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A Note on Matching Covered Graphs

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Abstract

A graph is called matching covered if for its every edge there is a maximum matching containing it. It is shown that line-extremal matching covered graphs contain a perfect matching.

Let Z^+ denote the set of nonnegative integers. We consider finite undirected graphs $G = (V(G), E(G))$ without multiple edges or loops [1], where $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. For a vertex $u \in V(G)$ define the set $N_G(u)$ as follows:

$$N_G(u) \equiv \{e \in E(G)/\ e \text{ is incident with } u\ \}.$$

The set of all maximum matchings [1,2] of the graph G is denoted by $M(G)$, and for $e \in E(G)$ define the set $M(e)$ as follows:

$$M(e) \equiv \{F \in M(G)/\ e \in F\ \}.$$

A graph G is called matching covered if for its every edge $e \in E(G)$ $M(e) \neq \emptyset$. A vertex $u \in V(G)$ is said to be covered (missed) by a matching $F \in M(G)$ if $N_G(u) \cap F \neq \emptyset$ ($N_G(u) \cap F = \emptyset$). A matching $F \in M(G)$ is called perfect if it covers every vertex $v \in V(G)$.

In a connected graph G the length of the shortest $u - v$ path [1] is denoted by $\rho(u, v)$, where u, v are vertices of the graph G. For a vertex $w \in V(G)$ and $U \subseteq V(G)$ set:

$$\rho(w, U) \equiv \min_{u \in U} \rho(w, u).$$

In this paper it is proved that every line-extremal matching covered graph contains a perfect matching. Non defined terms and conceptions can be found in [1,2,3].

Lemma. If G is a matching covered graph, which does not contain a perfect matching, then

(1) for every edge $e = (u, v) \in E(G)$ there is a $F \in M(G)$ such that F misses either u or v;

(2) if for edges $e, e' \in E(G)$ $M(e) = M(e')$ then $e = e'$.

Proof. (1) For every $F \in M(G)$ consider the sets $A(F)$ and $B(F)$ defined in the following way:

$$A(F) \equiv \{w \in V(G)/\ F \text{ covers } w\},$$
$$B(F) \equiv \{w \in V(G)/\ F \text{ misses } w\}.$$

Clearly, for each $F \in M(G)$ the following holds:

$$V(G) = A(F) \cup B(F),\ A(F) \cap B(F) = \emptyset,\ A(F) \neq \emptyset,\ B(F) \neq \emptyset.$$

For an edge $e = (u, v) \in E(G)$ define a mapping $\mu_e : M(G) \to Z^+$ as follows:

$$\mu_e(F) \equiv \min\{\rho(u, B(F)),\ \rho(v, B(F))\},\ \text{where } F \in M(G).$$

Choose $F_0 \in M(G)$ satisfying the condition:

$$\mu_e(F_0) \equiv \min_{F \in M(G)} \mu_e(F).$$

Let us show that F_0 misses either u or v. For the sake of contradiction assume F_0 to cover both u and v. Let $w_0, (w_0, w_1), w_1, ..., w_{k-1}, (w_{k-1}, w_k), w_k$ be a simple path of the graph G satisfying the conditions:

$$w_0 \in B(F),\ \{w_1, ..., w_k\} \subseteq A(F_0),\ \{w_{k-1}, w_k\} = \{u, v\},\ k = 1 + \mu_e(F_0),\ k \geq 2.$$

Set $e' \equiv (w_1, w_2)$. Let us prove that $e' \notin F_0$. If $e' \in F_0$ then consider the matching $F_1 \in M(G)$ defined as follows:

$$F_1 \equiv (F_0 \backslash \{e'\}) \cup \{e\}.$$

It is clear that $\mu_e(F_1) < \mu_e(F_0)$, which contradicts to the choice of F_0, therefore $e' \notin F_0$. Take a maximum matching $F_0' \in M(e')$ satisfying the condition:

$$|F_0 \cap F_0'| = \max_{F' \in M(e')} |F_0 \cap F'|.$$

Let us show that $w_0 \in A(F_0')$. If $w_0 \notin A(F_0')$ then assume:

$$F_0'' \equiv (F_0' \backslash \{e'\}) \cup \{e\}.$$

Note that $F_0'' \in M(G)$ and $\mu_e(F_0'') < \mu_e(F_0)$, which is impossible, therefore $w_0 \in A(F_0')$. It is not hard to see that the choice of F_0'' implies that there is a simple path $\nu_0, (\nu_0, \nu_1), \nu_1, ..., \nu_{2l-1}, (\nu_{2l-1}, \nu_{2l}), \nu_{2l}$ $(l \geq 1)$ of the graph G satisfying the conditions:

$$\begin{gathered} \{(\nu_0, \nu_1), ..., (\nu_{2l-2}, \nu_{2l-1})\} \subseteq F_0', \{(\nu_1, \nu_2), ..., (\nu_{2l-1}, \nu_{2l})\} \subseteq F_0, \\ e' \notin \{(\nu_0, \nu_1), ..., (\nu_{2l-2}, \nu_{2l-1})\}, \\ \nu_0 = w_0,\ \nu_{2l} \in \{w_1, w_2\}. \end{gathered}$$

Set:

$$\bar{F}_0 \equiv (F_0 \backslash \{(\nu_1, \nu_2), ..., (\nu_{2l-1}, \nu_{2l})\}) \cup \{(\nu_0, \nu_1), ..., (\nu_{2l-2}, \nu_{2l-1})\}.$$

Clearly $\bar{F}_0 \in M(G)$ and $\mu_e(\bar{F}_0) < \mu_e(F_0)$,which contradicts to the choice of F_0, therefore F_0 misses either u or v.

(2) Suppose $e, e' \in E(G)$, $e = (u, v)$ and $e \neq e'$.Let us show that $M(e) \neq M(e')$. Take a matching $F_1 \in M(G)$ missing either u or v. For the sake of definiteness let us assume that F_1 covers u and misses v. If $e' \in F_1$ then $M(e) \neq M(e')$, therefore without loss of generality we may assume that $e' \notin F_1$. As F_1 covers u, then there is a $w \in V(G)$ such that $(u, w) \in F_1$. Set:

$$F_2 \equiv (F_1 \backslash \{(u,w)\}) \cup \{(u,v)\}.$$

Clearly, $F_2 \in M(G)$, $e \in F_2$ and $e' \notin F_2$, therefore $M(e) \neq M(e')$. The proof of the **Lemma** is complete.

Theorem. Suppose that the graph G satisfies the following two properties:

(1) G is a matching covered graph,

(2) $G - e$ is not a matching covered graph for every edge $e \in E(G)$.

Then the graph G has a perfect matching.

Proof. Without loss of generality we may assume G to be connected. Let us show that there are two distinct edges e and e'such that $M(e) = M(e')$.

Take an arbitrary edge $e_0 \in E(G)$. Suppose that the edges $e_0, \dots, e_k$ $(k \geq 0)$ are already defined, and consider the graph $G - e_k$. As it is not a matching covered graph, then there exists an edge $\bar{e} \neq e_k$ such that $M(e_k) \supseteq M(\bar{e})$. Set $e_{k+1} \equiv \bar{e}$.

Consider the infinite sequence $\{e_k\}_{k=0}^{\infty}$of edges of the graph G. Clearly, there are numbers $i, j \in Z^+$, $i < j$ such that $e_i = e_j$. The construction of the sequence $\{e_k\}_{k=0}^{\infty}$implies that

$$M(e_i) \supseteq M(e_{j-1}) \supseteq M(e_j) = M(e_i), \text{ and } e_{j-1} \neq e_j,$$

therefore

$$M(e_{j-1}) = M(e_j).$$

Lemma implies that G has a perfect matching. The proof of the **Theorem** is complete.

References

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Գրառում զուգակցումներով ծածկված գրաֆների մասին

Վ.Վ. Մկրտչյան

Ամփոփում

Գրաֆը կոչվում է զուգակցումներով ծածկված, եթե նրա ցանկացած կողի համար գոյություն ունի այն պարունակող մաքսիմալ զուգակցում: Ցույց է տրվել, որ զուգակցումներով ծածկված գրաֆները, որոնք էքստրեմալ են կողի հեռացման գործողության նկատմամբ, պարունակում են կատարյալ զուգակցում: