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On Trees with a Special Proper Partial $0-1$ Colorings

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Abstract

It is proved that in a tree in which the distance between any two endpoints is even, there is a maximum proper partial $0-1$ coloring such that the edges colored by 0 form a maximum matching.

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex x in G is denoted by $d_G(x)$. If $X \subseteq E(G)$ then a mapping $f : X \to \{0,1\}$ is referred as a partial $0-1$ coloring of the graph G. For $i = 0,1$ and the partial $0-1$ coloring f of the graph G, denote $f_i \equiv \{e \in X / f(e) = i\}$. The partial $0-1$ coloring f is proper if the sets f_0 and f_1 are matchings of the graph G. Denote

$$\lambda(G) \equiv \max\{|f_0| + |f_1| \,/\, f \text{ is a proper partial } 0-1 \text{ coloring of the graph } G\}.$$

A proper partial $0-1$ coloring f of the graph G is maximum if $|f_0| + |f_1| = \lambda(G)$. Set: $\alpha(G) \equiv \max\{|f_i| \,/ i = 0,1$ and f is a maximum proper partial (shortly, MPP) $0-1$ coloring of the graph $G\}$. It is clear, that for every graph G $\alpha(G) \leq \beta(G)$, where $\beta(G)$ is the cardinality of a maximum matching of the graph G. In this paper I show that if G is a tree in which the distance between any two endpoints is even, the equality $\alpha(G) = \beta(G)$ holds. Non defined terms and conceptions can be found in [1,2].

Lemma1. Let G be a graph, $u \in V(G)$, $w \in V(G)$, $(u,w) \in E(G)$, $d_G(u) = 1$. Then there is a MPP $0-1$ coloring f of the graph G, such that $|f_0| = \alpha(G)$ and $(u,w) \in f_0$.

Proof. Let f be a MPP $0-1$ coloring of the graph G with $|f_0| = \alpha(G)$. Suppose $(u,w) \notin f_0$.

Case1. $(u,w) \notin f_1$. As f is a MPP $0-1$ coloring of the graph G, there is a $(w,w') \in E(G)$, such that $(w,w') \in f_0$. Consider the mapping $g : f_1 \cup (f_0 \backslash \{(w,w')\}) \cup \{(u,w)\} \to \{0,1\}$ defined in the following way:

$$g(e) = \begin{cases} 0, & \text{if } \ e \in (f_0 \backslash \{(w,w')\}) \cup \{(u,w)\} \\ 1, & \text{if } \ e \in f_1. \end{cases}$$

It is clear that g is a MPP $0-1$ coloring of the graph G, $(u,w) \in g_0$ and $|g_0| = |f_0| = \alpha(G)$.

Case2. $(u,w) \in f_1$. As f is a MPP $0-1$ coloring of the graph G, with $|f_0| = \alpha(G)$, then there is a $(w,w_1) \in f_0$. Consider the maximal alternating path
$u, (u,w), w, (w,w_1), w_1, ..., w_{k-1}, (w_{k-1}, w_k), w_k$, where k is odd,
$\{(u,w), (w_1,w_2), ..., (w_{k-2}, w_{k-1})\} \subseteq f_1$ and $\{(w,w_1), (w_2,w_3), ..., (w_{k-1}, w_k)\} \subseteq f_0$. Define a mapping $g : f_0 \cup f_1 \to \{0,1\}$ as follows:

$$g(e) = \begin{cases} f(e), & \text{if } \ e \notin \{(u,w),(w,w_1),...,(w_{k-1},w_k)\} \\ 1-f(e), & \text{if } \ e \in \{(u,w),(w,w_1),...,(w_{k-1},w_k)\}. \end{cases}$$

Clearly, g is a MPP $0-1$ coloring of the graph G with $(u,w) \in g_0$ and $|g_0| = |f_0| = \alpha(G)$. The proof is complete.

Lemma2. Let G be a graph, $u \in V(G)$, $v \in V(G)$, $w \in V(G)$, $d_G(u) = d_G(v) = 1$, $(u,w) \in E(G)$, $(v,w) \in E(G)$. Then

(a) there is a MPP $0-1$ coloring f of the graph G, such that $|f_0| = \alpha(G)$, $(u,w) \in f_0$ and $(v,w) \in f_1$;

(b) $\lambda(G) = 2 + \lambda(G\backslash\{u,v,w\})$, $\alpha(G) = 1 + \alpha(G\backslash\{u,v,w\})$.

Proof. (a) By **Lemma1**, there is a MPP $0-1$ coloring f of the graph G, such that $|f_0| = \alpha(G)$ and $(u,w) \in f_0$. Suppose $(v,w) \notin f_1$,then there is a $(w,w') \in E(G)$, such that $(w,w') \in f_1$. Consider a mapping $g : f_0 \cup (f_1\backslash\{(w,w')\}) \cup \{(v,w)\} \to \{0,1\}$ defined in the following way:

$$g(e) = \begin{cases} 0, & \text{if } \ e \in f_0 \\ 1, & \text{if } \ e \in (f_1\backslash\{(w,w')\}) \cup \{(v,w)\}. \end{cases}$$

Clearly, g is a MPP $0-1$ coloring of the graph G with $(u,w) \in g_0$, $(v,w) \in g_1$ and $|g_0| = \alpha(G)$.

(b) Let $w_1,...,w_r$ be vertices of the graph G such that $d_G(w) = r+2$ $(r \geq 0)$, $u \notin \{w_1,...,w_r\}$, $v \notin \{w_1,...,w_r\}$, $(w,w_i) \in E(G)$ for $i = 1,...,r$, and f be a MPP $0-1$ coloring f of the graph G, such that $|f_0| = \alpha(G)$, $(u,w) \in f_0$, $(v,w) \in f_1$. As $(w,w_i) \notin f_0 \cup f_1$ for $i = 1,...,r$,we have

$\lambda(G) = \lambda(G\backslash\{(w,w_1),...,(w,w_r)\}) = 2 + \lambda(G\backslash\{u,v,w\})$,

$\alpha(G) = \alpha(G\backslash\{(w,w_1),...,(w,w_r)\}) = 1 + \alpha(G\backslash\{u,v,w\})$.

The proof is complete.

Corollary. Let G be a graph, $U = \{u_0,u_1,u_2,u_3,u_4\}$ be a subset of the set of vertices of G satisfying the conditions: $d_G(u_0) = d_G(u_4) = 1$, $d_G(u_1) = d_G(u_3) = 2$, $(u_{i-1},u_i) \in E(G)$ for $i = 1,2,3,4$. Then the following is true:

$$\lambda(G) = \lambda(G\backslash U) + 4, \ \alpha(G) \geq 2 + \alpha(G\backslash U).$$

Proof. Lemma2 implies

$\lambda(G) = 2 + \lambda(G\backslash\{u_0,u_4\}) = \lambda(G\backslash U) + 4$, therefore $\alpha(G) \geq 2 + \alpha(G\backslash U)$.

Theorem. Let G be a tree in which the distance between any two endpoints is even. Then the equality $\alpha(G) = \beta(G)$ holds.

Proof. Clearly, the statement of the theorem is true for the case $|E(G)| \leq 6$. Assume that it holds for trees with $|E(G)| \leq t-1$, and let us prove that it will hold for the case $|E(G)| = t$, where $t \geq 7$.

Case1. There is a $U = \{u_0,u_1,u_2,u_3\} \subseteq V(G)$, such that $d_G(u_0) = 1$, $d_G(u_1) = d_G(u_2) = 2$, $(u_{i-1},u_i) \in E(G)$ for $i = 1,2,3$.Set $G' = G\backslash\{u_0,u_1\}$. Clearly, $\beta(G) = \beta(G') + 1$. As $d_G(u_0) = 1$, $d_G(u_1) = 2$ and $d_{G\backslash\{u_0\}}(u_1) = 1$, $d_{G\backslash\{u_0\}}(u_2) = 2$, we have $\lambda(G) = 1 + \lambda(G\backslash\{u_0\}) = \lambda(G') + 2$, thus if g is a MPP $0-1$ coloring of tree G', such that $|g_0| = \alpha(G')$ and $(u_2,u_3) \in g_0$, then the mapping $f : g_0 \cup g_1 \cup \{(u_0,u_1),(u_1,u_2)\} \to \{0,1\}$ defined as

$$f(e) = \begin{cases} g(e), & \text{if } \ e \notin \{(u_0,u_1),(u_1,u_2)\} \\ 1, & \text{if } \ e = (u_1,u_2) \\ 0, & \text{if } \ e = (u_0,u_1), \end{cases}$$

is a MPP $0-1$ coloring of the tree G, therefore $\alpha(G) \geq |f_0| = 1+|g_0| = 1+\alpha(G')$. As the distance between any two endpoints of G' is even and $|E(G')| < t$, we have $\alpha(G') = \beta(G')$, therefore

$$\alpha(G) \geq 1+\alpha(G') = 1+\beta(G') = \beta(G), \text{ or } \alpha(G) = \beta(G).$$

Case2. There is a $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_5) = 2$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1,2,3,4$, $(u_2, u_5) \in E(G)$, $(u_5, u_6) \in E(G)$.Set $G' = G\backslash\{u_5, u_6\}$. Clearly, $\beta(G) = \beta(G') + 1$. From **Corollary** follows that $\lambda(G) = \lambda(G\backslash\{u_0, u_1, u_2, u_3, u_4\})+4$, therefore $\lambda(G) = \lambda(G\backslash\{(u_2, u_5)\}) = \lambda(G')+1$ and $\alpha(G) \geq 1+\alpha(G')$. Note that the distance between any two endpoints of the tree G' is even and $|E(G')| < t$, thus the equality $\alpha(G') = \beta(G')$ holds, and therefore

$$\alpha(G) \geq 1+\alpha(G') = 1+\beta(G') = \beta(G), \text{ or } \alpha(G) = \beta(G).$$

Case3. There is a $U = \{u_0, u_1, u_2\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_2) = 1$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1, 2$. Let $D_1, ..., D_r$ be the connected components of $G\backslash U$. Clearly, $\beta(G) = 1 + \sum_{i=1}^{r} \beta(D_i)$. Note that D_i $i = 1, ..., r$ is a tree for which $|E(D_i)| < t$ and the distance between any two endpoints is even, thus $\alpha(D_i) = \beta(D_i)$, therefore, by **Lemma2**, we have

$$\alpha(G) = 1+\alpha(G\backslash U) = 1+\sum_{i=1}^{r}\alpha(D_i) = 1+\sum_{i=1}^{r}\beta(D_i) = \beta(G).$$

Case4. There is a $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_5) = 2$, $d_G(u_2) = 3$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1,2,3,4,6$, $(u_2, u_5) \in E(G)$. Set $G' = G\backslash\{u_0, u_1\}$. Clearly, $\beta(G) = \beta(G') + 1$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds.

Lemma1 implies, that there is a MPP $0-1$ coloring g of the tree $G\backslash\{u_0, u_1, u_2, u_5, u_6\}$ such that $(u_3, u_4) \in g_0$. Consider the mapping $f : g_0 \cup g_1 \cup \{(u_0, u_1), (u_2, u_3), (u_2, u_5), (u_5, u_6)\} \to \{0, 1\}$ defined as follows:

$$f(e) = \begin{cases} g(e), & \text{if} \quad e \notin \{(u_0, u_1), (u_2, u_3), (u_2, u_5), (u_5, u_6)\} \\ 0, & \text{if} \quad e \in \{(u_0, u_1), (u_2, u_5)\} \\ 1, & \text{if} \quad e \in \{(u_2, u_3), (u_5, u_6)\}. \end{cases}$$

The Corollary implies, that f is a MPP $0-1$ coloring of the tree G, therefore $\lambda(G) = \lambda(G\backslash\{(u_1, u_2)\}) = \lambda(G') + 1$ and $\alpha(G) \geq 1+\alpha(G') = \beta(G')+1 = \beta(G)$, or $\alpha(G) = \beta(G)$.

Case5. There is a $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_6) = 1$, $d_G(u_1) = d_G(u_3) = 2$, $d_G(u_2) = d_G(u_5) = 3, (u_{i-1}, u_i) \in E(G)$ for $i = 1,2,3,4,6$, $(u_2, u_5) \in E(G)$, $(u_5, u_7) \in E(G)$.Set $G' = G\backslash\{u_0, u_1, u_2, u_3, u_4\}$. Note that $\beta(G) = \beta(G') + 2$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds. From **Corollary** we have

$$\alpha(G) \geq 2+\alpha(G') = 2+\beta(G') = \beta(G), \text{ or } \alpha(G) = \beta(G).$$

Case6. There is a $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\} \subseteq V(G)$, such that $d_G(u_0) = d_G(u_4) = d_G(u_5) = d_G(u_9) = 1$, $d_G(u_1) = d_G(u_3) = d_G(u_6) = d_G(u_8) = 2$, $d_G(u_2) = d_G(u_7) = 3$, $(u_{i-1}, u_i) \in E(G)$ for $i = 1,2,3,4,6,7,8,9$, $(u_2, u_{10}) \in E(G)$, $(u_7, u_{10}) \in E(G)$. Set $G' = G\backslash\{u_0, u_1, u_2, u_3, u_4\}$. Clearly $\beta(G) = \beta(G') + 2$. As $|E(G')| < t$ and the distance between any two endpoints of the tree G' is even, the equality $\alpha(G') = \beta(G')$ holds, therefore from **Corollary** we have

$$\alpha(G) \geq 2 + \alpha(G') = 2 + \beta(G') = \beta(G), \text{ or } \alpha(G) = \beta(G) .$$

As every tree G, in which the distance between any two endpoints is even, and $|E(G)| \geq 7$, satisfies at least one of the conditions of the six cases considered above, the proof of the Theorem is complete.

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Հատուկ տիպի ճշգրիտ մասնակի 0-1 ներկումներ պարունակող ծառերի մասին

Վ.Վ. Մկրտչյան

Ամփոփում

Աշխատանքում ցույց է տրվել, որ այն ծառերում, որոնցում ցանկացած երկու կախված գագաթների հեռավորությունը զույգ թիվ է, գոյություն ունի մաքսիմալ ճշգրիտ մասնակի 0-1 ներկում, որ 0 գույնով ներկված կողերի բազմությունը կազմում է ծառի մաքսիմալ զուգակցում: