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On Interval Edge Colorings of Harary Graphs $H_{2n-2,2n}$*

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Abstract

The problems of existence and construction of interval edge colorings of Harary graphs $H_{2n-2,2n}$ are investigated. Bounds are found for the possible number of colors in interval edge colorings of $H_{2n-2,2n}$.

Let $G = (V(G), E(G))$ be an undirected graph without loops and multiple edges[1]. $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$.

A function $\alpha : E(G) \to \{1, 2, ..., t\}$ is a proper edge t-coloring of a graph G iff for each $i, 1 \le i \le t$ there is an edge $e \in E(G)$ with $\alpha(e) = i$ and $\alpha(e') \ne \alpha(e'')$ for any pair of adjacent edges $e' \in E(G)$ and $e'' \in E(G)$.

A proper edge t-coloring α of a graph G is an interval edge t-coloring of G iff for each vertex $x \in V(G)$ the edges incident to $x \in V(G)$ are colored by $d_G(x)$ consecutive colors.

$\mathcal{N}_t$ denotes the set of graphs, for which an interval edge t-coloring exists. Let $\mathcal{N} \equiv \bigcup\limits_{t \ge 1} \mathcal{N}_t$.

For $G \in \mathcal{N}$ the least and the greatest value of t, for which $G \in \mathcal{N}_t$, are denoted by $w(G)$ and $W(G)$, respectively.

Non-defined conceptions and terms can be found in [1-4].

Let us consider a graph K_{2n}, where $V(K_{2n}) = \{x_1, x_2, ..., x_{2n}\}$, $E(K_{2n}) = \{(x_i, x_j) \mid 1 \le i \le 2n,\ 1 \le j \le 2n,\ i < j\}$. Define the set $E_{0,2n} \subseteq E(K_{2n})$ as follows: $E_{0,2n} \equiv \{(x_i, x_{i+n}) \mid 1 \le i \le n\}$. Evidently, $E_{0,2n}$ is a perfect matching of K_{2n}. It is not difficult to see [2,3] that the graph $K_{2n} \,/\, E_{0,2n}$ is isomorphic to Harary graph $H_{2n-2,2n}$, and the graph K_{2n} is isomorphic to Harary graph $H_{2n-1,2n}$.

Theorem1[4]. For any $n \in N$ $\quad K_{2n} \in \mathcal{N}$.

Theorem2[5]. For any $n \in N$ $\quad W(K_{2n}) \ge 3n - 2$.

Theorem3. For $n \ge 2$ $\quad H_{2n-2,2n} \in \mathcal{N}_{3n-3}$.

Proof. The case $n = 2$ is evident.

Now assume that $n \ge 3$.

Define a proper edge $(3n-3)$-coloring α of the graph $H_{2n-2,2n}$ in the following way:

for $i = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor, j = 2, ..., n, i < j, i + j \le n + 1$ $\qquad \alpha((x_i, x_j)) = i + j - 2;$

for $i = 2, ..., n-1, j = \left\lfloor \frac{n}{2} \right\rfloor + 2, ..., n, i < j, i + j \ge n + 2$ $\qquad \alpha((x_i, x_j)) = i + j + n - 4;$

for $i = 3, ..., n, j = n + 1, ..., 2n - 2, j - i \le n - 2$ $\qquad \alpha((x_i, x_j)) = n + j - i - 1;$

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for $i=1,...,n-1,\ j=n+2,...,2n,\ j-i\geq n+1 \quad \alpha((x_i,x_j))=j-i-1$;
for $i=2,...,1+\left\lfloor\frac{n-1}{2}\right\rfloor,\ j=n+1,...,n+\left\lfloor\frac{n-1}{2}\right\rfloor,\ j-i=n-1 \quad \alpha((x_i,x_j))=2(i-1)$;
for $i=\left\lfloor\frac{n-1}{2}\right\rfloor+2,...,n,\ j=n+1+\left\lfloor\frac{n-1}{2}\right\rfloor,...,2n-1,\ j-i=n-1 \quad \alpha((x_i,x_j))=i+j-3$;
for $i=n+1,...,n+\left\lfloor\frac{n}{2}\right\rfloor-1,\ j=n+2,...,2n-2,\ i<j,\ i+j\leq 3n-1 \quad \alpha((x_i,x_j))= i+j-2n$;
for $i=n+1,...,2n-1,\ j=n+\left\lfloor\frac{n}{2}\right\rfloor+1,...,2n,\ i<j,\ i+j\geq 3n \quad \alpha((x_i,x_j))=i+j-n-2$.

It is not difficult to see that α is an interval edge $(3n-3)$-coloring of the graph $H_{2n-2,2n}$.

The proof is complete.

Corollary1. For $n\geq 2$ $H_{2n-2,2n}\in\mathcal{N}$.

Corollary2. For $n\geq 2$ $W(H_{2n-2,2n})\geq 3n-3$.

The results of [4], the **Corollary1** and the definition[2,3] of the graph $H_{2n-2,2n}$ imply

Corollary3. For $n\geq 2$ $w(H_{2n-2,2n})=2n-2$.

Corollary4. For $n\geq 2$ and $2n-2\leq t\leq 3n-3$ $H_{2n-2,2n}\in\mathcal{N}_t$.

Theorem4. For any $m\in N$ $W(H_{4m-2,4m})\geq W(K_{2m})+4m-2$.

Proof. Let us consider a graph K_{4m} with $V(K_{4m})=\{x_1,x_2,...,x_{4m}\}$. Assume $H_{4m-2,4m}\equiv K_{4m}\,/\,E_{0,4m}$. Let G be the subgraph of $H_{4m-2,4m}$, induced by the subset $\{x_1,x_2,...,x_{2m}\}$ of the set of its vertices. Clearly, G is isomorphic to the graph K_{2m} and, consequently, by the **Theorem1** there exists an interval edge $W(K_{2m})$-coloring α of G.

Let us define a proper edge $(W(K_{2m})+4m-2)$-coloring β of the graph $H_{4m-2,4m}$.

For $i=1,2,...,4m,\ j=1,2,...,4m,\ i\neq j$ and $i\neq j-2m$, we set:

$$\beta((x_i,x_j))=\begin{cases}\alpha((x_i,x_j)) & \text{if } 1\leq i\leq 2m, 1\leq j\leq 2m;\\ \alpha((x_i,x_{j-2m}))+2m-1 & \text{if } 1\leq i\leq 2m,\ 2m+1\leq j\leq 4m;\\ \alpha((x_{i-2m},x_{j-2m}))+4m-2 & \text{if } 2m+1\leq i\leq 4m,\ 2m+1\leq j\leq 4m.\end{cases}$$

It is not difficult to see that β is an interval edge $(W(K_{2m})+4m-2)$-coloring of the graph $H_{4m-2,4m}$.

The proof is complete.

Corollary5. If n is even and $n\geq 2$ then $W(H_{2n-2,2n})\geq 3,5n-4$.

Corollary6. If n is even, $n\geq 2$ and $2n-2\leq t\leq 3,5n-4$ then $H_{2n-2,2n}\in\mathcal{N}_t$.

References

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Խառարիի H2n-2,2n գրաֆների միջակայքային կողային ներկումների մասին

Ռ.Ռ. Քամալյան, Պ.Ա. Պետրոսյան

Ամփոփում

Դիտարկված են Խառարիի $H_{2n-2,2n}$ գրաֆների միջակայքային կողային ներկումների գոյության և կառուցման հարցեր, և ստացված են գնահատականներ այդ ներկումներում օգտագործվող գույների հնարավոր թվի համար: