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An Inequality Related to the Pairs of Matchings of a Graph

Rafayel R. Kamalian*, Vahan V. Mkrtchyan**

*Institute for Informatics and Automation Problems of NAS RA
*Department of Informatics and Applied Mathematics, Yerevan State University
e-mails rrkamalian@yahoo.com, vahanmkrtchyan2002@yahoo.com

Abstract

For a given graph disjoint pairs of matchings the union of which contains as many edges as possible are considered. It is shown that the relation of the cardinality of a maximum matching to the cardinality of the largest matching in those pairs does not exceed 3/2. A conjecture is posed which states that this coefficient can be replaced by 5/4 . Finally, a family of graphs is presented which shows that the abovementioned coefficient can not be replaced by a constant which is smaller than 5/4.

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The cardinality of a maximum matching of G will be denoted by $\beta(G)$. Define $\nu(G)$ as

$$\nu(G) \equiv \max\{|H| + |H'| \,/\, H, H' \text{ are disjoint matchings of } G\}.$$

Assume:

$$\alpha(G) \equiv \max\{|H|, |H'| \,/\, H, H' \text{ are disjoint matchings of } G \text{ such that } |H| + |H'| = \nu(G)\}.$$

It is clear, that for every graph G the following inequality is true:

$$\alpha(G) \leq \beta(G), \text{ or equivalently, } 1 \leq \beta(G)/\alpha(G) .$$

If we consider a matching covered tree G, i.e. every edge of G belongs to a maximum matching, then it can be shown that the equality $\beta(G)/\alpha(G) = 1$ holds [1]. Complete characterization of these trees can be found in [2,3].

A natural question that arises here is the following: how large the relation $\beta(G)/\alpha(G)$ can be for an arbitrary graph G? In this paper we show that for every graph G the inequality $\beta(G)/\alpha(G) \leq 3/2$ holds. After that a family $\{G_n\}$ of graphs is presented which satisfies the equality $\lim\limits_{n\to\infty} \beta(G_n)/\alpha(G_n) = 5/4$. This shows that the abovementioned constant 3/2 can not be replaced by a constant which is smaller than 5/4 no matter how large the graph is. Finally, in the end of the paper, we pose a conjecture which states that for any graph G the inequality $\beta(G)/\alpha(G) \leq 5/4$ is true. Non defined terms and conceptions can be found in [2,4,5].

Theorem. For every graph G the inequality $\beta(G)/\alpha(G) \le 3/2$ holds.

Proof. Let F be a maximum matching of the graph G. Among pairs (H, H') of disjoint matchings of G with $|H| + |H'| = \nu(G)$ and $|H| = \alpha(G)$, choose one which satisfies the condition

$$|F \cap (H \cup H')| \to \max.$$

It is not hard to see that there are $|F| - |H| = \beta(G) - \alpha(G)$ alternating paths $w_0, (w_0, w_1), w_1, ..., w_{2l-2}, (w_{2l-2}, w_{2l-1}), w_{2l-1}$ of the graph G with odd length such that

$$\{(w_0, w_1), ..., (w_{2l-2}, w_{2l-1})\} \subseteq F, \ \{(w_1, w_2), ..., (w_{2l-3}, w_{2l-2})\} \subseteq H.$$

The choice of the pair (H, H') implies that for each such a path we have:

$$l \ge 3, \ \{(w_0, w_1), (w_{2l-2}, w_{2l-1})\} \subseteq H', \ \{(w_0, w_1), ..., (w_{2l-2}, w_{2l-1})\} \not\subseteq H'.$$

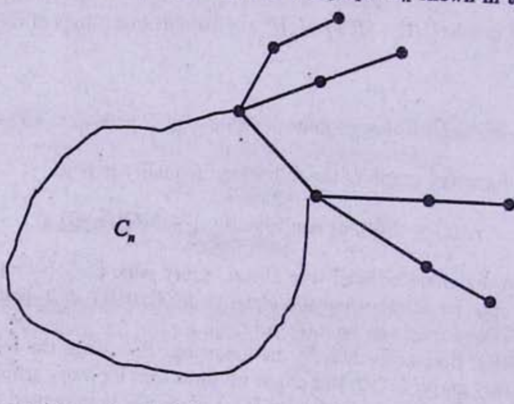
Clearly, these properties imply that

$$2(\beta(G) - \alpha(G)) \le |H'| \le \alpha(G), \text{ or}$$
$$\beta(G) \le 3\alpha(G)/2.$$

The proof of the Theorem is complete.

A natural question that arises at this point is the following: can the coefficient 3/2 in the above inequality be replaced by a smaller one? The following example shows that no matter how large the graph is, 3/2 can not be replaced by a constant which is smaller than 5/4.

For every natural number n, $n \ge 3$ consider the graph G_n shown in the figure below:



G_n $(n \ge 3)$

where C_n is the simple cycle of the length n. Note that

$$\beta(G_n) = \begin{cases} 5k+2, & \text{if } n = 2k+1, k \ge 1, \\ 5k, & \text{if } n = 2k, k \ge 2. \end{cases}$$

On the other hand,

$$\alpha(G_n) = \begin{cases} 4k+2, & \text{if } n = 2k+1, k \geq 1, \\ 4k, & \text{if } n = 2k, k \geq 2. \end{cases}$$

Therefore,

$$\lim_{n\to\infty} \beta(G_n)/\alpha(G_n) = 5/4.$$

It seems reasonable to pose the following
Conjecture: For every graph G the inequality $\frac{\beta(G)}{\alpha(G)} \leq \frac{5}{4}$ is true.

References

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Գրաֆում զուգակցումների զույգերին առնչվող մի անհավասարության մասին

Ռ. Քամալյան, Վ. Մկրտչյան

Ամփոփում

Դիտարկվել են գրաֆի չհատվող զուգակցումների այն զույգերը, որոնց միավորումը պարունակում է հնարավորին չափ շատ կող: Ցույց է տրվել, որ գրաֆի մաքսիմալ զուգակցման հզորության հարաբերությունը այդ զույգերում ամենաշատ թվով կողեր պարունակող զուգակցման հզորությանը չի գերազանցում 3/2-ը: Առաջարկվել է վարկած, համաձայն որի այս գործակիցը կարելի է փոխարինել 5/4-ով: Գտնվել է գրաֆների մի ընտանիք, որը ցույց է տալիս, որ որքան էլ մեծ լինի գրաֆը, վերոհիշյալ գործակիցը հնարավոր չէ փոխարինել 5/4-ից փոքր թվով: