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Interval Edge Colourings of Complete Graphs and n-cubes

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Abstract

For complete graphs and n-cubes bounds are found for the possible number of colours in an interval edge colourings.

Let $G = (V, E)$ be an undirected graph without loops and multiple edges [1], $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$, the maximum degree of a vertex of G-by $\Delta(G)$, and the chromatic index of G-by $\chi'(G)$ and the diameter of G-by $d(G)$. A graph is regular, if all its vertices have the same degree. If α is a proper edge colouring [2] of the graph G, then $\alpha(e)$ denotes the colour of an edge $e \in E(G)$ in the colouring α. For a proper edge colouring α of the graph G and for any $x \in V(G)$ we denote by $S(x, \alpha)$ the set of colours of edges incident with x.

An interval [3] t–colouring of G is a proper colouring of edges of G with colours $1, 2, \ldots, t$ such that at least one edge of G is coloured by colour $i, 1 \leq i \leq t$, and the edges incident with each vertex $x \in V(G)$ are coloured by $d_G(x)$ consecutive colours.

A graph G is interval-colourable if there is $t \geq 1$ for which G has an interval t–colouring. The set of all interval-colourable graphs is denoted by $\mathcal{N}$ [4].

For $G \in \mathcal{N}$ we denote by $w(G)$ and $W(G)$, respectively, the least and greatest value of t, for which G has an interval t–colouring.

The problem of deciding whether or not a bipartite graph belongs to $\mathcal{N}$ was shown in [5] to be NP-complete [6,7].

It was proved [4] that if G has no triangle and $G \in \mathcal{N}$ then $W(G) \leq |V(G)| - 1$. It follows from here that if G is bipartite and $G \in \mathcal{N}$ then $W(G) \leq |V(G)| - 1$.

Theorem 1 [8]. If G is a bipartite graph and $G \in \mathcal{N}$, then $W(G) \leq d(G)(\Delta(G) - 1) + 1$.

For graphs which can contain a triangle the following results hold:

Theorem 2 [4]. If $G \in \mathcal{N}$ is a graph with nonempty set of edges then $W(G) \leq 2|V(G)| - 3$.

Theorem 3 [9]. If $G \in \mathcal{N}$ and $|V(G)| \geq 3$ then $W(G) \leq 2|V(G)| - 4$.

In [4] there was proved the following

Theorem 4. Let G be a regular graph.

1) $G \in \mathcal{N}$ iff $\chi'(G) = \Delta(G)$.

2) If $G \in \mathcal{N}$ and $\Delta(G) \leq t \leq W(G)$ then G has an interval t–colouring.

From Theorem 4 and the result of [10] it follows that the problem "Does a given regular graph belongs to the set $\mathcal{N}$ or not?" is NP-complete.

In this paper interval edge colourings of complete graphs and n-cubes are investigated. Non-defined conceptions and notations can be found in [1,2,4,8].

From the results of [11] and Theorem 4 it follows that for any odd p $K_p \notin \mathcal{N}$. It's easy to see that $\chi'(K_{2n}) = \Delta(K_{2n}) = 2n-1$ [1] and, therefore for any $n \in N$ $K_{2n} \in \mathcal{N}$, $w(K_{2n}) = 2n-1$.

Theorem 5 [12]. For any $n \in N$ $W(K_{2n}) \geq 3n-2$.

Theorem 6. Let $n = p2^q$, where p is odd and q is nonnegative integer. Then $W(K_{2n}) \geq 4n-2-p-q$.

Proof. Let's prove that for any $m \in N$ $W(K_{4m}) - W(K_{2m}) \geq 4m-1$.

Consider a graph K_{4m} with $V(K_{4m}) = \{x_1, x_2, ..., x_{4m}\}$ and $E(K_{4m}) = \{(x_i, x_j) |\ x_i \in V(K_{4m}),\ x_j \in V(K_{4m}),\ i < j\}$.

Let G be a subgraph of the graph K_{4m}, induced by its vertices $x_1, x_2, ..., x_{2m}$. Evidently G is isomorphic to the graph K_{2m} and, consequently, there exists an interval $W(K_{2m})$–colouring α of G.

Now we define an edge colouring β of K_{4m}.

For $i = 1, 2, ..., 4m$ and $j = 1, 2, ..., 4m$, where $i \neq j$, we set:

$$\beta((x_i, x_j)) = \begin{cases} \alpha((x_i, x_j)) & \text{if } 1 \leq i \leq 2m,\ 1 \leq j \leq 2m; \\ \min S(x_i, \alpha) + 2m - 1 & \text{if } 1 \leq i \leq 2m,\ 2m+1 \leq j \leq 4m,\ i = j-2m; \\ \alpha((x_i, x_{j-2m})) + 2m & \text{if } 1 \leq i \leq 2m,\ 2m+1 \leq j \leq 4m,\ i \neq j-2m; \\ \alpha((x_{i-2m}, x_{j-2m})) + 4m - 1 & \text{if } 2m+1 \leq i \leq 4m,\ 2m+1 \leq j \leq 4m. \end{cases}$$

It is not difficult to see that β is an interval $(W(K_{2m}) + 4m - 1)$–colouring of K_{4m}. Now we can conclude:

$$W(K_{p2^{q+1}}) \geq W(K_{p2^q}) + p2^{q+1} - 1$$
$$W(K_{p2^q}) \geq W(K_{p2^{q-1}}) + p2^q - 1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$W(K_{p2^2}) \geq W(K_{p2}) + p2^2 - 1$$

Adding these inequalities we obtain

$$W(K_{2n}) \geq W(K_{2p}) + p\sum_{i=2}^{q+1} 2^i - q.$$

Now, using the result of Theorem 5, we have

$$W(K_{2n}) \geq 3p - 2 - q + p\sum_{i=2}^{q+1} 2^i = 3p - 2 - q + 4p(2^q - 1) = 4n - 2 - p - q.$$

The proof is complete.

Corollary 1. Let $n = p2^q$, where p is odd and q is nonnegative integer. If $2n-1 \leq t \leq 4n-2-p-q$ then there exists an interval t–colouring of K_{2n}.

Lemma. For any $n \in N$ $Q_n \in \mathcal{N}$ and $w(Q_n) = n$.

Proof. As for any $n \in N$ Q_n is a regular bipartite graph then $\chi'(Q_n) = \Delta(Q_n) = n$ and, from Theorem4, $Q_n \in \mathcal{N}$, $w(Q_n) = n$.

Theorem 7. For any $n \in N$ $W(Q_n) \geq \frac{n(n+1)}{2}$.

Proof. Let's prove that for $n \geq 2$ $W(Q_n) - W(Q_{n-1}) \geq n$.

Evidently, $Q_n = K_2 \times Q_{n-1}$, therefore there are two subgraphs $Q^{(1)}_{n-1}$ and $Q^{(2)}_{n-1}$ of Q_n, which satisfy conditions:

$$V(Q^{(1)}_{n-1}) \cap V(Q^{(2)}_{n-1}) = \emptyset,$$
$$Q^{(i)}_{n-1} \text{is isomorphic to } Q_{n-1},\ i = 1, 2.$$

It follows from Lemma that for $i = 1, 2$ $Q^{(i)}_{n-1} \in \mathcal{N}$.

Evidently $Q^{(1)}_{n-1}$ is isomorphic to $Q^{(2)}_{n-1}$, therefore there exists a bijection $f : V(Q^{(1)}_{n-1}) \longrightarrow V(Q^{(2)}_{n-1})$ such that $(x, y) \in E(Q^{(1)}_{n-1})$ iff $(f(x), f(y)) \in E(Q^{(2)}_{n-1})$. Let α be an interval $W(Q^{(1)}_{n-1})-$colouring of the graph $Q^{(1)}_{n-1}$.

Let's define an edge colouring β of the graph $Q^{(2)}_{n-1}$ in the following way: for every edge $(u, v) \in E(Q^{(2)}_{n-1})$ $\beta((u, v)) = \alpha((f^{-1}(u), f^{-1}(v))) + n$.

Now we define an edge colouring γ of the graph Q_n.

For every edge $(x, y) \in E(Q_n)$

$$\gamma((x,y)) = \begin{cases} \alpha((x,y)) & \text{if } x \in V(Q^{(1)}_{n-1}),\ y \in V(Q^{(1)}_{n-1}); \\ \min S(x,\alpha) + n - 1 & \text{if } x \in V(Q^{(1)}_{n-1}),\ y \in V(Q^{(2)}_{n-1}) \text{ and } y = f(x); \\ \beta((x,y)) & \text{if } x \in V(Q^{(2)}_{n-1}),\ y \in V(Q^{(2)}_{n-1}). \end{cases}$$

It is not difficult to see that γ is an interval $(W(Q_{n-1}) + n)-$colouring of Q_n.

For $n \geq 2$ we have

$$W(Q_n) \geq W(Q_{n-1}) + n$$
$$W(Q_{n-1}) \geq W(Q_{n-2}) + n - 1$$
$$\dots\dots\dots\dots\dots\dots$$
$$W(Q_2) \geq W(Q_1) + 2$$

Adding these inequalities we obtain $W(Q_n) \geq \frac{n(n+1)}{2}$.

The proof is complete.

Corollary 2. If $n \leq t \leq \frac{n(n+1)}{2}$ then Q_n has an interval $t-$colouring.

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Լրիվ գրաֆների և n-չափանի խորանարդների միջակայքային կողային ներկումներ

Պ. Պետրոսյան

Ամփոփում

Լրիվ գրաֆների և n-չափանի խորանարդների համար ստացված են գնահատականներ միջակայքային կողային ներկման մեջ օգտագործվող գույների հնարավոր թվի համար: