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On Restriction Optimal Fixpoints

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Abstract

Optimal fixpoints extract maximum consistent information from recursive programs. However, optimal fixpoints, although they always exist for a recursive operator, aren't necessarily computable.

We have introduced a modified notion of the optimal fixpoint, where the recursive operators are restricted to computable inputs. Existence results for and properties of this fixpoint are summarized in the article.

1 Definitions and Comments

The notation and comments differ somewhat from [2]. We explicitly deal with the restriction of recursive operator to computable inputs instead of introducing the notion of an optimal fixpoint of an extensional function.

Definition 1 *We say that partial function f precedes g symbolically, $f \preceq g$), if*

$$\forall x(g(x)\downarrow \Rightarrow (f(x)\downarrow \,\&\, g(x) = f(x))).$$

The following definitions are taken from [1].

Definition 2 *Partial functions f and g are called consistent if*

$$\forall x((f(x)\downarrow \ \&\ g(x)\downarrow) \Rightarrow f(x) = g(x)).$$

A set of functions is called consistent, if any two functions in the set are consistent.

For a consistent set A of functions we can define a *join*, i.e. a function f such that $f(x) = y \Leftrightarrow \exists g \in A$ and $g(x) = y$.

The set of fixpoints of a recursive operator Φ will be denoted as $Fix(\Phi)$. $Fix(\Phi)$ is not, generally speaking, consistent.

The set of computable fixpoints of a recursive operator Φ will be denoted as $CFix(\Phi)$:

Definition 3 *A fixpoint of Φ is called fxp-consistent if it is consistent with all the functions in $Fix(\Phi)$: A fixpoint is called fxc-consistent if it is consistent with all the functions in $CFix(\Phi)$:*

The set of all fxp-consistent fixpoints of a recursive operator will be denoted as $Fixc(\Phi)$: Similarly, the set of all fxc-consistent fixpoints will be denoted as $CFixc(\Phi)$:

Definition 4 *If the set of fxp-consistent fixpoints of* Φ *has a greatest element, it is called the* optimal fixpoint *of* Φ*:*

Theorem 1 (Z. Manna, and A. Shamir, [1]) *Every recursive operator has an optimal fixpoint.*

Thus, optimal fixpoint, like the least fixpoint, is uniquely determined for a recursive operator. Generally speaking, it is not computable.

Definition 5 *If the set of fxc-consistent fixpoints of* Φ *has a greatest element, it is called the* restriction optimal fixpoint *of* Φ*:*

In the sections below we will discuss the existence and behavior of the restriction optimal fixpoint.

2 Existence of restriction optimal fixpoint

In [2], the following propositions have been proved (formulated in terms of optimal fixpoints of extensional functions):

Proposition 2 *There exists a recursive operator that does not have a restriction optimal fixpoint.*

Proof. The proof uses a variant of "search operator", defined as follows:

$$\Phi[f](< x, y >) = \begin{cases} 1, & \text{if T(x, x, y);} \\ 2 * f(< x, y+1 >), & \text{otherwise.} \end{cases} \tag{1}$$

The operator essentially checks sequential step counts to see if the $\phi_x(x)$ converges. For $x \in K$, this reduces to

$$\Phi[f](< x, y >) = 2^{y_0 - y} \tag{2}$$

where y_0 is the exact step count at which $\phi_x(x)$ terminates. For $x \notin K$, we obtain

$$\Phi[f](< x, y >) = 2 * f(< x, y+1 >) \tag{3}$$

The solutions set for this has two elements: $\lambda y.0$ and $\lambda y.\bot$. The set of fixpoints is formed by a cartesian product over all x. $Fixc(\Phi) = Fix(\Phi)$. The greatest element is the function that has 0 for all $x \in \bar{K}$ and it is uncomputable.

Every computable fixpoint is formed by selecting a recursively enumerable subset of $\bar{K}$ and assuming the function 0 for $< x, y >$ pairs where x is in that subset.

There is no greatest recursively enumerable subset of $\bar{K}$ (there aren't even any maximal recursively enumerable subsets, since $\bar{K}$ is productive). This concludes the proof. ■

An uncomputable optimal fixpoint doesn't mandate having a large set of computable fixpoints (and lacking greatest computable element). We modify the definition of 1 as follows:

$$\Phi[f](<x,y>) = \begin{cases} 1 * defaround[f](<x,y>), & \text{if T(x, x, y);} \\ 2 * f(<x,y+1>) * defaround[f](<x,y>), & \\ \text{otherwise.} & \end{cases} \quad (4)$$

where *defaround* is a recursive operator thus defined:

$$defaround[f](x) = def[f](x+1) * def[f](x-1) \quad (5)$$

Every value of a fixpoint of Φ is only defined if two of its neighboring values are defined. So any fixpoint is either total or totally undefined (which now always is a fixpoint). So we have a total uncomputable (optimal) fixpoint and a totally undefined restriction optimal fixpoint.

Summarizing the behavior of uncomputable optimal fixpoints when we restrict the operator to computable functions, we conclude that the the greatest elements of consistent sets either disappear or give way to a computable restriction optimal fixpoint.

Several existence properties for restricted optimal fixpoints are given below.

Proposition 3 *If all the uncomputable fixpoints of recursive operator* Φ *belong to* $Fixc(\Phi)$, *then*

$$CFixc(\Phi) = Fixc(\Phi) \bigcap P \quad (6)$$

where P *is the set of all computable functions.*

Proof. When we restrict a recursive operator, the new set of fixpoints is the subset of the former. Therefore, the set of computable fixpoints consistent with all others can only become larger. Thus:

$$Fixc(\Phi) \bigcap \mathbf{P} \subset CFixc(\Phi) \quad (7)$$

We now show the opposite inclusion. Pick a function f from $CFixc(\Phi)$. Assuming $f \notin Fixc(\Phi)$, we conclude that f is inconsistent with an uncomputable fixpoint g. But $g \in Fixc(\Phi)$ according to the premise, so it is consistent with all fixpoints. The contradiction means that $f \in Fixc(\Phi)$: ■

Corollary 4 *If all the uncomputable fixpoints of* Φ *belong to* $Fixc(\Phi)$ *and the optimal fixpoint is computable, then restriction optimal fixpoint exists and coincides with the optimal fixpoint.*

Proof. According to the previous proposition, $CFixc(\Phi) = Fixc(\Phi) \bigcap \mathbf{P}$. Therefore, the greatest elements of these two sets coincide. ■

Proposition 5 *If the join of the set* $CFixc(\Phi)$ *is a computable function, then restriction optimal fixpoint exists.*

The proof uses a mix of ideas from the proofs of the first recursion theorem and the theorem about existence of optimal fixpoints.

Corollary 6 *If the set of functions in* $CFixc(\Phi)$ *is enumerable, then the restriction optimal fixpoint exists.*

Corollary 7 *If the optimal fixpoint is computable and can't be extended to an uncomputable function, then the restriction optimal fixpoint exists.*

Proof. The optimal fixpoint is in $Cfixc(\Phi)$, the join is equal to it. Whatever new functions get added to the set $Cfixc(\Phi)$, the join can only increase, and can not become uncomputable. ■

In fact a stronger statement is valid: when the computable optimal fixpoint can't be extended to an uncomputable function, the optimal and restricted optimal fixpoints coincide.

3 Ordering of optimal fixpoints

What are the possible ordering relations between the optimal fixpoint and the restriction optimal fixpoint?

Proposition 8 *If the restriction optimal fixpoint and the optimal fixpoint are computable, then the latter precedes the former.*

Proof. Consider the set $Fixc(\Phi)\bigcap \mathbf{P}$: We have that:

$$Fixc(\Phi)\bigcap \mathbf{P} \subset CFixc(\Phi) \tag{8}$$

The optimal fixpoint, since it is computable, belongs to $Fixc(\Phi)\bigcap\mathbf{P}$. Therefore, it must precede the greatest element of $CFixc(\Phi)$: ■

The case of uncomputable optimal fixpoint is different. The restricted optimal fixpoint may precede the optimal fixpoint (as in the case of operator 4). By definition, the optimal fixpoint is consistent with the restricted optimal fixpoint. However:

Proposition 9 *The optimal fixpoint and restricted optimal fixpoint of a recursive operator can generally be incomparable.*

Proof. The construction method used in this proof will be used later as well. Given two recursive operators, we can consider them acting simultaneously on a pair of functions, producing another pair. The set of fixpoints of the operator pair is the cartesian product of the Fix sets for the two operators. It is clearly possible to construct a single operator that acts on a disjoint union of functions and has the same behavior as the pair.

However, considering simple cartesian products of fixpoint sets isn't always useful, as their properties are straightforward: the optimal fixpoint is the pair of optimal fixpoints of the component operators, etc.

We introduce a kind of interaction whereby the fixpoints of one operator depend on the fixpoints of the other. This interaction is unidirectional, i.e. the first operator acts as normal, whereas the second operator gets as a parameter the function passed to the first operator. The behavior of the second operator thus depends on both input functions. When applied to fixpoints, this means that the second operator can adjust its set of fixpoints by checking properties of the fixpoints of the first operator.

The following example will clarify the construction.

$$\Psi[f,g](x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \text{ and } g(x) = 0; \\ 1, & \text{if } x = 0,\ g(x) = 1 \text{ and } def[f](r). \end{cases} \tag{9}$$

The main function here is g, the auxiliary parameter is f. The operator is recursive in their disjoint union. r is an arbitrary fixed number.

The fixpoints are defined as g such as $g = \Psi[f,g]$. Those are:

- $g_1 = \lambda x.0$;
- $g_2(x) = \begin{cases} 0, & \text{when } x \neq 0; \\ \perp, & \text{when } x = 0. \end{cases}$
- if f is defined at r, then also the following function:

$$g_3(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases} \tag{10}$$

Note that g_1 and g_3 are not consistent. We now use operator 4 as the first in the pair, and Ψ as defined above as the second. Ψ has two fixpoints: the undefined one and the totally uncomputable one. The undefined one (denoted Λ) pairs only with g_1 and g_2, whereas the total one (denoted Δ: with all three. The resultant set of fixpoints is:

- $h_1 =< \Lambda, g_1 >$
- $h_2 =< \Lambda, g_2 >$
- $h_3 =< \Delta, g_1 >$
- $h_4 =< \Delta, g_2 >$
- $h_5 =< \Delta, g_3 >$

The optimal fixpoint is h_4, it is uncomputable. Only h_1 and h_2 are computable, and the restricted optimal fixpoint is h_2. The two are incomparable.

■

A more elaborate construction shows that an uncomputable optimal fixpoint can precede the restriction optimal fixpoint as well.

4 Computable optimal fixpoints and restriction optimal fixpoints

We further consider the behavior of restricted optimal fixpoint in the presence of computable optimal fixpoint. The two optimal fixpoints can coincide (e.g. when there is only one fixpoint). The following proposition shows that they can also differ.

Proposition 10 *There exists a recursive operator such that its optimal fixpoint is computable, the restriction optimal fixpoint exists, but they differ.*

Proof. Fix r_0 and s_0 from $\bar{K}$ and denote $r =< r_0, 0 >$, $s < s_0, 0 >$. Consider the following operator:

$$\Phi[f](r) = \begin{cases} 1, & \text{if } f(r) = 1; \\ 0 * def[f](s), & \text{otherwise.} \end{cases} \tag{11}$$

For other points, Φ is defined as follows:

$$\Phi[f](< x, y >) = \begin{cases} def[f](r), & \text{if } T(x, x, y); \\ 2 * f(< x, y + 1 >) * def[f](r), & \text{otherwise.} \end{cases} \tag{12}$$

The definition apparently mimics the construction of 1. We now define a new operator Ψ: $\Psi[f](x) = \Phi[f](x) * defaround_r[f](x)$.

$defaround_r$ is a modification of the $defaround$ we used before. Like $defaround$, it makes each point dependent on its neighbors, but skips over r both as argument and as a neighbor.

The result is that any fixpoint of Ψ is either undefined outside r or total except possibly for r. For fixpoints total outside r we have $\Psi[f] = \Phi[f]$, so such fixpoints coincide with those of Φ. Fixpoints undefined outside r are those functions that keep r fixed.

Fixpoints in the latter category are:

- $f_1 = \lambda x.\bot$
- $f_2 = \lambda x. \begin{cases} 1, & \text{if } x = r; \\ \bot, & \text{otherwise.} \end{cases}$

The fixpoint can't be 0 in this case as it is undefined in s.

There are also two fixpoints in the first category. They are total, uncomputable and coincide outside r. One of them is 0 in r (we denote it by f_3), the other is 1 (we denote it f_4).

Thus the set of fixpoints has 4 elements. f_3 is not consistent with f_4 and f_2, so the optimal fixpoint is f_1 (computable). $CFix$ includes only f_1 and f_2 and the restricted optimal fixpoint is f_2. ■

Proposition 2 states that the restricted optimal fixpoint may not exist. However, the construction of the proof uses an operator that has an uncomputable optimal fixpoint. We ask whether the restricted optimal fixpoint may not exist if the optimal fixpoint is computable.

Proposition 11 *There exists a recursive operator with computable optimal fixpoint that does not have an optimal fixpoint.*

Proof. We will construct a witness recursive operator.

To do this, we combine two operator constructions from previous proofs.

Ψ operator from the previous proof has four fixpoints (denoted f_1 to f_4). We will also reuse the r point mentioned there.

Φ operator (1) has its all fixpoints consistent. They fall into two classes:

- U - uncomputable functions. This class has a greatest element.
- C - computable functions. There is no greatest element in this class.

We define a new operator in terms of Ψ: $\Omega[f,g]\Psi[g] * def[f](r)$. Here, we again use a parameter to modify the set of fixpoints of Ψ. We now combine Φ with Ω. This results in the following set of fixpoints:

- $S_1 = < f_1, \lambda.\bot >$
- $S_2 = < f_2, U >$
- $S_3 = < f_2, C >$
- $S_4 = < f_3, U >$
- $S_5 = < f_3, C >$

- $S_6 = < f_4, U >$
- $S_7 = < f_4, C >$

Only S_1 and S_3 consist of computable functions. Inconsistencies between f_2, f_3 and f_4 means $Fixc = S_1$. $CFixc = S_1 \cup S_3$.

The optimal fixpoint is totally undefined. The restriction optimal fixpoint doesn't exist. ■

5 Complexity of fixpoint-related sets

Since we deal with computable functions, it is natural to consider the corresponding sets of Gödel numbers and their complexity in the arithmetical hierarchy.

The following main results have been obtained in this regard.

Theorem 12 *The set* $CFXP = \{ < x, y > \mid \phi_y$ *is an element of the set of fxc-consistent fixpoints of recursive operator determined by enumeration operator* $\Phi_x\}$ *is* Π_3*-complete.*

Theorem 13 *The set* $CGFP = \{ < x, y > \mid \phi_y$ *is the greatest element of the set of fxc-consistent fixpoints of recursive operator determined by enumeration operator* $\Phi_x\}$ *is* Π_3*-complete.*

Theorem 14 *The set* $COPT = \{ < x, y > \mid \phi_y$ *is the restricted optimal fixpoint of recursive operator determined by enumeration operator* $\Phi_x\}$ *is* Π_4*-complete.*

We omit the proofs of these statements here as they are rather lengthy.

References

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Սահմանափակման օպտիմալ անշարժ կետերի մասին

Վ. Մարգարյան

Ամփոփում

Օպտիմալ անշարժ կետերը առավելագույն ինֆորմացիան են պարունակում ռեկուրսիվ ծրագրի մասին: Սակայն դրանք կարող են լինել ոչ հաշվարկելի: Մենք ներմուծել ենք օպտիմալության փոփոխված գաղափար, որի դեպքում դիտարկվում է միայն հաշվարկելի ֆունկցիաների վրա օպերատորի գործողությունը: Հոդվածում ներկայացված են այն անշարժ կետի գոյության եվ հատկությունների վերաբերյալ արդյունքների ամփոփում: