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On Logarithmically Asymptotically Optimal Hypothesis Testing of Three Distributions for Pair of Independent Objects

Evgueni A. Haroutunian and Parandzem M. Hakobyan

Institute for Informatics and Automation Problems of NAS of RA
e-mail evhar@ipia.sci.am, par_h@ipia.sci.am

Abstract

The problem of hypotheses testing for a model consisting from two independent objects is considered. It is supposed that three probability distributions are known and objects independently from each other follow to one of them. The matrix of asymptotic interdependencies (reliability–reliability functions) of all possible pairs of the error probability exponents (reliabilities) in optimal testing for this model is studied.

The case with two independent objects and two given probability distributions was elaborated by Haroutunian and Ahlswede.

1 Problem statement and Preliminary Result.

We consider a generalization of the problem of many hypotheses testing concerning one object [1] to the similar problem for a model with two independent objects. This problem was introduced in the book of Ahlswede and Vegener [2] and studied for the case of two known distributions in [3].

Let X_1 and X_2 be independent random variables taking values in the finite set $\mathcal{X}$, and presenting characteristics of two independent objects. The cardinality of $\mathcal{X}$ is K. Suppose that X_i, $i = 1,2$, assumes values x_{k_i}, $k_i = \overline{1,K}$, $i = 1,2$. The random vector (X_1, X_2) assumes values $(x_{k_1}, x_{k_2}) \in \mathcal{X} \times \mathcal{X}$, $k_1, k_2 = \overline{1,K}$.

Let $\mathcal{P}(\mathcal{X})$ be the space of all possible distributions on $\mathcal{X}$. There are given three probability distributions G_1, G_2 and G_3 from $\mathcal{P}(\mathcal{X})$. We consider the model with two objects characteristics of which X_i, $i = 1,2$, independently follow to one of the three distributions. Let $\{(x_{k_1^n}, x_{k_2^n})\}$, $k_i^n = \overline{1,K}$, $i = 1,2$, $n = \overline{1,N}$, be a sequence of results of N independent observations of the vector (X_1, X_2). The statistician must define a pair of distributions corresponding to obtained data. The selection is made from three given distributions, i.e. we have three hypotheses: $H_1 : G = G_1$, $H_2 : G = G_2$ and $H_3 : G = G_3$. We call the procedure of making decision on the base of N observations the test and denote it by φ_N. The test φ_N of this model may be composed by the pair of the tests φ_N^1 and φ_N^2 for the respective object: $\varphi_N = (\varphi_N^1, \varphi_N^2)$.

Let $\alpha_{m_1,m_2|l_1,l_2}(\varphi_N)$, be the probability of the erroneous acceptance by the test φ_N of hypotheses pair (H_{l_1}, H_{l_2}) provided that the pair (H_{m_1}, H_{m_2}) is true, where (m_1, m_2)

$, l_2)$, $m_i, l_i = \overline{1,3}$, $i = 1, 2$. The probability to reject a true pair of hypotheses (H_{m_1}, H_{m_2}), is defined by:

$$\alpha_{m_1,m_2|m_1,m_2}(\varphi_N) = \sum_{(l_1,l_2)\neq(m_1,m_2)} \alpha_{m_1,m_2|l_1,l_2}(\varphi_N). \quad (1)$$

We have to consider corresponding error probability exponents of the sequence of tests $\varphi = \{\varphi_N\}$, which is convienient to call "reliabilities" of φ,

$$E_{m_1,m_2|l_1,l_2}(\varphi) \triangleq \overline{\lim_{N\to\infty}} - \frac{1}{N} \log \alpha_{m_1,m_2|l_1,l_2}(\varphi_N), \quad m_i, l_i = \overline{1,3}, \quad i = 1, 2. \quad (2)$$

It follows from (1) and (2) that

$$E_{m_1,m_2|m_1,m_2}(\varphi) = \min_{(l_1,l_2)\neq(m_1,m_2)} E_{m_1,m_2|l_1,l_2}(\varphi). \quad (3)$$

The matrix $\mathbf{E}(\varphi) = \{E_{m_1,m_2|l_1,l_2}(\varphi)\}$ we call the reliability matrix of the sequence φ of tests.

Definition 1: *The test sequence φ^* is called logarithmically asymptotically optimal (LAO) for this model if for given values of the elements $E_{1,1|1,3}$, $E_{1,1|3,1}$, $E_{2,2|2,3}$ and $E_{2,2|3,2}$ it provides maximal values for all other elements of $\mathbf{E}(\varphi^*)$.*

Our aim is to define conditions on $E_{1,1|1,3}$, $E_{1,1|3,1}$, $E_{2,2|2,3}$ and $E_{2,2|3,2}$ under which there exists LAO sequence of tests φ^*, and to show how other elements $E_{m_1,m_2|l_1,l_2}(\varphi^*)$, $m_i, l_i = 1, 2, 3$, $i = 1, 2$, of the matrix $\mathbf{E}(\varphi^*)$ can be found from them.

The divergences (Kullback-Leibler distances) $D(G_{l_1}||G_{l_2})$ and $D(Q||G_l)$, $Q \in \mathcal{P}(\mathcal{X})$, $l, l_i = \overline{1,3}$, $i = 1, 2$, are defined as usual [4]- [6].

$$D(G_{l_1}||G_{l_2}) = \sum_x G_{l_1}(x) \log \frac{G_{l_1}(x)}{G_{l_2}(x)},$$

$$D(Q||G_l) = \sum_x Q(x) \log \frac{Q(x)}{G_l(x)}.$$

In paper [1] the case of one object and M hypothetical distributions was considered. Let us recall main definitions for the case $M = 3$. The random variable X taking values in the finite set $\mathcal{X}$ and follow to one of the three distributions G_1, G_2 and G_3. The statistcian must select one among 3 hypotheses $H_l : G = G_l$, $l = \overline{1,3}$. Let $\mathbf{x} = (x_1, ..., x_N)$ be a sequence of results of N observations of the object. The procedure of decision making is a non-randomized test φ_N, it can be defined by division of the sample space $\mathcal{X}^N$ on 3 disjoint subsets $\mathcal{A}_l^N = \{\mathbf{x} : \varphi^N(\mathbf{x}) = l\}$, $l = \overline{1,3}$. The set $\mathcal{A}_l^N$ consist of all vectors $\mathbf{x}$ for which the hypothesis H_l is adopted. The probabilities of the erroneous acceptance of hypothesis H_l provided that H_m is true is

$$\alpha_{m|l}(\varphi_N) = G_m^N(\mathcal{A}_l^N).$$

The probability to reject H_m, when it is true, is

$$\alpha_{m|m}(\varphi_N) = \sum_{l\neq m} \alpha_{m|l}(\varphi_N).$$

Corresponding error probability exponents, called "reliability", are defined as

$$E_{m|l}(\varphi) \triangleq \overline{\lim_{N\to\infty}} - N^{-1} \log \alpha_{m|l}(\varphi_N), \ m, l = \overline{1,3}.$$

It follows (see (3)) that
$$E_{m|m}(\varphi) = \min_{l \neq m} E_{m|l}(\varphi), \ \ m = \overline{1,3}. \tag{4}$$

The matrix $\mathbf{E}(\varphi) = \{E_{m|l}(\varphi)\}$ is called reliability matrix of the sequence φ of tests.

Definition 2: *We call the sequence of tests logarithmically asymptotically optimal* (*LAO*) *if for given positive values of 2 diagonal elements of the matrix* $\mathbf{E}$ *the procedure provides maximal values for other elements of it.*

Now in Theorem 1 we reformulate the result of [1] for the case $M = 3$. For given positive elements $E_{1|1}$ and $E_{2|2}$ let us denote

$$E^*_{1|1} \triangleq E_{1|1}, \quad E^*_{2|2} \triangleq E_{2|2}, \tag{5.a}$$

$$E^*_{m|l} \triangleq \inf_{Q:\, D(Q||G_l) \leq E_{l|l}} D(Q||G_m), \ \ m = \overline{1,3}, \ \ l = 1,2, \ \ m \neq l, \tag{5.b}$$

$$E^*_{m|3} \triangleq \inf_{Q:\, D(Q||G_1) > E_{1|1}, D(Q||G_2) > E_{2|2}} D(Q||G_m), \ \ m = 1,2, \tag{5.c}$$

$$E^*_{3|3} \triangleq \min_{l=1,2} E_{3|l}. \tag{5.d}$$

Theorem 1[1]: *If different distributions* G_1, G_2 *and* G_3, $D(G_l||G_m) < \infty$, $l \neq m$, *and positive numbers* $E_{1|1}$, $E_{2|2}$ *are given and the following inequalities take place:*

$$E_{1|1} < \min[D(G_3||G_1), D(G_2||G_1)], \ \ E_{2|2} < \min[E^*_{2|1}, \ D(G_3||G_2)], \tag{6}$$

then

a) there exists a LAO sequence of tests φ^*, *such that all elements of the reliability matrix* $\mathbf{E}(\varphi^*)$ *are defined in* (5),

b) if one of the inequalities (6) *is violated, then at least one element of the matrix* $\mathbf{E}(\varphi^*)$ *is equal to* 0.

Remark 1: From the definition of $E^*_{1|3}$ and $E^*_{2|3}$ in (5) it follows, that $E^*_{m|3} \geq E_{m|m}$, $m = 1,2$. Taking into account the latter and (4) we obtain that

$$E^*_{1|1} = E^*_{1|3}, \quad E^*_{2|2} = E^*_{2|3}. \tag{7}$$

2 Results and Proofs

Proceeding to the case of two objects and the sequence of tests φ^i, let the reliability matrix $\mathbf{E}(\varphi^i)$, of the i-th object, $i = 1,2$, be denoted

$$\mathbf{E}(\varphi^i) = \begin{pmatrix} E_{1|1}(\varphi^i) & E_{1|2}(\varphi^i) & E_{1|3}(\varphi^i) \\ E_{2|1}(\varphi^i) & E_{2|2}(\varphi^i) & E_{2|3}(\varphi^i) \\ E_{3|1}(\varphi^i) & E_{3|2}(\varphi^i) & E_{3|3}(\varphi^i) \end{pmatrix}.$$

We begin by proving the following

Lemma: *If positive elements* $E_{1|1}(\varphi^i)$, $E_{2|2}(\varphi^i)$, $i = 1,2$, *satisfy the condition* (6), *then the following equalities hold true for the test* $\varphi = (\varphi^1, \varphi^2)$ *for two objects:*

$$E_{m_1,m_2|l_1,l_2}(\varphi) = E_{m_1|l_1}(\varphi^1) + E_{m_2|l_2}(\varphi^1), \ \text{if} \ \ m_1 \neq l_1, \ \ m_2 \neq l_2, \tag{8.a}$$

$$E_{m_1,m_2|l_1,l_2}(\varphi) = E_{m_i|l_i}(\varphi_i), \text{ if } m_{3-i} = l_{3-i},\ m_i \neq l_i,\ i = 1,2. \quad (8.b)$$

Proof: From the independence of the objects it follows that

$$\alpha_{m_1,m_2|l_1,l_2}(\varphi_N) = \alpha_{m_1|l_1}(\varphi_N^1)\ \alpha_{m_2|l_2}(\varphi_N^2), \text{ if } m_1 \neq l_1,\ m_2 \neq l_2, \quad (9.a)$$

$$\alpha_{m_1,m_2|l_1,l_2}(\varphi_N) = \alpha_{m_i|l_i}(\varphi_N^i)\ [1 - \alpha_{m_{3-i}|l_{3-i}}(\varphi_N^{3-i})], \text{ if } m_{3-i} = l_{3-i},\ m_i \neq l_i. \quad (9.b)$$

According to (2), (9.a), (9.b) and Theorem 1, we obtain (8.a) and (8.b).

For given $E_{1,1|1,3}$, $E_{1,1|3,1}$, $E_{2,2|2,3}$, and $E_{2,2|3,2}$ let us consider the following sets:

$$R_1^1 \triangleq \{Q : D(Q||G_1) \leq E_{1,1|3,1}\}, \quad R_1^2 \triangleq \{Q : D(Q||G_1) \leq E_{1,1|1,3}\},$$

$$R_2^1 \triangleq \{Q : D(Q||G_2) \leq E_{2,2|3,2}\}, \quad R_2^2 \triangleq \{Q : D(Q||G_2) \leq E_{2,2|2,3}\},$$

$$R_3^1 \triangleq \{Q : D(Q||G_1) > E_{1,1|3,1},\ D(Q||G_2) > E_{2,2|3,2}\},$$

$$R_3^2 \triangleq \{Q : D(Q||G_1) > E_{1,1|1,3},\ D(Q||G_2) > E_{2,2|2,3}\}.$$

The optimal values of the reliabilities of the LAO test sequence will be the following:

$$E^*_{1,1|1,3} \triangleq E_{1,1|1,3},\ E^*_{1,1|3,1} \triangleq E_{1,1|3,1},\ E^*_{2,2|2,3} \triangleq E_{2,2|2,3},\ E^*_{2,2|3,2} \triangleq E_{2,2|3,2}, \quad (10.a)$$

$$E^*_{m_1,m_2|l_1,l_2} \triangleq \inf_{Q:Q\in R^i_{l_i}} D(Q||G_{m_i}), \quad m_i \neq l_i,\ m_{3-i} = l_{3-i},\ i = 1,2, \quad (10.b)$$

$$E^*_{m_1,m_2|l_1,l_2} \triangleq E^*_{m_1,m_2|m_1,l_2} + E^*_{m_1,m_2|l_1,m_2}, \quad m_i \neq l_i,\ i = 1,2, \quad (10.c)$$

$$E^*_{m_1,m_2|m_1,m_2} \triangleq \min_{(l_1,l_2)\neq(m_1,m_2)} E^*_{m_1,m_2|l_1,l_2}. \quad (10.d)$$

The main result of the present paper is

Theorem 2: *Let all distributions G_l, $l = \overline{1,3}$, are different, and absolutely continuous each relative to others $0 < D(G_l||G_m) < \infty$, $l \neq m$. If positive elements $E_{1,1|1,3}$, $E_{1,1|3,1}$, $E_{2,2|2,3}$ and $E_{2,2|3,2}$ are given and the following inequalities hold*

$$E_{1,1|1,3} < \min[D(G_3||G_1), D(G_2||G_1)], \quad (11.a)$$

$$E_{1,1|3,1} < \min[D(G_3||G_1), D(G_2||G_1)], \quad (11.b)$$

$$E_{2,2|2,3} < \min[E^*_{2,2|2,1},\ D(G_3||G_2)], \quad (11.c)$$

$$E_{2,2|3,2} < \min[E^*_{2,2|1,2},\ D(G_3||G_2)], \quad (11.d)$$

then:

a) there exists a LAO test sequence φ^*, the reliability matrix of which $\mathbf{E}(\varphi^*) = \{E_{m_1,m_2|l_1,l_2}(\varphi^*)\}$ is defined in (10),

b) if one of the inequalities (11) is violated, then there exists at least one element equal to 0 in the matrix $\mathbf{E}(\varphi^*)$.

Proof: Conditions (11) imply that inequalities (6) hold simultaneously for the both objects. Using (7), we can write inequalities (6) for both objects as follows:

$$E^1_{1|3} < \min(D(G_3||G_1), D(G_2||G_1)), \quad (12.a)$$

$$E^2_{1|3} < \min(D(G_3||G_1), D(G_2||G_1)), \tag{12.b}$$

$$E^2_{2|3} < \min[E^{*,2}_{2|1},\ D(G_3||G_2)], \tag{12.c}$$

$$E^1_{2|3} < \min[E^{*,1}_{2|1},\ D(G_3||G_2)]. \tag{12.d}$$

We shall prove, for example, (12c), which is the consequence of the inequality (11.c). Let us consider a test $\varphi = (\varphi^1, \varphi^2)$, such that $E_{2,2|2,3}(\varphi) = E_{2,2|2,3}$ and $E_{2,2|2,1}(\varphi) = E^*_{2,2|2,1}$. The corresponding error probabilities $\alpha_{2,2|2,3}(\varphi_N)$ and $\alpha_{2,2|2,1}(\varphi_N)$ are given as products defined by (9.b). According to (2) and (9) we obtain that

$$E_{2,2|2,1}(\varphi) = E^{2,*}_{2|1} + \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{2|2}(\varphi^1_N)), \tag{13.a}$$

$$E_{2,2|2,3}(\varphi) = E^2_{2|3} + \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{2|2}(\varphi^1_N)), \tag{13.b}$$

where $E^{2,*}_{2|1} = E_{2|1}(\varphi^2)$ and $E^2_{2|3} = E_{2|3}(\varphi^2)$.

When $\min[E^*_{2,2|2,1},\ D(G_3||G_2)] = E^*_{2,2|2,1}$, from (13) and (11.c) we conclude that $E^2_{2|3} < E^{*,2}_{2|1}$. We shall show that also $E^{2,*}_{2|1} < D(G_3||G_2)$, that again follows from (13) and $E^*_{2,2|2,1} \leq D(G_3||G_2)$. Hence in this case the inequality (12c) holds.

Let us assume now that $\min[E^*_{2,2|2,1},\ D(G_3||G_2)] = D(G_3||G_2)$. First we shall prove, that when the reliability $E_{2,2|2,3}$ satisfies the condition (11.c), then $E^2_{2|3} < D(G_3||G_2)$. Assume that the opposite statement is true, i.e. $E^2_{2|3} \geq D(G_3||G_2)$. In this case using (13.b) and (11.c) we can derive

$$D(G_3||G_2) + \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{2|2}(\varphi^1_N)) \leq E^2_{2|3} + \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{2|2}(\varphi^1_N)) < D(G_3||G_2).$$

$$\overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{2|2}(\varphi^1_N)) < 0.$$

The last inequality states that an index N_0 exists, such that for subsequence of $N > N_0$ we will have $1-\alpha_{2|2}(\varphi^1_N) > 1$, i.e the assumption was wrong.

Now we will prove that also $E^2_{2|3} < E^{2,*}_{2|1}$. From (11.c) it follows that $E_{2,2|2,3} < E^*_{2,2|2,1}$. Using (13) we can derive $E^2_{2|3} < E^{2,*}_{2|1}$. Hence in this case (12c) also takes place.

It follows from (7) and (12) that conditions (6) of Theorem 1 hold for both objects. According to Theorem 1 there exist LAO sequences of tests $\varphi^{1,*}$ and $\varphi^{2,*}$, for the first and the second objects, , such that the elements of the matrices $\mathbf{E}(\varphi^{1,*})$ and $\mathbf{E}(\varphi^{2,*})$ are determined in (5). We will take $\varphi^* = (\varphi^{1,*}, \varphi^{2,*})$ as a test for the model and show that it is LAO and that other elements of the matrix $\mathbf{E}(\varphi^*)$ are determined in (10).

It follows from (12) and (7) that the requirements of Lemma are accomplished. Applying Lemma we can deduce that the reliability matrix $\mathbf{E}(\varphi^*)$ can be obtained from matrices $\mathbf{E}(\varphi^{1,*})$ and $\mathbf{E}(\varphi^{2,*})$ as in (8).

Thus, we obtain that

$$E_{1,1|1,3} = E^2_{1|3},\ \ E_{1,1|3,1} = E^1_{1|3},\quad E_{2,2|1,3} = E^2_{2|3},\ \ E_{2,2|3,2} = E^1_{2|3}. \tag{14}$$

When (11) takes place, according to (8b), (5), (7) and (14) we obtain, that the elements $E_{m_1,m_2|l_1,l_2}(\varphi^*)$ $\ m_i \neq l_i,\ m_{3-i} = l_{3-i},\ i = 1,2$, of the matrix $\mathbf{E}(\varphi^*)$ are determined by relations (10.b). From (8.a) and (10.b) we obtain (10.c). The equality in (10.d) is the particular case of (3). When one of the inequalities (11) is violated, then from (9) and (10.b) we see, that some elements in the matrix $\mathbf{E}(\varphi^*)$ must be equal to 0.

Now let us show that the compound test for two objects is LAO, that is it is optimal. Suppose that for given $E_{1,1|1,3}$, $E_{1,1|3,1}$, $E_{2,2|2,3}$ and $E_{2,2|3,2}$ there exists a test $\varphi' = (\varphi'_1, \varphi'_2)$ with matrix $\mathbf{E}(\varphi')$, such that it has at least one element exceeding the respective element of the matrix $\mathbf{E}(\varphi^*)$. It is contradiction to the fact, that LAO tests $\varphi^{1,*}$ and $\varphi^{2,*}$ have been used for the objects X_1 and X_2.

Remark 2: The similar result may be received if we take alternatively:

$E_{2,1|2,3}$ or $E_{3,1|3,3}$, instead $E_{1,1|1,3}$,
$E_{1,2|3,2}$ or $E_{1,3|3,3}$, instead $E_{1,1|3,1}$,
$E_{1,2|1,3}$ or $E_{3,2|3,3}$, instead $E_{1,1|3,1}$,
$E_{2,1|3,1}$ or $E_{2,3|3,3}$ instead $E_{1,1|3,1}$.

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Երկու անկախ օբյեկտների զույգի նկատմամբ երեք վարկածների լոգարիթմորեն ասիմպտոտորեն օպտիմալ ստուգում

Ե. Ա. Հարությունյան և Փ. Մ. Հակոբյան

Ամփոփում

Դիտարկված են երկու անկախ օբյեկտներից կազմված մոդելի համար վարկածների ստուգման խնդիրը: Հայտնի են երեք հավանականային բաշխումներ, և օբյեկտներից յուրաքանչյուրը անկախորեն ընդունում է դրանցից մեկը: Այս մոդելի համար ուսումնասիրվել է օպտիմալ տեստավորման արդյունքում բոլոր հնարավոր զույգերի սխալների հավանականությունների ցուցիչների (հուսալիությունների) փոխկախվածությունը: Երկու հավանականային բաշխումներով դեպքը ուսումնասիրվել է Հարությունյանի և Ալսվեդեի կողմից [3]: