), \tag{1.6}$$

where the function $\chi(y,\tau)$ satisfies the Schrödinger equation for a harmonic oscillator on the stochastic space-time $\{y,\tau\}$ continuum:

$$i\frac{\partial\chi}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\chi}{\partial y^2} + \frac{\Omega_0^2 y^2}{2}\chi, \tag{1.7}$$

where

$$y = \frac{x}{r}, \quad \xi(t) = r(t)e^{i\gamma(t)}, \quad \tau = \frac{\gamma(t)}{\Omega_0}, \quad \gamma(t) = \Omega_0\int\limits_{-\infty}^{t}\frac{dt'}{r^2(t')}.$$

The function $\xi(t)$ is defined from the classical homogenous equation of motion for the oscillator with the frequency $\Omega(t)$

$$\ddot{\xi} + \Omega^2(t)\xi = 0. \tag{1.8}$$

Taking into account (1.6) and well known solution of autonomous quantum harmonic oscillator [10] for stochastic complex processes which are described closed strong interacting system "quantum object+environment" we can write following expression:

$$\Psi_{stc}(m|x,t;\{\xi\}) = \Big(\frac{1}{2^n n!}\sqrt{\frac{\Omega_0}{\pi r(t)}}\Big)^{1/2}\times$$

$$\exp\Big[-i\Big(n+\frac{1}{2}\Big)\Omega_0\int\limits_{t_0}^{t}\frac{dt'}{r^2(t')} + i\frac{\dot{r}_t(t)}{2r(t)}x^2 - \frac{\Omega_0}{2r^2(t)}x^2\Big]H_n\Big(\sqrt{\Omega_0}\frac{x}{r(t)}\Big). \tag{1.9}$$

The solution of (1.9) is a random complex process defined on the extended space $\Xi = R^1 \otimes R_{\{\xi\}}$, where R^1 is one dimensional euclidian space and $R_{\{\xi\}}$ is the corresponding functional space. It is easy to show that the mentioned complex random processes are orthogonal. Taking integral over the space R^1 we get:

$$\int\limits_{-\infty}^{+\infty}\Psi_{stc}(n|x,t;\{\xi\})\Psi^*_{stc}(m|x,t;\{\xi\})\,dx = \delta_{nm}, \tag{1.10}$$

where the symbol $*$ means the complex conjugation. The relation (1.10) shows that the closed odd dimensional system "quantum oscillator+environment" is described in terms of the full orthogonal basis of quadratically integrable functionals of the space L^2. The last fact is very important for the further strong mathematical constructions for the statistical parameters of the system.

2 Stochastic density matrix method

The quantum system is impossible to isolate from the environment. This is a principal problem if taking into account that even for the ideal isolation nevertheless any system is located

in the fundamental physical vacuum. Remember that many important features of atomic systems, like Lamb shift of energy levels, spontaneous transitions, etc., are explained by vacuum fluctuations. So, the processes in quantum systems are to some extent irreversible. For the investigation of irreversible processes the non-stationary density matrix representation based on quantum Liouvile equation [11] is often used. However the application of this representation has restrictions. It is used for the cases when the system before the interaction switch on was on thermodynamic equilibrium state and after the interaction was applied, its evolution is adiabatic. Below in the frames of considered model the new approach is used for the investigation of statistical properties of irreversible quantum system without any restriction on the quantities and the rate of interaction change. This method is based on bilinear form is constructed from orthogonal complex random processes. Below it is referred as the *stochastic density matrix method.*

Definition 2.1. *The stochastic density matrix is defined by the expression:*

$$\rho_{stc}(x,t;\{\xi\}|x',t';\{\xi'\}) = \sum_{m=0}^{\infty} w_0^{(m)} \rho_{stc}^{(m)}(x,t;\{\xi\}|x',t';\{\xi'\}), \tag{2.1}$$

where $\rho_{stc}^{(m)}(x,t;\{\xi\}|x',t';\{\xi'\})$ *is partial stochastic density matrix and defined by help of bilinear form:*

$$\rho_{stc}^{(m)}(x,t;\{\xi\}|x',t';\{\xi'\}) = \Psi_{stc}(m|x,t;\{\xi\})\Psi_{stc}^{*}(m|x',t';\{\xi'\}). \tag{2.2}$$

In the expression of (2.1) $w_0^{(m)}$ *has the meaning of the initial distribution over quantum states with energies* $E_m = (m+1/2)\Omega_0$, *until the moment when the generator of random excitations is activated. Integrating (2.1) over euclidian space and taking into account (1.10), we obtain the normalization condition for the weight functions:*

$$\sum_{m=0}^{\infty} w_0^{(m)} = 1, \quad w_0^{(m)} \geq 0. \tag{2.3}$$

Below we'll define the mean values of various operators. Note that when averaging on extended space Ξ the order of the integration is important. If the integral is taken first on R^1 space then on $R_{\{\xi\}}$, the stochastic density matrix becomes equal to unity. This means that in the extended space all conservations laws are valid, in other words the stochastic matrix in this space is unitary. Else if we take the integration in the inverse order, we get another picture. After the integration on $R_{\{\xi\}}$ the obtained density matrix describes quantum processes in Euclidean space R^1. Its trace is not unity, in general. This means that the conservation laws are not valid already. This can be explained by the fact that the system has been in a strongly non-equilibrium state and after relaxation its parameters have been significantly changed.

Below we'll be interested in quantum subsystem processes, hence the integration first on $R_{\{\xi\}}$ and then on R^1 is supposed.

Definition 2.2. *The expected value of the operator* $\hat{A}(x,t|\{\xi\})$ *in quantum state with the index m is:*

$$A_m = \lim_{t\to+\infty}\left\{\frac{1}{N_m(t)} Sp_x\left[Sp_{\{\xi\}}\hat{A}\rho_{stc}^{(m)}\right]\right\}, \quad N_m(t) = Sp_x\left[Sp_{\{\xi\}}\rho_{stc}^{(m)}\right]. \tag{2.4}$$

The mean value of the operator $\hat{A}(x,t|\{\xi\})$ over the whole ensemble of states will respectively be given by:

$$A = \lim_{t\to+\infty}\Big\{\frac{1}{N(t)}Sp_x\big[Sp_{\{\xi\}}\hat{A}\rho_{stc}\big]\Big\}, \quad N(t) = Sp_x\big[Sp_{\{\xi\}}\rho_{stc}\big]. \tag{2.5}$$

The operation $Sp_{\{\xi\}}$ in (2.3) and (2.4) is defined by functional integral representation

$$Sp_{\{\xi\}}\big\{K(x,t;\{\xi\}|x',t';\{\xi'\})\big\} = \sqrt{\frac{\Omega_0}{\pi}}\int K(x,t;\{\xi\}|x',t;\{\xi\})\,D\{\xi\}, \tag{2.6}$$

and correspondingly the operation Sp_x is defined as a simple integration

$$Sp_x\big\{K(x,t;\{\xi\}|x',t';\{\xi'\}\big\} = \sqrt{\frac{\Omega_0}{\pi}}\int K(x,t;\{\xi\}|x,t;\{\xi'\})\,dx. \tag{2.7}$$

If one wishes to have the quantity describing irreversible behavior of the system, it is necessary to change definition of entropy.

Definition 2.3. *The von Neumann entropy, the standard measure of randomness of statistical ensemble described by density matrix , which is defined as:*

$$S_N(\epsilon,t) = -\frac{1}{N(t)}Sp_x\big\{\rho\ln\rho\big\}, \tag{2.8}$$

where $N(t) = Sp_x\rho$ and $\rho = Sp_{\{\xi\}}\{\rho_{stc}\}$.

The definition (2.8) of entropy is correct for quantum information theory and also is agree with the Shannon entropy in the classical limit.

Often it is interesting to know entropies of isolated quantum state (partial entropy)

$$S_N^{(m)}(\epsilon,t) = -\frac{1}{N_m(t)}Sp_x\big\{\rho^{(m)}\ln\rho^{(m)}\big\}, \tag{2.9}$$

where $N_m(t) = Sp_x\rho^{(m)}$ and $\rho^{(m)} = Sp_{\{\xi\}}\big\{\rho_{stc}^{(m)}\big\}$.

Definition 2.4. *The entropies can be defined another forms. The total entropy may be calculated by formula*

$$S_G(\epsilon,t) = -\frac{1}{N(t)}Sp_x\Big\{Sp_{\{\xi\}}\big[\rho_{stc}\ln\rho_{stc}\big]\Big\}, \tag{2.10}$$

and the partial entropy correspondingly by formula

$$S_G^{(m)}(\epsilon,t) = -\frac{1}{N_m(t)}Sp_x\Big\{Sp_{\{\xi\}}\big[\rho_{stc}^{(m)}\ln\rho_{stc}^{(m)}\big]\Big\}. \tag{2.11}$$

Before proceeding further to the calculations of the physical parameters let us write down the general form of the partial stochastic density matrix:

$$\rho_{stc}^{(m)}\big(x,t;\{\xi\}|x',t';\{\xi'\}\big) = \sqrt{\frac{\Omega_0}{\pi r(t)r(t')}}\exp\Big\{-i\big(m+\frac{1}{2}\big)\Omega_0\Big[\int_{t_0}^{t}\frac{d\mu}{r^2(\mu)} - \int_{t_0}^{t'}\frac{d\mu}{r^2(\mu)}\Big]+$$

$$\frac{i}{2}\Big[\frac{r_t(t)}{r(t)}x^2 - \frac{r_{t'}(t')}{r(t')}x'^2\Big] - \frac{1}{2}\Omega_0\Big[\frac{1}{r^2(t)}x^2 + \frac{1}{r^2(t')}x'^2\Big]\Big\}H_m\big(\sqrt{\Omega_0}\frac{x}{r(t)}\big)H_m\big(\sqrt{\Omega_0}\frac{x'}{r(t')}\big). \tag{2.12}$$

3 Calculation of thermodynamic potentials

Now let us turn to the calculation of the ground state equilibrium entropy, which is defined by the expression (2.11):

$$S_G^{(0)}(\lambda) = \lim_{t\to+\infty} S^{(0)}(\lambda, t). \tag{3.1}$$

The solution of equation (1.8) may be presented to the form:

$$\xi(t) = \begin{cases} \xi_0(t) \equiv \exp(i\Omega_0 t), & t \le t_0 = -\infty, \\ \xi_0(t_0)\exp\left\{\int_{t_0}^{t} \Phi(t')dt'\right\}, & t > t_0, \end{cases} \tag{3.2}$$

where $\Phi(t)$ is some complex function.

After the substitution (3.2) in the (1.8) we can define the following nonlinear SDE for a function $\Phi(t)$:

$$\dot{\Phi} + \Phi^2 + \Omega_0^2 + \sqrt{2\epsilon} f(t) = 0, \quad \Phi(t_0) = \dot{\xi}_0(t_0)/\xi_0(t_0) = i\Omega_0, \tag{3.3}$$

where $\dot{\Phi} = d_t\Phi$. The second equation in the (3.2) expresses a condition which guarantees continuity of the function $\xi(t)$ and its first derivative at the $t = t_0$. The function $\Phi(t)$ is described a complex-valued random process due to the initial condition. As a result the SDE (3.3) is equivalent to a set of two SDE for real-valued random processes. Namely, introducing real and imaginary parts of $\Phi(t)$

$$\Phi(t) = u_1(t) + iu_2(t),$$

we finally obtain the following set of SDE for the components of random vector process $\vec{u} \equiv \vec{u}(u_1, u_2)$:

$$\begin{cases} \dot{u}_1 = -u_1^2 + u_2^2 - \Omega_0^2 - \sqrt{2\epsilon} f(t), \\ \dot{u}_2 = -2u_1u_2, \end{cases} \quad \begin{cases} u_1(t_0) = Re[\dot{\xi}_0(t_0)/\xi_0(t_0)] = 0, \\ u_2(t_0) = Im[\dot{\xi}_0(t_0)/\xi_0(t_0)] = \Omega_0. \end{cases} \tag{3.4}$$

The pair of random processes (u_1, u_2) are not independent, because their evolution is influenced by the common random force $f(t)$. This means that the joint probability distribution:

$$P_0(\vec{u}, t|\vec{u}_0, t_0) = \left\langle \prod_{i=1}^{2} \delta(u_i(t) - u_{0i}) \right\rangle, \quad u_{0i} = u_i(t_0),$$

is a non-factorable function. Proceeding from the known evolution equations (3.4), we obtain by the standard method the Fokker-Planck equation for P_0 (see for ex. [12] or [13]) which has the form:

$$\frac{\partial P_0}{\partial t} = \hat{L}_0 P_0, \tag{3.5}$$

$$\hat{L}_0(u_1, u_2) \equiv \epsilon\frac{\partial^2}{\partial u_1^2} + (u_1^2 - u_2^2 + \Omega_0^2)\frac{\partial}{\partial u_1} + 2u_1u_2\frac{\partial}{\partial u_2} + 4u_1, \tag{3.6}$$

with the initial and border conditions:

$$P_0(u_1, u_2; t)\Big|_{t=t_0} = \delta(u_1 - u_{01})\delta(u_2 - u_{02}), \quad P_0(u_1, u_2; t)\Big|_{||\vec{u}||\to+\infty} \to 0. \tag{3.7}$$

Proceeding from (2.8), (2.9) and (2.10), we can write down the expression for nonequilibrium entropy of ground state:

$$S_G^0(\epsilon,t) = -\frac{1}{2} + \frac{N_{\alpha;\alpha}(t)}{N_\alpha(t)}\Big|_{\alpha=0}, \tag{3.8}$$

where

$$N_\alpha(t) = Sp_{\{\xi\}}\{I_\alpha(t;\{\xi\})\}, \quad N_{\alpha;\alpha}(t) = \partial_\alpha Sp_{\{\xi\}}\{I_\alpha(t;\{\xi\})\}. \tag{3.9}$$

In (3.9) the following notation have been made:

$$I_\alpha(t;\{\xi\}) = \frac{\theta(u_2(t))}{\sqrt{u_2(t)}}\exp\Big(-(\alpha+1)\int\limits_{t_0}^{t} u_1(t')dt'\Big), \quad \theta(u_2) = \begin{cases} 1, & u_2 > 0, \\ 0, & u_2 < 0, \end{cases}$$

Now we are ready to calculate the functional traces in expression (3.9). Using the distribution P_0 it is easy to construct continuum measures in the expressions for $N_0(t) = N_\alpha(t)|_{\alpha=0}$ and $N_{\alpha;\alpha}(t)$, which define the entropy of non-equilibrium quantum systems. These integrals may be calculated with the use of the generalized Feynman-Kac formula (see for example [9]). So as give the following representation:

$$N_\alpha(t) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{1}{\sqrt{u_2}}\theta(u_2)Q_\alpha(u_1,u_2;t)du_1du_2, \tag{3.10}$$

where the function $Q_\alpha(u_1,u_2,t)$ is a solution of the equation:

$$\partial_t Q_\alpha = \hat{L}_0 Q_\alpha - (\alpha+1)u_1 Q_\alpha, \tag{3.11}$$

which is satisfied the following initial and border conditions:

$$Q_\alpha(u_1,u_2;t)\Big|_{t=t_0} = \delta(u_1-u_{01})\delta(u_2-u_{02}), \quad Q_\alpha(u_1,u_2;t)\Big|_{\|\vec{u}\|\to+\infty} \to 0.$$

If in the expression (3.10) the substitution $\alpha = 0$ is put, the normalization constant $N_0(t)$ will be obtained. If we can calculate the quantity $Q_\alpha(u_1,u_2;t)$ then obviously we can be able calculate the function $D_\alpha(u_1,u_2,t) \equiv \partial_\alpha Q_\alpha(u_1,u_2,t)$. It is easy to obtain the equation for the latter by differentiae the equation (3.11) with respect to α:

$$\partial_t D_\alpha = \hat{L}_0 D_\alpha - (\alpha+1)u_1 D_\alpha - u_1 Q_\alpha, \tag{3.12}$$

correspondingly with conditions:

$$D_\alpha(u_1,u_2;t)\Big|_{t=t_0} = 0, \quad D_\alpha(u_1,u_2;t)\Big|_{\|\vec{u}\|\to+\infty} \to 0.$$

Introducing the designations $D_0(u_1,u_2;t) \equiv D_\alpha(u_1,u_2;t)\Big|_{\alpha=0}$, we obtain the representation:

$$N_{0;0}(t) = N_{\alpha;\alpha}(t)\Big|_{\alpha=0} = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{1}{\sqrt{u_2}}\theta(u_2)D_0(u_1,u_2;t)du_1du_2. \tag{3.13}$$

Now the expression for the equilibrium entropy can be obtained:

$$S_G^{(0)}(\lambda) = -\frac{1}{2} + \frac{N_{0,0}^{st}(\lambda)}{N_0^{st}(\lambda)}, \tag{3.14}$$

where

$$N_0^{st}(\lambda) = \lim_{t\to\infty} N_0(t), \quad N_{0,0}^{st}(\lambda) = \lim_{t\to\infty} N_{0,0}(t), \quad \lambda = (\Omega_0/\epsilon^{1/3})^2.$$

Note that at the limit $t \to +\infty$ the solutions $Q_a(u_1, u_2; t)$ and $D_0(u_1, u_2; t)$ turn to their stationary limits $Q_a^{st}(u_1, u_2)$ and $D_0^{st}(u_1, u_2)$.

Note that the von Neumann entropy (2.8) coincides with the entropy expressions in (2.10) when the interaction with the environment is weak, i.e. then $\epsilon \ll 1$.

To have in view the expressions (2.5) and (2.12) by a little transformation for the average energy it can be defined:

$$E_{osc}^{(0)}(\lambda) = \lim_{t\to+\infty}\left\{\frac{1}{N_0(t)} Sp_z Sp_{(\xi)}\left[\hat{H}_0 \rho_{slc}^{(0)}\right]\right\}, \tag{3.15}$$

where operator $\hat{H}_0$ is determined from (1.2) by averaging of $\hat{H}$ over all:

$$\hat{H}_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega_0^2 x^2. \tag{3.16}$$

Substituting (3.16) in (3.15) after simple algebra we obtain for the ground state energy:

$$E_{osc}^{(0)}(\lambda) = \frac{1}{2}\big(1 + K(\lambda)\big)\Omega_0, \tag{3.17}$$

with the expressions:

$$K(\lambda) = \frac{1}{N_0^{st}(\lambda)} \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{1}{\sqrt{\bar{u}_2}}\left\{-1 + \frac{\bar{u}_1^2 + \bar{u}_2^2 + \lambda}{2\sqrt{\lambda}\bar{u}_2}\right\} Q_0^{st}(\lambda; \bar{u}_1, \bar{u}_2)\, d\bar{u}_1\, d\bar{u}_2,$$

$$N_0^{st}(\lambda) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{1}{\sqrt{\bar{u}_2}} Q_0^{st}(\lambda; \bar{u}_1, \bar{u}_2)\, d\bar{u}_1\, d\bar{u}_2, \quad \bar{u}_1 = \frac{u_1}{\epsilon^{1/3}}, \quad \bar{u}_2 = \frac{u_2}{\epsilon^{1/3}}. \tag{3.18}$$

In the expressions (3.18) the function $Q_0(\bar{u}_1, \bar{u}_2; t)$ is a solution of the equation:

$$\frac{\partial Q_0(\bar{u}_1, \bar{u}_2; t)}{\partial t} = \left\{\frac{\partial^2}{\partial \bar{u}_1^2} + (\bar{u}_1^2 - \bar{u}_2^2 + \lambda)\frac{\partial}{\partial \bar{u}_1} + 2\bar{u}_1\bar{u}_2\frac{\partial}{\partial \bar{u}_2} + 3\bar{u}_1\right\} Q_0(\bar{u}_1, \bar{u}_2; t), \tag{3.19}$$

which in the limit of stationary possesses is equal $Q_0^{st}(\lambda; \bar{u}_1, \bar{u}_2) = \lim_{t\to+\infty} Q_0(\bar{u}_1, \bar{u}_2; t)$. The energy of n-th quantum state in the thermodynamic limit is calculated similarly:

$$E_{osc}^{(n)}(\lambda) = (n + 1/2)(1 + K(\lambda))\Omega_0. \tag{3.20}$$

As evident from expression (3.20) after the relaxation all energetic levels are equidistant.

4 Uncertain relations, Weyl transformation and Wigner function for the ground state

According to the uncertainty relation, in quantum system the coordinates and momentums can't have arbitrary small dispersions. This principle is experimentally verified many times. However at the present time as a result of recent quantum technology development the necessity in the overcoming of this fundamental restriction and taking the control over the uncertainty relation arises.

The dispersion of the operator $\hat{A}_i$ is determined by:

$$\Delta\hat{A}_i(t) \equiv \left\{Sp_x\left(\rho\hat{A}_i^2\right) - \left[Sp_x\left(\rho\hat{A}_i\right)\right]^2\right\}^{1/2}. \tag{4.1}$$

Now using the expression (4.1) it is easy to calculate the dispersions for the operator $\hat{A}_i$ in the extended space $\Xi = R^1 \otimes R_{\{\xi\}}$ at the time point t:

$$\Delta\hat{A}_i(t) \equiv \frac{1}{N_0(t)}\left\{Sp_xSp_{\{\xi\}}\left(\rho_{stc}^{(0)}\hat{A}_i^2\right) - \left[Sp_xSp_{\{\xi\}}\left(\rho_{stc}^{(0)}\hat{A}_i\right)\right]^2\right\}^{1/2}. \tag{4.2}$$

Now, with help of expression (4.2) we can calculate the dispersions for the coordinate $\hat{x}$ and momentum $\hat{p}$:

$$\Delta\hat{x}(t) = \left\{\frac{1}{2N_0(t)}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\frac{1}{\bar{u}_2^{3/2}}Q_0(\bar{u}_1,\bar{u}_2;t)\,d\bar{u}_1\,d\bar{u}_2\right\}^{1/2}, \tag{4.3}$$

$$\Delta\hat{p}(t) = \left\{\frac{1}{2N_0(t)}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\frac{\bar{u}_1^2+\bar{u}_2^2}{\bar{u}_2^{3/2}}Q_0(\bar{u}_1,\bar{u}_2;t)\,d\bar{u}_1\,d\bar{u}_2\right\}^{1/2}. \tag{4.4}$$

The product of dispersions for the operators at time point t_0, when the interaction with the environment is switch on describes the standard Hisenberg relation $[\Delta\hat{x}(t)\Delta\hat{p}(t)]\big|_{t=t_0} = 1/2$. It is interesting to calculate the uncertainty relation for large time values in the equilibrium limit. By averaging $t \to \infty$ we obtain:

$$\Delta\hat{x}_{st}\Delta\hat{p}_{st} = \lim_{t\to+\infty}\left[\Delta\hat{x}(t)\Delta\hat{p}(t)\right] = \frac{1}{2}\frac{\sqrt{A_x(\lambda)A_p(\lambda)}}{N_0^{st}(\lambda)}, \tag{4.5}$$

where the following notations have been made:

$$A_x(\lambda) = \int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\frac{1}{\bar{u}_2^{3/2}}Q_0^{st}(\lambda;\bar{u}_1,\bar{u}_2)\,d\bar{u}_1\,d\bar{u}_2,$$

$$A_p(\lambda) = \int\limits_{-\infty}^{+\infty}\int\limits_{0}^{+\infty}\frac{\bar{u}_1^2+\bar{u}_2^2}{\bar{u}_2^{3/2}}Q_0^{st}(\lambda;\bar{u}_1,\bar{u}_2)\,d\bar{u}_1\,d\bar{u}_2. \tag{4.6}$$

where $Q_0^{st}(\lambda;\bar{u}_1,\bar{u}_2)$ is the stationary limit of solution of equation (3.11) at $\alpha = 0$.

It is easy to check that the relations (4.5)-(4.6) differ substantially from the Hisenberg uncertainty relations. In particular, it allows to control the fundamental relation (4.5) using the power parameter λ, which characterizes the fluctuations of the environment.

Definition 4.1. *We call the expression*

$$W_{stc}(p,x,t;\{\xi\}) = \sum_{m=0}^{\infty} w_0^{(m)} W_{stc}(m|p,x,t;\{\xi\}), \tag{4.7}$$

stochastic Winger function and correspondingly $W_{stc}(m|p,x,t;\{\xi\})$ the partial stochastic Winger function. In particular, for the partial stochastic Winger function we get:

$$W_{stc}(m|p,x,t;\{\xi\}) = \int_{-\infty}^{+\infty} e^{ipv}\Psi_{stc}(m|(x-v/2),t;\{\xi\})\Psi^*_{stc}(m|(x+v/2),t;\{\xi\})\,dv, \tag{4.8}$$

Using the stochastic Winger function it is possible to calculate the mean values of the physical quantities, which correspond to the operators $\hat{A}$:

$$\bar{A} = \bar{a} \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Sp_{\{\xi\}}\left\{a(p,x,t;\{\xi\})\rho^W_{stc}(p,x,t;\{\xi\})\right\}dp\,dx, \tag{4.9}$$

where stochastic function $a(p,x,t;\{\xi\})$ is defined by Weyl transformation of operator $\hat{A}$:

$$a(p,x,t;\{\xi\}) = \int_{-\infty}^{+\infty} e^{ipv}\Psi_{stc}(m|(x-v/2),t;\{\xi\})\hat{A}\Psi^*_{stc}(m|(x+v/2),t;\{\xi\})\,dv. \tag{4.10}$$

Note that the Weyl transformation of quantum operator $\hat{A}$ after averaging over the random process $Sp_{\{\xi\}}\{a(p,x,t;\{\xi\})\}$ can be used in order to obtain the classical quantity corresponding to the quantum operator $\hat{A}$. This function is the classical limit $\hbar \to 0$ (the system of units $\hbar = 1$ and $c = 1$ is used, so the limit $\hbar \to 0$ actually corresponds to infinite action limit):

$$\hat{A} \to \mathrm{cla}_{cl}(p,x,t) = \lim_{\hbar\to 0} Sp_{\{\xi\}}\left\{a(p,x,t;\{\xi\})\right\}.$$

Now we can construct the Winger function for the ground state:

$$W^0(x,p;t) = Sp_{\{\xi\}}\left\{W_{stc}(0|p,x,t;\{\xi\})\right\} =$$

$$2\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\frac{1}{\sqrt{\bar{u}_2}}\exp\left\{-\frac{(p-\bar{u}_1 x)^2-\bar{u}_2^2x^2}{\bar{u}_2}\right\}Q(\bar{u}_1,\bar{u}_2;t)\,d\bar{u}_1 d\bar{u}_2. \tag{4.11}$$

It is easy to see that function (4.11) make sense coordinate x and momentum p simultaneous distribution in the phase space at the time t. Particularly with the help of this expression may be investigated the relaxation processes and average distribution in the limit of stationary processes. Note that such as in the regular case the integration of function $W_{stc}(n|p,x,t;\{\xi\})$ by phase space is normalized per unit:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_{stc}(n|p,x,t;\{\xi\})\,dpdx = 1. \tag{4.12}$$

Finally it is important to note that for averaging function (4.11) it is not to hold expression type of (4.12).

Conclusion

There are three different reasons which may cause a chaos in the basic quantum mechanical object, i.e. the wave function. The first reason refers to measurements performed over a quantum system [4, 5]. The second reason consists in the more fundamental openness of any quantum system resulting from the fact that all the beings are immersed into a physical vacuum [14]. In the third place, as it follows from the recent papers [6, 7, 8], a chaos may also appear in the wave function even in a closed dynamical system. Many of the fundamentally important questions of the quantum physics such as the Lamb shift of energy levels, spontaneous transitions between the atom levels, quantum Zeno effect [15], processes of chaos and self-organization in quantum systems, especially those where the phenomena of phase transitions type may occur, can be described qualitatively and quantitatively in a rigorous way only within the nonperturbative approaches. The Lindblad representation [16, 17] for the density matrix of the system "quantum object + thermostat" describes *a priori* the most general situation which may appear in the nonrelativistic quantum mechanics. Nevertheless, we need to consider a reduced density matrix on a semi-group [4], when investigating a quantum subsystem. This is quite an ambiguous procedure and moreover its technical realization is possible only in the framework of a particular perturbation scheme.

A crucially new approach to constructing the quantum mechanics of the closed nonrelativistic system "quantum object + thermostat" has been developed recently by the authors of [6, 7] from the principle of "local correctness of Schrödinger representation". To put it differently, it has been assumed that the evolution of the quantum system is such that it may be described by the Schrödinger equation on any small time interval, while the motion as a whole is described by a SDE for the wave function. In this case, however, there emerges not a simple problem to find a measure for calculating the average values of the physical system parameters. Nevertheless, there exists a certain class of models for which all the derivations can be made not applying the perturbation theory [7].

In the present paper we explore further the possibility of building the nonrelativistic quantum mechanics of closed system "quantum object + thermostat" within the framework of the model of one-dimensional randomly wandering QHO (with a random frequency and subjected to a random external force). Mathematically the problem is formulated in terms of SDE for a complex-valued probability process defined on the extended space $R^1 \otimes R_{\{\xi\}}$. The initial SDE is reduced to the Schrödinger equation for an autonomous oscillator defined on a random space-time continuum, with the use of a nonlinear transformation and one-dimensional etalon nonlinear equation of the Langevin type defined on the functional space $R_{\{\xi\}}$. It is possible to find for any fixed $\{\xi\}$ an orthonormal basis of complex-valued random functionals in the space $L_2(R^1)$ of square-integrable functions. With the assumption that the random force generator is described by a white noise correlator, the Fokker-Planck equation for a conditional probability is found. From the solutions of this equation on an infinitely small time interval a measure of the functional space $R_{\{\xi\}}$ can be constructed. Then by averaging an instantaneous value of the transition probability over the space $R_{\{\xi\}}$, the mean value of the transition probability is represented by a functional integral. Using the generalized Feynman-Kac theorem, it is possible to reduce the functional integral in the most general case, where both frequency and force are random, to a multiple integral of the fundamental solution of some parabolic partial differential equation. The qualitative analysis of the parabolic equation shows that it may have discontinuous solutions. This is equivalent to the existence of phenomena like the phase transitions in the microscopic transition probabilities.

In the context of the developed approach the representation of the stochastic density matrix is introduced, which allows to build a closed scheme for both nonequilibrium and equilibrium thermodynamics. The analytic formulas for the ground energy level broadening and shift are obtained, as well as for the entropy of the ground quantum state. The important results of the work are calculation of expressions for generalized quantum entropy, uncertain relations and Wigner function for the quantum subsystem which is strong interacting with the environment.

The further development of the considered formalism in application to exactly solvable many-dimensional models may essentially extend our understanding of the quantum world and lead us to the new nontrivial discoveries.

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Պատահական քվանտային ընթացքները և նրանց ղեկավարման հնարավորությունները

Ա. Գևորգյան

Ամփոփում

Մարումն ու դեկոհերենտությունը, փոխազդեցությունը թերմոստատի կամ ընդհանուր դեպքում ֆիզիկական վակումի հետ, չափումները և բաց քվանտային համակարգերի այլ բարդ պրոբլեմները, վերջին հաշվով քվանտային համակարգերի նրանց շրջակայքի հետ փոխազդեցության արդյունք են: Այս պրոբլեմները մաթեմատիկորեն նկարագրվում են կոմպլեքս հավանականային ընթացքների լեզվով (ԿՊԸ): Մասնավորապես նկարագրելով շրջակայքը, ինչպես Մարկովյան ընթացք, քվանտային համակարգի շրջակայքի հետ փոխազդեցության նկարագրության համար արտածել ենք Լանժեվեն-Շրեդինգեր տիպի պատահական դիֆերենցիալ հավասարում (ՊԴՀ). Ցույց է տրված, որ պատահական 1D քվանտային ներդաշնակ տատանակը (ՔՆՏ) Լ.-Շրեդ հավասարման շրջանակներում ունի լուծումներ՝ օրթոգոնալ ԿՀԸ տեսքով: Օրթոգոնալ ԿՀԸ-ի հենքի վրա զարգացված է պատահական խտության մատրիցայի մեթոդ (ՊԽՄ), որի շրջանակներում կատարված է անհաշվելի չափողականության փակ համակարգի ՆՔՆՏ+Շրջակայքե հետազոտությունը: ՊԽՄ մեթոդի օգնությամբ թերմոդինամիկական պոտենցիալները՝ նման ոչհավասարակշիռ էնտրոպիայի և հիմնական վիճակի էներգիայի, ճշգրիտ կառուցված են: Հաշված են օպերատորների դիսպերսիաները: Մասնավորապես անորոշությունների առնչությունների համար, կախված ՔՆՏ-ի շրջակայքի հետ փոխազդեցության պարամետրից, ստացված են արտահայտություններ: Որոշված է Վեյլի ձևափոխությունը պատահական օպերատորների համար: Հիմնական վիճակի Վիգների ֆունկցիան ուսումնասիրված է մանրամասն: