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Analysis of Case Splitting in an Arithmetical System

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Abstract

Investigations in this paper concern the analysis of case splitting $[(\neg A \to B) \to (A \to B) \to B]$ in an arithmetical system. It is shown that the case splitting can be done according to a thicker class of formulas (quantifier-free formulas) instead of decidable formulas. The latter includes the class of quantifier-free formulas. Some approaches to the case splitting, promoted by other authors, have been investigated, and some corrections concerning the selection of quantifier-free formulas instead of decidable formulas have been done. According to these corrections, a new derivation of case splitting is suggested.

Introduction to minimal logic and arithmetic (Necessary Definitions)

At first, let us give required definitions of notions in minimal logic. Thereby, providing essential foundation, further we shall extend our considerations to arithmetic. Let us first fix our language L. *Types* are built from ground types (ι - for the natural numbers and o - for the boolean objects) by the operations $\rho \to \sigma$ and $\rho \times \sigma$. For any type ρ let a countable infinite set of variables of type ρ (denoted by $x^\rho, y^\rho, \ldots$) and a set C of constants of type ρ (denoted by c^ρ) be given.

Terms and their types are defined inductively by

$$\forall x^\rho, c^\rho \in Terms$$

$$r^{\rho\to\sigma},\ s^\rho \in Terms \Rightarrow (rs)^\sigma \in Terms$$

$$r^\sigma \in Terms \Rightarrow (\lambda x^\rho r)^{\rho\to\sigma} \in Terms$$

$$t_0^{\rho_0}, t_1^{\rho_1} \in Terms \Rightarrow \langle t_0, t_1\rangle^{\rho_0\times\rho_1} \in Terms$$

$$t^{\rho_0\times\rho_1} \in Terms \ \Rightarrow \ (\pi_i(t))^{\rho_i} \in Terms,\ \ i \in \{0,1\}$$

Since any term has a unique normal form with respect to $\beta\eta$- conversion, in the sequel we will identify terms with the same $\beta\eta$- normal form. Recall the definitions of β and η reductions

$$\beta = \{ \, ((\lambda x.t)\tau,\ t[x := \tau]) \, \},\ \eta = \{ \, (\lambda x.\, Mx, M) \ / \ x \notin FV(M) \, \}.$$

The set $FV(r)$ of *free variables* of a term r is defined as usual.

Assume that a set P of predicate symbols R of arities $\rho_1, \ldots, \rho_n$ is given. 0-ary relation symbols are called propositional symbols. *Formulas* are defined by:

If $t_1^{\rho_1}, \ldots, t_n^{\rho_n}$ are terms and $R \in P$ is a relation symbol of arity $\rho_1, \ldots, \rho_n$, then $R(t_1, \ldots, t_n)$ is a formula.

$\bot$(to be read "falsity") is a formula.

If A and B are formulas, then $A \to B$ is a formula.

If A and B are formulas, then $A \wedge B$ is a formula.

If A is a formula and x^ρ is a variable, then $\forall x^\rho A$ is a formula.

$R(t_1, \ldots, t_n)$and $\bot$ are called *atomic formulas* or *atoms* (further we will return to this notion).

A term t is called closed, if $FV(t) = \emptyset$. We write $t[x_1, \ldots, x_n]$ to indicate that $x_1, \ldots, x_n$ compose the list of all free variables in t.

Derivations are within minimal logic.

For any L-formula A let countably many *assumption variables* of type A be given (denoted by u^A, v^A, ...). The notions of a *derivation term* d^A in minimal logic and its set $FA(d^A)$ of *free assumption variables* are defined inductively by

(A)	u^A is a derivation term with $FA(u^A) = \{u^A\}$.
$(\to^+)$	If d^B is a derivation term, then $(\lambda u^A d^B)^{A \to B}$ is a derivation term with $FA(\lambda u^A d^B) = FA(d^B) \backslash \{u^A\}$.
$(\to^-)$	If $d^{A \to B}$ and e^A are derivation terms, then $(d^{A \to B} e^A)^B$ is a derivation term with $FA(d^{A \to B} e^A) = FA(d^{A \to B}) \cup FA(e^A)$.
$(\wedge^+)$	If d^A and e^B are derivation terms, then $\langle d^A, e^B \rangle^{A \wedge B}$ is a derivation term with $FA(\langle d^A, e^B \rangle^{A \wedge B}) = FA(d^A) \cup FA(e^B)$.
$(\wedge^-)$	If $d^{A_0 \wedge A_1}$ is a derivation term, then $\pi_i(d^{A_0 \wedge A_1})^{A_i}$ is a derivation term with $FA(\pi_i(d^{A_0 \wedge A_1})^{A_i}) = FA(d^{A_0 \wedge A_1})$, $i \in \{0, 1\}$..
$(\forall^+)$	If d^A is a derivation term and $x^\rho \notin \cup \left\{ FV(B) \mid u^B \in FA(d^A) \right\}$, then $(\lambda x^\rho d^A)^{\forall x^\rho A}$ is a derivation term with $FA(\lambda x^\rho d^A) = FA(d^A)$.
$(\forall^-)$	If $d^{\forall x^\rho A}$ is a derivation term and t^ρ is a term, then $(d^{\forall x^\rho A} t^\rho)^{A[t/x^\rho]}$ is a derivation term with $FA(d^{\forall x^\rho A} t^\rho) = FA(d^{\forall x^\rho A})$.

A term d^A is called closed, if $FA(d^A) = \emptyset$. We write $d^B[u_1^{A_1}, \ldots, u_n^{A_n}]$ to indicate that $u_1^{A_1}, \ldots, u_n^{A_n}$ compose the list of all free assumption variables in d^A. Further we also use the notation $d : A$ instead of d^A.

For any derivation d we define its set $FV(d)$ of *free (object) variables* by

$$FV(u^A) := FV(A) \quad ,$$

$$FV(\lambda u^A d^B) := FV(A) \cup FV(d^B) \quad ,$$

$$FV(d^{A\to B}e^{A}) := FV(d^{A\to B}) \cup FV(e^{A}) \quad ,$$

$$FV(\langle d^{A}, e^{B}\rangle) := FV(d^{A}) \cup FV(e^{B}) \quad ,$$

$$FV(\pi_i(d^{A\wedge B})) := FV(d^{A\wedge B}) \quad ,$$

$$FV(\lambda x d^{A}) := FV(d^{A})\setminus\{x\} \quad ,$$

$$FV(d^{\forall x A}t) := FV(d^{\forall x A}) \cup FV(t).$$

Two kinds of substitution are provided for derivation terms d: we can substitute a derivation term f^A for a free assumption variable u^A, denoted $d[f/u]$; we can substitute an object term t for a free object variable x, denoted $d[t/x]$.

Negation and the existential quantifier are defined by

$$\neg A := A \to \bot,$$
$$\exists x\, A := \neg\forall x \neg\, A.$$

Derivation terms in *intuitionistic* and in *classical* logic are obtained by adding to the first (assumption-) clause of the definition:

in the case of intuitionistic logic: For any $R \in P$ relation symbol $\mathrm{Efq}_R : \forall \vec{x}.\bot \to R(\vec{x})$ is a derivation term with $FA(\mathrm{Efq}_R) = \emptyset$ (**Ex-falso-quodlibet axiom**);
in the case of classical logic: For any $R \in P$ relation symbol $\mathrm{Stab}_R : \forall \vec{x}.\neg\neg R(\vec{x}) \to R(\vec{x})$ is a derivation term with $FA(\mathrm{Stab}_R) = \emptyset$ (**Stability axiom**).

Hence, from the assumptions given above, we can prove: for any relation symbol R occurring in a formula A we can derive $\neg\neg A \to A$ and $\bot \to A$ in classical and intuitionistic logic, respectively.

From $\neg\neg A \to A$ one can clearly derive $\bot \to A$. Therefore, any formula derivable in intuitionistic logic is also derivable in classical logic.

Let us now extend our L-language by a strong existential quantifier (the term a constructive existential quantifier is also used in literature) written $\exists^*$ (as opposed to $\exists$ defined by $\neg\forall\neg$). There are two approaches to deal with formulas containing $\exists^*$ in a constructive setting (e.g. in minimal or intuitionistic logic): Weyl's approach and Heyting's approach [3].

In this paper we consider only the Weyl's approach, that is:
a formula containing $\exists^*$ is considered not to be an entity the deduction system can deal with: some "realizing terms" are required to turn it into a "judgment".

Let us now describe Weyl's approach. To every formula A and terms $\vec{r} = r_1^{\rho_1}, \ldots, r_m^{\rho_m}$ we associate a judgment $\vec{r}$ mr A (to be read $\vec{r}$ modified realizes A), which will be a formula not containing $\exists^*$. The list of types $\rho_1, \ldots, \rho_m = \tau(A)$ is defined as follows: (by ε is denoted the empty list)

$$\tau(R(\vec{t})) := \varepsilon \qquad \text{(inparticular } \tau(\bot) = \varepsilon) \tag{1}$$

$$\tau(A\to B) := \begin{cases} \tau(B) & \text{if } \tau(A)=\varepsilon \\ \varepsilon & \text{if } \tau(B)=\varepsilon \\ \tau(A)\to\tau(B) & \text{otherwise} \end{cases} \tag{2}$$

$$\tau(A\wedge B) := \begin{cases} \tau(A) & \text{if } \tau(B)=\varepsilon \\ \tau(B) & \text{if } \tau(A)=\varepsilon \\ \tau(A)\times\tau(B) & \text{otherwise} \end{cases} \tag{3}$$

$$\tau(\forall x^\rho A) := \begin{cases} \varepsilon & \text{if } \tau(A)=\varepsilon \\ \rho\to\tau(A) & \text{otherwise} \end{cases} \tag{4}$$

$$\tau(\exists^* x^\rho A) := \begin{cases} \rho & \text{if } \tau(A)=\varepsilon \\ \rho\times\tau(A) & \text{otherwise} \end{cases} \tag{5}$$

Now judgments $\vec{r}^{\tau(A)}$ mr A are defined by

$$\begin{aligned} \varepsilon \operatorname{mr} R(\vec{t}) &:= R(\vec{t})\,, \\ r_1,\dots,r_n \operatorname{mr} (A\to B) &:= \forall\vec{x}.\vec{x} \operatorname{mr} A \to r_1\vec{x},\dots,r_n\vec{x} \operatorname{mr} B\,, \\ \vec{r},\ \vec{s} \operatorname{mr} (A\wedge B) &:= \vec{r} \operatorname{mr} A \ \wedge\ \vec{s} \operatorname{mr} B\,, \\ r_1,\dots,r_n \operatorname{mr} \forall x^\rho B &:= \forall x^\rho r_1 x,\dots,r_n x \operatorname{mr} B\,, \\ r,\ \vec{s} \operatorname{mr} \exists^* x^\rho B &:= \vec{s} \operatorname{mr} B[r/x]\,. \end{aligned}$$

Assume that to any assumption variable u^B we have assigned a list $\vec{x}_u^{\tau(B)} = x_{u_1}^{\rho_1},\dots,x_{u_n}^{\rho_n}$ of distinct variables, where $\rho_1,\dots,\rho_n = \tau(B)$. Relative to this assignment we define for any derivation d^A its extracted terms ets (d^A), by induction on d^A. If $\tau(A)=\sigma_1,\dots,\sigma_k$, then ets (d^A) will be a list $r_1^{\sigma_1},\dots,r_k^{\sigma_k}$.

$$\begin{aligned} \operatorname{ets}(u^A) &= \vec{x}_u^{\tau(A)}\,, \\ \operatorname{ets}(\lambda u^A d^B) &= \lambda\vec{x}_u^{\tau(A)}\operatorname{ets}(d^B)\,, \\ \operatorname{ets}(d^{A\to B}e^A) &= \operatorname{ets}(d)\operatorname{ets}(e)\,, \\ \operatorname{ets}(\langle d^A,\ e^B\rangle) &= \operatorname{ets}(d^A),\operatorname{ets}(e^B)\,, \\ \operatorname{ets}(\pi_0(d^{A\wedge B})) &= \left(\textit{the head of } \operatorname{ets}(d^{A\wedge B}) \textit{ of same length as } \tau(A)\right)\,, \\ \operatorname{ets}(\pi_1(d^{A\wedge B})) &= \left(\textit{the tail of } \operatorname{ets}(d^{A\wedge B}) \textit{ of same length as } \tau(B)\right)\,, \\ \operatorname{ets}(d^{\forall x^\rho A}t^\rho) &= \operatorname{ets}(d)\,t\,. \end{aligned} \tag{6}$$

Note that if ets $(d) = r_1,\dots,r_k$ and ets $(e) = \vec{s}$, then ets (d) ets $(e) = r_1\vec{s},\dots,r_k\vec{s}$ and $\lambda\vec{x}$ ets $(d) = \lambda\vec{x}r_1,\dots,\lambda\vec{x}r_k$.

Let us now extend these considerations to arithmetic. It is based on Gödel's system T and just adds the corresponding arithmetical apparatus to it. Here we identify terms with the same normal forms.

The constants are

$$\text{true}^o,\ \text{false}^o,\ 0^\iota,\ S^{\iota\to\iota},\ R_{o,\rho},\ R_{\iota,\rho}.$$

$R_{\iota,\rho}$ is the *primitive recursion operator* of type $\rho\to(\iota\to\rho\to\rho)\to\iota\to\rho$ and $R_{o,\rho}$ is the recursion operator for the type o of booleans, i.e. is of type $\rho\to\rho\to o\to\rho$ and represents definition by cases. Terms have already been defined at the beginning of the paper. We add the following *conversion rules* (writing $t+1$ for $S^{\iota\to\iota}t$).

$$
\begin{aligned}
&R_{\iota,\rho}\, r\, s\, 0 \to_R r,\\
&R_{\iota,\rho}\, r\, s\, (t+1) \to_R s\, t\, (R_{\iota,\rho}\, r\, s\, t),\\
&R_{o,\rho}\, r\, s\, \mathrm{true} \to_R r,\\
&R_{o,\rho}\, r\, s\, \mathrm{false} \to_R s.
\end{aligned}
$$

For this system of terms every term strongly normalized, and that the normal form is uniquely determined. By identifying $=_{\beta R}$-equal terms we can greatly simplify many formal derivations.

Let **atom** be an unary predicate symbol taking one argument of type o. The intended interpretation of **atom** is the set {true}; hence "**atom**(t)" means "t = true". Formulas are built from atomic formulas by means of $\to$, $\wedge$, $\forall$ and $\exists^*$. Recall that $\bot$ is considered as atomic formula, since it can be defined $\bot := \mathrm{atom}(flase)$.

Our *induction schemata* are the universal closures of

$$
\begin{aligned}
&A[0/n] \to (\forall n.A \to A[n+1/n]) \to \forall n A\ ,\\
&A[true/p] \to A[false/p] \to \forall p(A).
\end{aligned}
$$

We also extend the notion of a derivation term by constants for the "truth axiom" $\mathrm{ax}_{\mathrm{true}}$, induction axioms ($\mathrm{Ind}_{n,A}$ and $\mathrm{Ind}_{p,A}$) and $\mathrm{ax}_{\mathrm{false},A}$ axiom. Hence derivation terms in arithmetic are obtained by adding the clauses

$$
\begin{aligned}
&\mathrm{ax}_{\mathrm{true}} : \mathrm{atom}(true), \quad \text{with } FA(ax_{\mathrm{true}}) = \emptyset \qquad (T : \mathrm{atom}(true) \text{ form is also usable}),\\
&\mathrm{ax}_{\mathrm{false},A} : \mathrm{atom}(false) \to A, \quad \text{with } FA(ax_{\mathrm{false},A}) = \emptyset,\\
&\mathrm{Ind}_{n,A} : \forall.\, A[0/n] \to (\forall n.\, A \to A\,[n+1/n]) \to \forall n\, A, \quad \text{with } FA(\mathrm{Ind}_{n,A}) = \emptyset,\\
&\mathrm{Ind}_{p,A} : \forall.\, A\,[true/p] \to A\,[false/p] \to \forall p\, A, \quad \text{with } FA(\mathrm{Ind}_{p,A}) = \emptyset.
\end{aligned} \tag{7}
$$

Clearly $FV(T) := FV(\mathrm{ax}_{\mathrm{false},A}) := FV(\mathrm{Ind}_{n,A}) := FV(\mathrm{Ind}_{p,A}) = \emptyset$.

The notion of extracted terms can straightforwardly be extended to this situation. In the case of $\mathrm{Ind}_{n,A}$ we have to prove

$$\mathrm{ets}(\mathrm{Ind}_{n,A}) \text{ mr } \forall \vec{x}.\, A\,[0/n] \to (\forall n.\, A \to A\,[n+1/n]) \to \forall n\, A,$$

i.e.

$$
\begin{aligned}
\forall \vec{x} \forall \vec{y} \forall \vec{f} \forall n. \vec{y} \text{ mr } A[0/n] \to (\forall n \forall y_1. \vec{y}_1 \text{ mr } A \to \vec{f} n \vec{y}_1 \text{ mr } A[n+1/n]) \to \\
\to \mathrm{ets}(\mathrm{Ind}_{n,A}) \vec{x} \vec{y} \vec{f} n \text{ mr } A\ .
\end{aligned}
$$

Hence we let

$$\mathrm{ets}(\mathrm{Ind}_{n,A}) := \lambda \vec{x}. R_1, \ldots, R_k\ , \tag{8}$$

where k is the length of $\tau(A) \neq \varepsilon$ ($\tau(A) = \rho_1, \ldots, \rho_k$) and $R_1, \ldots, R_k$ are simultaneous primitive recursion operators of type $R_i : \vec{\rho} \to (\iota \to \vec{\rho} \to \vec{\rho}) \to \iota \to \rho_i$ satisfying

$$
\begin{aligned}
&R_i \vec{y} \vec{f}\, 0 = y_i\\
&R_i \vec{y} \vec{f}\, (z+1) = f_i\, z\, (R_1 \vec{y} \vec{f} z) \ldots (R_k \vec{y} \vec{f} z)
\end{aligned}
$$

where = denotes equality of $\beta\eta R$- normal forms. Using these equations the above claim will be easily proven (recall that terms with the same normal form are identified). The operators $R_1, \ldots, R_k$ can be defined from the recursion constant $R_{\iota, \rho_1 \times \cdots \times \rho_k}$.

Boolean induction (i.e. case analysis) is treated similarly. We let

$$\text{ets}(\text{Ind}_{p,A}) := \lambda\vec{x}.R_1, \ldots, R_k \ , \tag{9}$$

(here again $\tau(A) = \rho_1, \ldots, \rho_k \neq \varepsilon$), where now $R_1, \ldots, R_k$ are simultaneous primitive recursion (or case splitting) operators of type $R_i : \vec{\rho} \to \vec{\rho} \to o \to \rho_i$ satisfying

$$R_i : \vec{y}\,\vec{z} \text{ true} = y_i \ ,$$
$$R_i : \vec{y}\,\vec{z} \text{ false} = z_i \ .$$

2 Analysis of case splitting

Definition: We call a formula A decidable, if there is a term t_A such that $\vdash A \longleftrightarrow \text{atom}(t_A)$.

Remark: The following remark will be helpful later: every quantifier-free formula is decidable.

Let denote $\supset := \lambda p \lambda q.R\, q \text{ true } p$ and $\& := \lambda p \lambda q.R\, (R \text{ true false}\, q) \text{ false } p$. So, clearly

$$\forall p, q.(\text{atom}(p) \to \text{atom}(q)) \leftrightarrow \text{atom}(\supset pq)$$
$$\forall p, q.(\text{atom}(p) \ \wedge \ \text{atom}(q)) \leftrightarrow \text{atom}(\& pq)$$

are provable. Hence we let

$$t_{\text{atom}(r)} := r \ ,$$
$$t_{A \to B} := \supset t_A t_B \ ,$$
$$t_{A \wedge B} := \& \ t_A t_B \ .$$

It must be mentioned that the inverse, i.e. every decidable formula is quantifier-free, in general is not correct. For eaxample the formula $\forall x \exists y(x = y)$ is decidable, but not quantifier-free.

Lemma: *(Cases [2])* We can do case splitting according to a quantifier-free L-formula A, i.e. for every formula B the following takes place:

$$\vdash (\neg A \to B) \to (A \to B) \to B. \tag{10}$$

Proof:
Recall that $\bot = \text{atom}(false)$ and $\neg A = A \to \bot$.
We shall use Boolean Induction schema:

$$\text{Ind}_{p,C} : C\,[p/true] \to C\,[p/false] \to \forall p\, C. \tag{11}$$

Taking into account the fact, that every quantifier-free formula is decidable (see remark mentioned above), we can easily construct a boolean term t_A such, that $\vdash A \leftrightarrow \text{atom}(t_A)$ takes place. Hence it suffices to derive

$$\forall p.((\text{atom}(p) \to \text{atom}(false)) \to B) \to (\text{atom}(p) \to B) \to B.$$

This is done by boolean induction on p, using the truth axiom $\text{ax}_{\text{true}} : \text{atom}(true)$.

After we take $C = (\neg A \to B) \to (A \to B) \to B$, the Boolean Induction schema (formula) will look like this (recall that $\vdash A \leftrightarrow \text{atom}(t_A)$ and $\neg A \equiv A \to \text{atom}(false)$):

$$\begin{aligned} \text{Ind}_{p,C} \; : \; & ([(\text{atom}(true) \to \text{atom}(false)) \to B] \to (\text{atom}(true) \to B) \to B) \to \\ & \to ([(\text{atom}(false) \to \text{atom}(false)) \to B] \to (\text{atom}(false) \to B) \to B) \to \\ & \to \forall p. \underbrace{([(\text{atom}(p) \to \text{atom}(false)) \to B] \to (\text{atom}(p) \to B) \to B)}_{C} \end{aligned} \tag{12}$$

Thus, it is sufficient to show the following

$$\vdash [(\text{atom}(true) \to \text{atom}(false)) \to B] \to [(\text{atom}(true) \to B) \to B] \tag{13}$$

nd

$$\vdash [(\text{atom}(false) \to \text{atom}(false)) \to B] \to [(\text{atom}(false) \to B) \to B]. \tag{14}$$

So, let us prove these derivations (13 and 14), using the following axioms and derivation ule [1]

$$\begin{array}{ll} \text{axiom 1} : A \to (B \to A) & \\ \text{axiom 2} : (A \to B) \to [(A \to (B \to C)) \to (A \to C)] & \\ \text{ax}_{\text{true}} : \text{atom}(true) & \text{Truth axiom} \\ \text{ax}_{\text{false},A} : \text{atom}(false) \to A & \text{Ex-falso-quodlibet} \\ \text{m.p.} : \text{derivation rule - modus ponens} & \end{array} \tag{15}$$

The proof of the first one (see 13) is given below

1	$\vdash \text{atom}(true)$	truth axiom
2	$\vdash \text{atom}(true) \to [(\text{atom}(true) \to B) \to \text{atom}(true)]$	axiom 1
3	$\vdash (\text{atom}(true) \to B) \to \text{atom}(true)$	1,2 m.p.
	let denote $C \equiv \text{atom}(true) \to B$	
4	$\vdash C \to (C \to C)$	axiom 1
5	$\vdash (C \to (C \to C)) \to \{[C \to ((C \to C) \to C)] \to (C \to C)\}$	axiom 2
6	$\vdash [C \to ((C \to C) \to C)] \to (C \to C)$	4,5 m.p.
7	$\vdash C \to ((C \to C) \to C)$	axiom 1
8	$\vdash C \to C$	6,7 m.p.
	according to the denotation of C, the step 8 will have the following look	
9	$\vdash (\text{atom}(true) \to B) \to (\text{atom}(true) \to B)$	opened step 8
10	$\vdash [(\text{atom}(true) \to B) \to \text{atom}(true)] \to$ $\{[(\text{atom}(true) \to B) \to (\text{atom}(true) \to B)] \to$ $[(\text{atom}(true) \to B) \to B]\}$	axiom 2
11	$\vdash [(\text{atom}(true) \to B) \to (\text{atom}(true) \to B)] \to$ $[(\text{atom}(true) \to B) \to B]$	3,10 m.p.
12	$\vdash (\text{atom}(true) \to B) \to B$	9,11 m.p.
	let denote $D \equiv (\text{atom}(true) \to B) \to B$	
13	$\vdash D \to \{[(\text{atom}(true) \to \text{atom}(false)) \to B] \to D\}$	axiom 1

		12,13 m.p.
14	$\vdash [(\text{atom}(true) \to \text{atom}(false)) \to B] \to D$	
	according to the denotation of D, the step 14 will have the following look	
15	$\vdash [(\text{atom}(true) \to \text{atom}(false)) \to B] \to [(\text{atom}(true) \to B) \to B]$	opened step 14

The proof of the second one (see 14) is given below

	let denote $C \equiv [\text{atom}(false) \to \text{atom}(false)] \to B$	
1	$\vdash C \to (C \to C)$	axiom 1
2	$\vdash (C \to (C \to C)) \to \{[C \to ((C \to C) \to C)] \to (C \to C)\}$	axiom 2
3	$\vdash [C \to ((C \to C) \to C)] \to (C \to C)$	1,2 m.p.
4	$\vdash C \to ((C \to C) \to C)$	axiom 1
5	$\vdash C \to C$	3,4 m.p.
6	$\vdash \text{atom}(false) \to \text{atom}(false)$	$\text{ax}_{\text{false, atom}(false)}$
7	$\vdash (\text{atom}(false) \to \text{atom}(false)) \to$ $[C \to (\text{atom}(false) \to \text{atom}(false))]$	axiom 1
8	$\vdash C \to (\text{atom}(false) \to \text{atom}(false))$	6,7 m.p.
	according to the denotation of C, the step 5 will have the following look (only the right appearance of the C is restored)	
9	$\vdash C \to [(\text{atom}(false) \to \text{atom}(false)) \to B]$	semi opened step 5
10	$\vdash [C \to (\text{atom}(false) \to \text{atom}(false))] \to$ $\to \{[C \to ((\text{atom}(false) \to \text{atom}(false)) \to B)] \to (C \to B)\}$	axiom 2
11	$\vdash [C \to ((\text{atom}(false) \to \text{atom}(false)) \to B)] \to (C \to B)$	8,10 m.p.
12	$\vdash C \to B$	9,11 m.p.
13	$\vdash [B \to ((\text{atom}(false) \to B) \to B)] \to$ $\to \{C \to [B \to ((\text{atom}(false) \to B) \to B)]\}$	axiom 1
14	$\vdash B \to ((\text{atom}(false) \to B) \to B)$	axiom 1
15	$\vdash C \to [B \to ((\text{atom}(false) \to B) \to B)]$	13,14 m.p.
16	$\vdash (C \to B) \to \{[C \to [B \to ((\text{atom}(false) \to B) \to B)]] \to$ $\to [C \to ((\text{atom}(false) \to B) \to B)]\}$	axiom 2
17	$\vdash \{C \to [B \to ((\text{atom}(false) \to B) \to B)]\} \to$ $\to [C \to ((\text{atom}(false) \to B) \to B)]$	12,16 m.p.
18	$\vdash C \to ((\text{atom}(false) \to B) \to B)$	15,17 m.p.
	according to the denotation of C, the step 18 is given by the following expression	
19	$\vdash [(\text{atom}(false) \to \text{atom}(false)) \to B] \to [(\text{atom}(false) \to B) \to B]$	opened step

It is all that we had to show.

Let us now construct derivation terms corresponding to the derivations (13) and (14). Then, using the obtained result, we shall finally construct derivation term for case splitting.

We denote by $\mathbf{ax1}_{A,B} : A \to (B \to A)$ and $\mathbf{ax2}_{A,B,C} : (A \to B) \to [(A \to (B \to C)) \to (A \to C)]$ the derivation terms for axiom1 and axiom2 (see 15), respectively. Then, from derivation of $\vdash C \to C$ we obtain it's derivation term (let denote it by OI_C):

$$\text{OI}_C = [(\text{ax1}_{C,\,C}\ \text{ax2}_{C,\,C\to C,\,C})\ \text{ax1}_{C,\,C\to C}]^{C\to C}.$$

Now, the following term will be derivation term, which corresponds to the derivation (13) (writing p for $\text{atom}(p)$):

$$\mathbf{P1} \equiv \left\{[\mathrm{OI}_{true \to B}\,((\mathrm{T}^{true}\,\mathrm{ax1}_{true,\ true \to B})\,\mathrm{ax2}_{true \to B,\ true,\ B})]\ \mathrm{ax1}_{D,\ [(true \to false) \to B] \to D}\right\}^{E},$$

where $D \equiv (true \to B) \to B$ and $E \equiv [(true \to false) \to B] \to D$.

We denote this derivation term by **P1**: $[(true \to false) \to B] \to (true \to B) \to B$.

Similarly, we obtain a derivation term, which corresponds to the derivation (14).

$$\left[\mathrm{OI}_C\,((\mathrm{ax}_{false,\ false}\ \mathrm{ax1}_{false \to false,\ C})^{C \to (false \to false)}\,\mathrm{ax2}_{C,\ false \to false,\ B})\right]^{C \to B} \equiv \mathbf{K}\,,$$

so, derivation term for (14) is

$$\mathbf{P2} \equiv \left[(\mathbf{K}^{C \to B}\,\mathrm{ax2}_{C,\ B,\ (false \to B) \to B})\,(\mathrm{ax1}_{B \to (false \to B) \to B,\ C}\ \mathrm{ax1}_{B,\ false \to B})\right]^{C \to ((flase \to B) \to B)},$$

where $C \equiv (false \to false) \to B$ and $\mathrm{ax}_{false,\ false} = atom(false) \to atom(false)$ (see 7).

We denote this derivation term by **P2**: $[(false \to false) \to B] \to (flase \to B) \to B$.

Now, we can construct the resulting derivation term for case splitting:

$$((\mathrm{Ind}_{p,C}\ \mathbf{P1})\,\mathbf{P2})^{\forall p.C},$$

and recall, that from now we have $C(p) = [(\mathrm{atom}(p) \to \mathrm{atom}(false)) \to B] \to (\mathrm{atom}(p) \to B) \to B$ (see 12).

Taking into account the fact $\vdash A \leftrightarrow atom(t_A)$ we obtain

$$((\mathrm{Ind}_{t_A,\ C(t_A)}\,\mathbf{P1})\,\mathbf{P2}),$$

as a derivation for case splitting $[(\neg A \to B) \to (A \to B) \to B]$.

Now let us analyze the approach of case splitting included in [3].

We can do case splitting according to decidable formulas A, i.e. for every formula $B[\vec{x}]$ we can prove

$$Cases_{A,\ B} : (A \to B) \to (\neg A \to B) \to B. \tag{16}$$

The derivation $Cases_{A,B}$ is given by

$$\lambda u_1, u_2.\,\mathrm{Ind}\,\vec{x}(\lambda u_3 \lambda u_4.\,u_3\ \mathrm{T})(\lambda u_5 \lambda u_6.\,u_6 \neg\,\mathrm{F})\,t_A(\lambda u_7.\,u_1(d_1 u_7))(\lambda u_8.\,u_2(d_2 u_8)), \tag{17}$$

where $d_1^{\mathrm{atom}(t_A) \to A}$ and $d_2^{\neg\,\mathrm{atom}(t_A) \to \neg A}$ are derivations, which exist according to the remark (mentioned at the beginning of this section) and the axioms and assumption variables with indices are (writing t for $\mathrm{atom}(t)$)

$$\mathrm{Ind}_{p,(p \to B) \to (\neg p \to B) \to B} \equiv \mathrm{Ind}_{p,C_1}, \qquad C_1 \equiv (p \to B) \to (\neg p \to B) \to B\ , \tag{*}$$

and

$$u_1^{A \to B},\ u_2^{\neg A \to B},\ u_3^{\mathrm{atom}(true) \to B},\ u_4^{\neg\,\mathrm{atom}(true) \to B},$$

$$u_5^{\mathrm{atom}(false) \to B},\ u_6^{\neg\,\mathrm{atom}(false) \to B},\ u_7^{\mathrm{atom}(t_A)},\ u_8^{\neg\,\mathrm{atom}(t_A)}.$$

We denote by T, F, ¬T, ¬F the $\mathrm{T}^{\mathrm{atom}(true)}$, $\mathrm{F}^{\mathrm{atom}(false)}$, $\neg\mathrm{T}^{\neg\,\mathrm{atom}(true)}$, $\neg\mathrm{F}^{\neg\,\mathrm{atom}(false)}$ terms respectively.

At first, let us show that the selected $Cases_{A,B}$ term is correct, i.e. it serves as a derivation for the formula $(A \to B) \to (\neg A \to B) \to B$.

$$u_3^{\mathrm{atom}(true)\to B}\ \mathrm{T}^{\mathrm{atom}(true)} = (u_3\mathrm{T})^B \quad \Rightarrow$$

$$(\lambda u_4^{\neg\,\mathrm{atom}(true)\to B}.\,(u_3\mathrm{T})^B)^{(\neg\,\mathrm{atom}(true)\to B)\to B} \quad \Rightarrow$$

$$(\lambda u_3 \lambda u_4.\,u_3\mathrm{T})^{(\mathrm{atom}(true)\to B)\to(\neg\,\mathrm{atom}(true)\to B)\to B} \quad \Rightarrow$$

$$\lambda u_3 \lambda u_4.\,u_3\mathrm{T}:\ (\mathrm{atom}(true) \to B) \to (\neg\mathrm{atom}(true) \to B) \to B \tag{18}$$

Similarly

$$u_6^{\neg\,\mathrm{atom}(false)\to B}(\neg\mathrm{F})^{\neg\,\mathrm{atom}(false)} = (u_6\neg\mathrm{F})^B \quad \Rightarrow \quad \cdots \quad \Rightarrow$$

$$\lambda u_5 \lambda u_6.\,u_6\neg\,\mathrm{F}:\ (\mathrm{atom}(false) \to B) \to (\neg\,\mathrm{atom}(false) \to B) \to B \tag{19}$$

From (18) and with regard to the fact that $\mathrm{Ind}_{p,A}\ :\ \forall.\,A[true/p] \to A[false/p] \to \forall pA$ is a derivation term for any A formula, we get

$$\begin{aligned}\mathrm{Ind}_{p,C_1}\ :\ &\forall.\,(\,(\mathrm{atom}(true) \to B) \to (\neg\,\mathrm{atom}(true) \to B) \to B) \to\\ &\to ((\mathrm{atom}(false) \to B) \to (\neg\,\mathrm{atom}(false) \to B) \to B) \to\\ &\to \forall p\underbrace{((\mathrm{atom}(p) \to B) \to (\neg\mathrm{atom}(p) \to B) \to B[\bar{x}])}_{C_1}\end{aligned}$$

Using (17) and (18) and taking into account definition of the derivation term $d^{\forall x^p A}\,t^p$: $A[t/x]$, we get

$$\underbrace{\mathrm{Ind}_{p,C_1}\,\bar{x}\,(\lambda u_3\lambda u_4.\,u_3\mathrm{T})\,(\lambda u_5\lambda u_6.\,u_6\neg\mathrm{F})}_{\equiv C_2}\ :\ \forall p\, C_1[\bar{x}/\bar{x}] \quad ,$$

and let denote it by C_2.

So, we have $C_2 : \forall p.(\mathrm{atom}\,(p) \to B) \to (\neg\,\mathrm{atom}\,(p) \to B) \to B$, and taking into account the fact, that p^o, t_A^o both are of type o, then the following is correct

$$C_2\,t_A\ :\ (\mathrm{atom}(t_A) \to B) \to (\neg\,\mathrm{atom}(t_A) \to B) \to B\,. \tag{**}$$

$$\left.\begin{aligned}d_1^{\mathrm{atom}(t_A)\to A}\,u_7^{\mathrm{atom}(t_A)} = (d_1u_7)^A\\ u_1^{A\to B}\end{aligned}\right\} \ \Rightarrow\ (u_1(d_1u_7))^B \ \Rightarrow \lambda u_7.\,u_1(d_1u_7):\ \mathrm{atom}(t_A) \to B\,. \tag{20}$$

Similarly

$$d_2^{\neg \mathrm{atom}(t_A)\to\neg A}\, u_8^{\neg \mathrm{atom}(t_A)} = \left.\begin{matrix}(d_2 u_8)^{\neg A}\\ u_2^{\neg A\to B}\end{matrix}\right\} \Rightarrow (u_2(d_2 u_8))^B \Rightarrow \lambda u_8.u_2(d_2\, u_8) : \neg \mathrm{atom}(t_A) \to B. \tag{21}$$

Using obtained (20), (21) and (20), we get

$$\underbrace{C_2\, t_A\, (\lambda u_7.\, u_1(d_1 u_7))\, (\lambda u_8.\, u_2(d_2 u_8))}_{C_3} \;:\; B,$$

and let denote it by C_3.

So, $Cases_{A,B} = \lambda u_1, u_2.\, C_3$ and thus we get correctness for $Cases_{A,B}\ : (A \to B) \to (\neg A \to B) \to B$.

Let us now construct extracted terms for $Cases_{A,B}$ derivation term.

$$\left.\begin{matrix}\mathrm{ets}(\lambda u^A d^B) = \lambda \bar{x}_u^{\tau(A)}\, \mathrm{ets}\,(d^B)\\ Cases_{A,B} = \lambda u_1, u_2.\, C_3^B\end{matrix}\right\} \;\Rightarrow\; \mathrm{ets}(\,Cases_{A,B}) = \lambda \bar{y}, \bar{z}.\mathrm{ets}(C_3^B) \quad ,$$

where $\bar{y}_{u_1}^{\tau(A\to B)},\ \bar{z}_{u_2}^{\tau(\neg A\to B)}$.

By definition of the extracted terms for the $d^{\forall x^p A}\, t^p$ derivation term (see 6) we obtain

$$\mathrm{ets}(d^{\forall x^p A}\, t^p) = \mathrm{ets}(d)\, t \quad\Rightarrow\quad \mathrm{ets}(C_2^{\forall p C_1}\, t_A^o) = \mathrm{ets}(C_2)\; t_A. \tag{22}$$

After some notations we come to

$$\left.\begin{matrix}C_4 \equiv \lambda u_7.\, u_1(d_1 u_7) & \Rightarrow & C_4^{\mathrm{atom}(t_A)\to B}\\ C_5 \equiv \lambda u_8.\, u_2(d_2 u_8) & \Rightarrow & C_5^{\neg\,\mathrm{atom}(t_A)\to B}\end{matrix}\right\} \;\Rightarrow\; C_3 = C_2\, t_A C_4 C_5 \quad .$$

Recall that

$$\mathrm{ets}(d^{A\to B} e^A) = \mathrm{ets}(d^{A\to B})\, \mathrm{ets}(e^A) \tag{23}$$

by definition of extracted terms (see 6).

Applying twice the definition (23) of extracted terms in the case for $d^{A\to B}\, e^A$ (see 6), we get

$$\mathrm{ets}(\ (C_2\, t_A C_4) C_5\ ) = \mathrm{ets}(C_2\, t_A C_4)\, \mathrm{ets}(C_5) = \mathrm{ets}(C_2\, t_A)\, \mathrm{ets}(C_4)\, \mathrm{ets}(C_5) = \\ = \mathrm{ets}(C_2)\, t_A\, \mathrm{ets}(C_4)\, \mathrm{ets}(C_5)$$

Since $\tau(\mathrm{atom}(p)) = \varepsilon$ (see 1), from definition of $\tau(A \to B)$ (see 2) we can write

$$\mathrm{ets}(C_4^{\mathrm{atom}(t_A)\to B}) = \bar{y}_1^{\tau(\mathrm{atom}(t_A)\to B)} = \bar{y}_1^{\tau(B)}$$
$$\mathrm{ets}(C_5^{\neg\,\mathrm{atom}(t_A)\to B}) = \bar{z}_1^{\tau(\neg\,\mathrm{atom}(t_A)\to B)} = \bar{z}_1^{\tau(B)}$$

So, we have $\mathrm{ets}(C_3) = \mathrm{ets}(C_2)\, t_A\, \bar{y}_1\, \bar{z}_1$.

Assume that $\bar{x} = x_1, \ldots, x_n$, then applying n times the definition (22) and twice the definition (23), we get

$$\mathrm{ets}(C_2) = \mathrm{ets}(\mathrm{Ind}_{p,C_1})\ \bar{x}\ \mathrm{ets}(C_6)\, \mathrm{ets}(C_7),$$

where by C_6, C_7 are denoted the following terms

$$C_6 \equiv \lambda u_3 \lambda u_4 .\, u_3\, \mathrm{T}, \quad C_7 \equiv \lambda u_5 \lambda u_6 .\, u_6 \neg \mathrm{F}.$$

Taking into account the fact that $\mathrm{ets}(\mathrm{Ind}_{p,C_1}) = \lambda \bar{x}.R_1, \ldots R_k$ (see 9) we obtain

$$\mathrm{ets}(C_2) = \bar{R}\, \mathrm{ets}(C_6)\, \mathrm{ets}(C_7),$$

where $\bar{R} = R_1, \ldots, R_k$.
Since $\mathrm{ets}(\mathrm{T}) = \varepsilon$ and $\mathrm{ets}(\mathrm{atom}(p)) = \varepsilon$ (see 1), we get the following form for $\mathrm{ets}(C_6)$

$$\mathrm{ets}(C_6) = \mathrm{ets}(\lambda u_3 \lambda u_4 .u_3 \mathrm{T}) = \lambda \bar{y}_2, \bar{z}_2 .\, \mathrm{ets}(u_3 \mathrm{T}) = \lambda \bar{y}_2, \bar{z}_2 .\, \mathrm{ets}(u_3) = \lambda \bar{y}_2, \bar{z}_2 .\, \bar{y}_2 \quad ,$$

as well as
$\bar{y}_2^{\tau(\mathrm{atom}(true) \to B)} = \bar{y}_2^{\tau(B)}$
$\bar{z}_2^{\tau(\neg \mathrm{atom}(true) \to B)} = \bar{z}_2^{\tau(B)}$, correspondingly for u_3 and u_4 .

Similarly

$$\mathrm{ets}(C_7) = \mathrm{ets}(\lambda u_5 \lambda u_6 .u_6 \neg \mathrm{F}) = \lambda \bar{y}_3, \bar{z}_3 .\mathrm{ets}(u_6 \neg \mathrm{F}) = \lambda \bar{y}_3, \bar{z}_3 .\mathrm{ets}(u_6) = \lambda \bar{y}_3, \bar{z}_3 .\bar{z}_3 \quad ,$$

as well as
$\bar{y}_3^{\tau(\mathrm{atom}(false) \to B)} = \bar{y}_3^{\tau(B)}$
$\bar{z}_3^{\tau(\neg \mathrm{atom}(false) \to B)} = \bar{z}_3^{\tau(B)}$, correspondingly for u_5 and u_6 .
So, we obtain $\mathrm{ets}(C_3) = \bar{R}\ (\lambda \bar{y}_2, \bar{z}_2 .\bar{y}_2)\ (\lambda \bar{y}_3, \bar{z}_3 .\bar{z}_3)\ t_A\ \bar{y}_1\, \bar{z}_1$, and finally

$$\mathrm{ets}(Cases_{A,B}) = \lambda \bar{y}, \bar{z} .\, \bar{R}\ (\lambda \bar{y}_2, \bar{z}_2 .\bar{y}_2)\, (\lambda \bar{y}_3, \bar{z}_3 .\bar{z}_3)\ t_A\ \bar{y}_1\ \bar{z}_1 = \lambda \bar{y}, \bar{z} .\ \mathbf{if}\ t_A\ \bar{y}_1\, \bar{z}_1 \quad ,$$

where $\mathbf{if} \equiv \bar{R}\, (\lambda \bar{y}_2, \bar{z}_2 .\bar{y}_2)\, (\lambda \bar{y}_3, \bar{z}_3 .\bar{z}_3)$ and all the $\bar{y}_1$, $\bar{z}_1$, $\bar{y}_2$, $\bar{z}_2$, $\bar{y}_3$, $\bar{z}_3$ are lists of variables of the same type $\tau(B)$, but $\bar{y}$ and $\bar{z}$ are lists of variables of type $\tau(A \to B)$ and $\tau(\neg A \to B)$, respectively.

That is why $\mathrm{ets}(Cases_{A,B}) = \lambda \vec{y}, \vec{z} .\mathbf{if}\ t_A\ \vec{y}\ \vec{z} =_\eta \mathbf{if}\ t_A$ in [3], will be proper, if we take quantifier-free formulas instead of decidable formulas in formulation of case splitting. In the case of quantifier-free formula A, $\tau(A)$ will be ε (see 1). Consequently, from definition of $\tau(A \to B)$ (see 2), we get

$$\tau(A \to B) = \tau(\neg A \to B) = \tau(B),$$

so, all the $\bar{y}$, $\bar{z}$, $\bar{y}_1$, $\bar{z}_1$ are lists of variables of the same type $\tau(B)$, and we can properly apply η-equality.
Clearly $\mathbf{if}\ true\ \vec{r}\ \vec{s} =_{\beta R} \vec{r}$ and $\mathbf{if}\ false\ \vec{r}\ \vec{s} =_{\beta R} \vec{s}$.
For better readability the following notation is often used for $\mathbf{if}\ t_A\ \vec{r}\ \vec{s}$

$$\mathbf{if}\ A\ \mathbf{then}\ \vec{r}\ \mathbf{else}\ \vec{s}\ \mathbf{fi}.$$

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Դեպքերի տրոհման վերլուծությունը թվաբանական համակարգում

Տ. Գալոյան

Ամփոփում

Ուսումնասիրությունները նվիրված են դեպքերի տրոհման $[(\neg A \to B) \to (A \to B) \to B]$ վերլուծությանը թվաբանական համակարգում: Ցույց է տրվել, որ դեպքերի տրոհումը բավական է կատարել համաձայն բանաձևերի ավելի նեղ դասի՝ քվանտորներից ազատ բանաձևերի, որոշելի բանաձևերի դասի փոխարեն: Վերջինս ավելի լայն դաս է և ընդգրկում է քվանտորներից ազատ բանաձևերի դասը: Ուսումնասիրվել են նաև այլ հեղինակների մոտեցումները դեպքերի տրոհմանը և կատարվել են որոշ ճշգրտումներ կապված որոշելի բանաձևի փոխարինմանը քվանտորներից ազատ բանաձևով: Համաձայն այս ամենի առաջարկվել է դեպքերի տրոհման նոր արտածում: