Mathematical Problems of Computer Science 25, 2006, 53–56.

Interval Colourings of Some Regular Graphs

Rafayel R. Kamalian, Petros A. Petrosyan

Institue for Informatics and Automation Problems of NAS of RA
e-mails rrkamalian@yahoo.com, pet_petros@yahoo.com

Abstract

A lower bound is obtained for the greatest possible number of colors in an interval colourings of some regular graphs.

Let $G = (V, E)$ be an undirected graph without loops and multiple edges [1], $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$, the maximum degree of a vertex of G-by $\Delta(G)$, and the chromatic index [2] of G-by $\chi'(G)$. A graph is regular, if all its vertices have the same degree. If α is a proper edge colouring of the graph G [3], then the color of an edge $e \in E(G)$ in the colouring α is denoted by $\alpha(e, G)$, and by $\alpha(e)$ if from the context it is clear to which graph it refers. For a proper edge colouring α, the set of colors of the edges that are incident to a vertex $x \in V(G)$, is denoted by $S(x, \alpha)$.

A proper colouring α of edges of G with colors $1, 2, \dots, t$ is interval [4], if for each color $i, 1 \leq i \leq t$, there exists at least one edge $e_i \in E(G)$ with $\alpha(e_i) = i$ and the edges incident with each vertex $x \in V(G)$ are colored by $d_G(x)$ consecutive colors.

For $t \geq 1$ let $\mathcal{N}_t$ denote the set of graphs which have an interval t-colouring, and assume: $\mathcal{N} \equiv \bigcup\limits_{t \geq 1} \mathcal{N}_t$. For $G \in \mathcal{N}$ the least and the greatest values of t, for which $G \in \mathcal{N}_t$, is denoted by $w(G)$ and $W(G)$, respectively.

In [5] it is proved:

Theorem 1. Let G be a regular graph.

1) $G \in \mathcal{N}$ iff $\chi'(G) = \Delta(G)$.

2) If $G \in \mathcal{N}$ and $\Delta(G) \leq t \leq W(G)$, then $G \in \mathcal{N}_t$.

[6] and theorem 1 imply that for regular graphs the problem of deciding whether $G \in \mathcal{N}$ or $G \notin \mathcal{N}$, is NP-complete [7,8].

In this paper we will consider regular graphs $G = (V, E)$, where

$$V(G) = \left\{x_j^{(i)} \mid 1 \leq i \leq k, 1 \leq j \leq n\right\},$$
$$E(G) = \left\{\left(x_p^{(i)}, x_q^{(i+1)}\right) \mid 1 \leq i \leq k-1, 1 \leq p \leq n, 1 \leq q \leq n\right\} \cup$$
$$\cup \left\{\left(x_p^{(k)}, x_q^{(1)}\right) \mid 1 \leq p \leq n, 1 \leq q \leq n\right\}, \; k \geq 3.$$

It is not hard to see that $\Delta(G) = 2n$. Let $\mathcal{R}(n, k)$ be the set of all those graphs.

In [9] it is shown that if $G \in \mathcal{R}(n, k)$ then

$$\chi'(G) = \begin{cases} 2n, & \text{if } n \cdot k \text{ is even,} \\ 2n+1, & \text{if } n \cdot k \text{ is odd.} \end{cases}$$

Theorem 1 implies:

Corollary 1. Let $G \in \mathcal{R}(n,k)$. Then:
1) $G \in \mathcal{N}$, if $n \cdot k$-is even;
2) $G \notin \mathcal{N}$, if $n \cdot k$-is odd.

Corollary 2. If $G \in \mathcal{R}(n,k)$ and $n \cdot k$-is even, then $w(G) = 2n$.

Theorem 2. If $G \in \mathcal{R}(n,k)$ and k-is even, then $W(G) \geq 2n + \frac{n \cdot k}{2} - 1$.

Proof. Suppose:

$$V(G) = \left\{x_j^{(i)} | 1 \leq i \leq k, 1 \leq j \leq n\right\},$$
$$E(G) = \left\{\left(x_p^{(i)}, x_q^{(i+1)}\right) \mid 1 \leq i \leq k-1, 1 \leq p \leq n, 1 \leq q \leq n\right\} \cup$$
$$\cup \left\{\left(x_p^{(k)}, x_q^{(1)}\right) \mid 1 \leq p \leq n, 1 \leq q \leq n\right\}.$$

Let G_1 be the subgraph of the graph G induced by the vertices $x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}, x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$. It is clear that $G_1$is a regular complete bipartite graph of the degree n. Therefore $\chi'(G_1) = \Delta(G_1) = n$, and due to theorem 1, $G_1 \in \mathcal{N}$.

Consider a proper edge colouring α of the edges of G_1 defined as follows:

$$\alpha\left(\left(x_p^{(k)}, x_q^{(1)}\right)\right) = p + q - 1 \text{ for } p = 1, 2, \dots, n \text{ and } q = 1, 2, \dots, n .$$

It is not hard to check that α is an interval $(2n-1)$-colouring of the graph G_1.

Define an edge colouring β of the graph G in the following way:

1) $\beta\left(\left(x_p^{(k)}, x_q^{(1)}\right), G\right) = \alpha\left(\left(x_p^{(k)}, x_q^{(1)}\right), G_1\right)$ for $p = 1, 2, \dots, n$ and $q = 1, 2, \dots, n$;

2) for $i = 1, 2, \dots, \frac{k}{2} - 1$ and $p = 1, 2, \dots, n$, $q = 1, 2, \dots, n$

$$\beta\left(\left(x_p^{(i)}, x_q^{(i+1)}\right), G\right) = \beta\left(\left(x_p^{(k-i)}, x_q^{(k-i+1)}\right), G\right) = \alpha\left(\left(x_p^{(k)}, x_q^{(1)}\right), G_1\right) + i \cdot n;$$

3) $\beta\left(\left(x_p^{(\frac{k}{2})}, x_q^{(\frac{k}{2}+1)}\right), G\right) = \alpha\left(\left(x_p^{(k)}, x_q^{(1)}\right), G_1\right) + \frac{n \cdot k}{2}$ for $p = 1, 2, \dots, n$ and $q = 1, 2, \dots, n$.

Let us show that β is an interval $\left(2n + \frac{n \cdot k}{2} - 1\right)$-colouring of the graph G.

First of all note that for i, $1 \leq i \leq 2n-1$ there is an edge $e_i \in E(G)$ such that $\beta(e_i) = i$.

Now let us show that for j, $2n \leq j \leq 2n + \frac{n \cdot k}{2} - 1$ there is an edge $e_j \in E(G)$ with $\beta(e_j) = j$.

Consider the vertices $x_n^{(2)}, x_n^{(3)}, \dots, x_n^{(\frac{k}{2})}$. The definition of β implies that

$$\bigcup_{i=2}^{\frac{k}{2}} S\left(x_n^{(i)}, \beta\right) = \left\{2n, 2n+1, \dots, 2n + \frac{n \cdot k}{2} - 1\right\}.$$

This proves that for j, $2n \leq j \leq 2n + \frac{n \cdot k}{2} - 1$ there is an edge $e_j \in E(G)$ with $\beta(e_j) = j$.

Let us show that the edges that are incident to a vertex $v \in V(G)$ are colored by $2n$ consecutive colors.

Let $x_j^{(i)} \in V(G)$, $1 \leq i \leq k, 1 \leq j \leq n$.

Case 1: $i = 1, 1 \leq j \leq n$.

The definition of β implies that:

$$S\left(x_j^{(1)},\beta\right)=\left(\bigcup_{l=1}^{n}\beta\left(\left(x_l^{(k)},x_j^{(1)}\right)\right)\right)\cup\left(\bigcup_{m=1}^{n}\beta\left(\left(x_j^{(1)},x_m^{(2)}\right)\right)\right)=$$
$$=\left(\bigcup_{l=1}^{n}\alpha\left(\left(x_l^{(k)},x_j^{(1)}\right),G_1\right)\right)\cup\left(\bigcup_{m=1}^{n}\left(\alpha\left(\left(x_j^{(k)},x_m^{(1)}\right),G_1\right)+n\right)\right)=$$
$$=\{j,j+1,...,j+n-1\}\cup\{j+n,j+n+1,...,j+2n-1\}=\{j,j+1,...,j+2n-1\}.$$

Case 2: $2\leq i\leq k-1, 1\leq j\leq n$.
The definition of β implies that:

$$S\left(x_j^{(i)},\beta\right)=\left(\bigcup_{l=1}^{n}\beta\left(\left(x_l^{(i-1)},x_j^{(i)}\right)\right)\right)\cup\left(\bigcup_{m=1}^{n}\beta\left(\left(x_j^{(i)},x_m^{(i+1)}\right)\right)\right)=$$
$$=\left(\bigcup_{l=1}^{n}\left(\alpha\left(\left(x_l^{(k)},x_j^{(1)}\right),G_1\right)+(i-1)\cdot n\right)\right)\cup\left(\bigcup_{m=1}^{n}\left(\alpha\left(\left(x_j^{(k)},x_m^{(1)}\right),G_1\right)+i\cdot n\right)\right)=$$
$$=\{j+(i-1)\cdot n,...,j+i\cdot n-1\}\cup\{j+i\cdot n,...,j+(i+1)\cdot n-1\}=$$
$$=\{j+(i-1)\cdot n,...,j+(i+1)\cdot n-1\}.$$

Case 3: $i=k, 1\leq j\leq n$.
The definition of β implies that:

$$S\left(x_j^{(k)},\beta\right)=\left(\bigcup_{l=1}^{n}\beta\left(\left(x_l^{(k-1)},x_j^{(k)}\right)\right)\right)\cup\left(\bigcup_{m=1}^{n}\beta\left(\left(x_j^{(k)},x_m^{(1)}\right)\right)\right)=$$
$$=\left(\bigcup_{l=1}^{n}\left(\alpha\left(\left(x_l^{(k)},x_j^{(1)}\right),G_1\right)+n\right)\right)\cup\left(\bigcup_{m=1}^{n}\alpha\left(\left(x_j^{(k)},x_m^{(1)}\right),G_1\right)\right)=$$
$$=\{j+n,j+n+1,...,j+2n-1\}\cup\{j,j+1,...,j+n-1\}=\{j,j+1,...,j+2n-1\}.$$

Theorem 2 is proved.

Corollary 3. If $G\in\mathcal{R}(n,k)$, k-is even and $2n\leq t\leq 2n+\frac{n\cdot k}{2}-1$, then $G\in\mathcal{N}_t$.

Let us note that if $G\in\mathcal{R}(n,4)$, then the lower bound of the proved theorem is the exact value of $W(G)$, that is $W(G)=4n-1$.

References

[1] F. Harary, "Graph Theory", Addison-Wesley, Reading, MA,1969.

[2] V.G. Vizing, The chromatic index of a multigraph, *Kibernetika* 3 (1965), pp. 29-39.

[3] A.A. Zykov, Theory of finite graphs, Novosibirsk, Nauka, 1969.

[4] A.S. Asratian, R.R. Kamalian, Interval colourings of edges of a multigraph, *Appl. Math.* 5 (1987), Yerevan State University, pp. 25-34.

[5] R.R. Kamalian, "Interval Edge colourings of Graphs", Doctoral dissertation, The Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, 1990.

[6] S.V. Sevastianov, On interval colorability of edges of a bipartite graph, *Methods of Discr. Anal.* In solution of extremal problems. The Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, N50 (1990), pp. 61-72.

[7] I. Holyer, The *NP*-completeness of edge colouring, *SIAM J. Comput.* 10, N4 (1981), pp. 718-720.

[8] S. Cook, The complexity of theorem-proving procedures. In *Proc.3rd ACM Symp.* on Theory of Computing, 1971, pp. 151-158.

[9] R.M. Karp, Reducibility among Combinatorial Problems, in "Complexity of Computer Computations" (R.E. Miller and J.W. Thatcher, Eds.), New York, 1972, pp. 85-103.

[10] E. T. Parker, Edge-colouring numbers of some regular graphs, *Proc. Amer. Math. Soc.* 37, (1973), pp. 423-424.

Որոշ համասեռ գրաֆների միջակայքային ներկումներ

Ռ. Քամալյան, Պ. Պետրոսյան

Ամփոփում

Աշխատանքում ստացված է ստորին գնահատական որոշ համասեռ գրաֆների միջակայքային ներկման մեջ օգտագործվող գույների առավելագույն հնարավոր թվի համար: