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On Special Maximum Matchings Constructing*

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Abstract

For bipartite graphs the NP-completeness is proved for the problem of existence of maximum matching which removal leads to a graph with given lower(upper) bound for the cardinality of its maximum matching.

1 Notations and definitions

We consider finite undirected graphs $G = (V(G), E(G))$ without multiple edges or loops [1], where $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. The greatest degree of a vertex of G is denoted by $\Delta(G)$. The cardinality of a maximum matching [1] of G is denoted by $\beta(G)$.

Let $X = \{x_1, ..., x_n\}$ be a set of boolean variables, $D = \{D_1, ..., D_r\}$ be a set of disjunctions, each of which consists of the literals of variables of X. For $j = 1, ..., r$ let us denote by $\tau(D_j)$ the set of indices of the variables which literals are in D_j. Define a conjunctive normal form $K(x_1, ..., x_n)$ in the following way: $K(x_1, ..., x_n) = D_1 \& ... \& D_r$. For $i = 1, ..., n$ we denote by $M(i, K)$ the set $\{D_{m(1,i)}, ..., D_{m(s(i),i)}\}$ of disjunctions from D, which contain a literal of x_i, without loss of generality supposing that $m(1, i) < ... < m(s(i), i)$. For $\sigma \in \{0, 1\}$ and $i = 1, ..., n$ define:

$$x_i^\sigma \equiv \begin{cases} x_i, & \text{if } \sigma = 1 \\ \bar{x}_i, & \text{if } \sigma = 0, \end{cases}$$

$$M_\sigma(i, K) \equiv \{D_j | 1 \le j \le r, D_j \in M(i, K), D_j \text{ contains } x_i^\sigma\}.$$

For a sequence $\tilde{\alpha} = (\alpha_1, ..., \alpha_n)$ in which $\alpha_i \in \{0, 1\}$, $i = 1, ..., n$ and a conjunctive normal form $K(x_1, ..., x_n)$ we define the set $Sat(K, \tilde{\alpha}) \subseteq D$ in the following way:

$$Sat(K, \tilde{\alpha}) \equiv \{D_j | 1 \le j \le r, D_j(\alpha_1, ..., \alpha_n) = 1\}.$$

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DEFINITION. The graph of the conjunctive normal form $K(x_1,...,x_n) = D_1\&...\&D_r$ is a graph $G(K(x_1,...,x_n))$ for which the sets $V(G(K(x_1,...,x_n)))$ and $E(G(K(x_1,...,x_n)))$ of vertices and edges, respectively, are defined as follows:

$V(G(K(x_1,x_2,...,x_n))) = \bigcup_{j=1}^{r} V(D_j)$, where for $j = 1,...,r$

$V(D_j) = \left\{v_i^{(j)} | 1 \le i \le n,\ i \in \tau(D_j)\right\}$;

$E(G(K(x_1,x_2,...,x_n))) = E_1(G(K(x_1,x_2,...,x_n))) \cup E_2(G(K(x_1,x_2,...,x_n)))$,

where $E_1(G(K(x_1,x_2,...,x_n))) = \bigcup_{j=1}^{r} E_{1j}(G(K(x_1,x_2,...,x_n)))$, and for $j = 1,...,r$

$E_{1j}(G(K(x_1,x_2,...,x_n))) = \{(u,w) \mid \{u,w\} \subseteq V(D_j), u \ne w\}$,

$E_2(G(K(x_1,x_2,...,x_n))) = \bigcup_{p=1}^{n} E_{2p}(G(K(x_1,x_2,...,x_n)))$, and for $p = 1,...,n$

$E_{2p}(G(K(x_1,x_2,...,x_n))) = \left\{\left(v_p^{(s)}, v_p^{(t)}\right) | 1 \le s \le r, 1 \le t \le r, s \ne t, p \in \tau(D_s) \cap \tau(D_t)\right\}$.

Non defined conceptions, terms and notations can be found in [2-5].

2 Auxiliary results

Let θ be a rectangular coordinate system which is defined on a plane γ. We denote by $P_\gamma(\theta)$ the set of all points of γ which coordinates are integer. For any graph G with $V(G) \subset P_\gamma(\theta)$ we imagine a vertex of G as an ordered pair (x,y) where x and y are its abscissa and ordinate in θ, respectively.

Lemma. Let G satisfy the condition $V(G) \subset P_\gamma(\theta)$. If the number of edges $((x',y'),(x'',y''))$ for which $|x'-x''| \equiv |y'-y''| \pmod 2$ is even on an arbitrary simple cycle of G then G is bipartite.

Proof is evident.

PROBLEM1. Maximal 2- Satisfiability.

CONDITION. Given a set X of boolean variables, a conjunctive normal form $K(x_1,...,x_n) = D_1\&...\&D_r$, for which $|\tau(D_j)| = 2$, $j = 1,...,r$, and a positive integer l, $l \le r$.

QUESTION. Does there exist a sequence $\tilde{\alpha} = (\alpha_1,...,\alpha_n)$ with $\alpha_i \in \{0,1\}$ for $i = 1,...,n$ such that $|Sat(K,\tilde{\alpha})| \ge l$?

Theorem1. [6] The **PROBLEM1** is NP-complete.

Now let us define the following particular case of the **PROBLEM1**

PROBLEM2. Connected Maximal 2- Satisfiability.

CONDITION. Given a set X of boolean variables, a conjunctive normal form $K(x_1,...,x_n) = D_1\&...\&D_r$, for which $|\tau(D_j)| = 2$, $j = 1,...,r$, the graph $G(K(x_1,...,x_n))$ is connected, and a positive integer l, $l \le r$.

QUESTION. Does there exist a sequence $\tilde{\alpha} = (\alpha_1,...,\alpha_n)$ with $\alpha_i \in \{0,1\}$ for $i = 1,...,n$ such that $|Sat(K,\tilde{\alpha})| \ge l$?

Theorem2. The **PROBLEM2** is NP-complete.

Proof. Evidently, the **PROBLEM2** belongs to the class NP. Let us describe a polynomial algorithm, reducing the **PROBLEM1** to the **PROBLEM2**.

Suppose, that in the individual problem I' of the **PROBLEM1** $X' = \{x_1,...,x_n\}$ is the set of boolean variables, $K(x_1,...,x_n) = D_1\&...\&D_r$ is the conjunctive normal form, for which $|\tau(D_j)| = 2$, $j = 1,...,r$ and l' is the positive integer, $l' \le r$.

Set:

$$X'' = X' \cup \{x_{n+1}\}, \text{ where } x_{n+1} \notin X'.$$

It is not difficult to see that for every $j, 1 \le j \le r$, there exist $x_{i(1,j)} \in X', x_{i(2,j)} \in X', \sigma_{i(1,j)} \in \{0,1\}, \sigma_{i(2,j)} \in \{0,1\}$ which satisfy the conditions:

$$1 \le i(1,j) \le n, 1 \le i(2,j) \le n;$$
$$\tau(D_j) = \{i(1,j), i(2,j)\};$$
$$D_j = x_{i(1,j)}^{\sigma_{i(1,j)}} \vee x_{i(2,j)}^{\sigma_{i(2,j)}}.$$

For $j = 1, ..., r$ assume:

$$K''(D_j) =$$
$$(x_{i(1,j)}^{\bar{\sigma}_{i(1,j)}} \vee x_{i(2,j)}^{\bar{\sigma}_{i(2,j)}}) \& (x_{i(1,j)}^{\sigma_{i(1,j)}} \vee x_{n+1}) \& (x_{i(2,j)}^{\sigma_{i(2,j)}} \vee x_{n+1}) \& (x_{i(1,j)}^{\sigma_{i(1,j)}} \vee \bar{x}_{n+1}) \& (x_{i(2,j)}^{\sigma_{i(2,j)}} \vee \bar{x}_{n+1}).$$

Consider the conjunctive normal form $K''(x_1, ..., x_n, x_{n+1})$ and the positive integer l'' defined as follows:

$$K''(x_1, ..., x_n, x_{n+1}) \equiv \&_{j=1}^{r} K''(D_j);$$

$$l'' = l' + 3r.$$

Note that every disjunction of the conjunctive normal form $K''(x_1, ..., x_n, x_{n+1})$ contains exactly two literals of the variables of the set X'' and $l'' \le 5r$. Moreover, as for every $K''(D_j), 1 \le j \le r$, there exists a disjunction containing a literal of the variable $x_{n+1} \in X''$, it is not hard to see that the graph $G(K''(x_1, ..., x_n, x_{n+1}))$ is connected.

Consider the individual problem I'' of the **PROBLEM2**, for which the set of variables is X'', the conjunctive normal form is $K''(x_1, ..., x_n, x_{n+1})$, and the positive integer is l''.

Let us show that the individual problem I' has a positive answer if and only if the individual problem I'' has a positive answer.

Suppose that the sequence $\tilde{\beta} = (\beta_1, ..., \beta_n)$ ($\beta_i \in \{0,1\}$, $i = 1, ..., n$) of the values of the variables $x_1, ..., x_n$ satisfies the inequality $\left|Sat(K', \tilde{\beta})\right| \ge l'$. Consider the sequence $\tilde{\alpha} = (\alpha_1, ..., \alpha_n, \alpha_{n+1})$, where $\alpha_i \equiv \beta_i$ for $i = 1, ..., n$, and $\alpha_{n+1} \equiv 0$. Let us show that $|Sat(K'', \tilde{\alpha})| \ge l''$. Note that for $j = 1, ..., r$

$$|Sat(K''(D_j), \tilde{\alpha})| = \begin{cases} 4, & \text{if } D_j \in Sat(K', \tilde{\beta}) \\ 3, & \text{if } D_j \notin Sat(K', \tilde{\beta}). \end{cases}$$

Therefore
$$|Sat(K'', \tilde{\alpha})| = 4\left|Sat(K', \tilde{\beta})\right| + 3(r - \left|Sat(K', \tilde{\beta})\right|) = 3r + \left|Sat(K', \tilde{\beta})\right| \ge 3r + l' = l''.$$

Now, suppose that the sequence $\tilde{\alpha} = (\alpha_1, ..., \alpha_n, \alpha_{n+1})$ $(\alpha_i \in \{0,1\}, i = 1, ..., n+1)$ of the values of the variables $x_1, ..., x_n, x_{n+1}$ satisfies the inequality $|Sat(K'', \tilde{\alpha})| \ge l''$. Consider the sequence $\tilde{\beta} = (\beta_1, ..., \beta_n)$, where $\beta_i \equiv \alpha_i$ for $i = 1, ..., n$. Let us show that $\left|Sat(K', \tilde{\beta})\right| \ge l'$.

First of all note that for $j = 1, ..., r$

$$3 \le |Sat(K''(D_j), \tilde{\alpha})| \le 4.$$

Assume

$$T(\bar{\alpha}) \equiv \{D_j | 1 \leq j \leq r, |Sat(K''(D_j), \bar{\alpha})| = 4\}.$$

Now we have

$$l' + 3r = l'' \leq |Sat(K'', \bar{\alpha})| = 4|T(\bar{\alpha})| + 3(r - |T(\bar{\alpha})|) = 3r + |T(\bar{\alpha})|,$$

therefore

$$|T(\bar{\alpha})| \geq l'.$$

It is not hard to see that

$$Sat(K', \bar{\beta}) \equiv T(\bar{\alpha}),$$

so

$$\left|Sat(K', \bar{\beta})\right| = |T(\bar{\alpha})| \geq l'.$$

As the individual problem I'' is constructed from the individual problem I' by a polynomial algorithm, the proof of the **Theorem2** is complete.

§3. Formulation of the main problems and investigation of their complexity

PROBLEM3.
CONDITION. Given a graph G and a positive integer k.
QUESTION. Does there exist a maximum matching $F_0(G)$ of G such that $\beta(G \backslash F_0(G)) \geq k$?
PROBLEM4.
CONDITION. Given a graph G and a positive integer k.
QUESTION. Does there exist a maximum matching $F_0(G)$ of G such that $\beta(G \backslash F_0(G) \leq k$?

Theorem3. The **PROBLEM3** is NP-complete for connected bipartite graphs G with $\Delta(G) = 3$.

Proof. Evidently [7,8], the **PROBLEM3** belongs to NP. Let us describe a polynomial algorithm which reduces the **PROBLEM2** to the **PROBLEM3** restricted to connected bipartite graphs G with $\Delta(G) = 3$.

Consider an individual problem I of the **PROBLEM2**, in which $X = \{x_1, ..., x_n\}$ is the set of variables, $K(x_1, ..., x_n) = D_1 \& ... \& D_r$ is the conjunctive normal form and l is the positive integer.

For $i = 1, ..., n$ and $j = 1, ..., r$ define the set $V(i, j)$ as follows:

$$V(i,j) \equiv \begin{cases} \{v_{11}(i,j), v_{12}(i,j), v_{21}(i,j), v_{22}(i,j), u_{11}(i,j), u_{12}(i,j), u_{21}(i,j), u_{22}(i,j)\}, & \text{if } D_j \in M(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

Assume:

$$V_1(X, K) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} V(i, j).$$

For $j = 1, ..., r$ define the set $V(j)$ in the following way:

$$V(j) \equiv \{w_1(j), w_2(j)\},$$

and let

$$V_2(X, K) \equiv \bigcup_{j=1}^{r} V(j).$$

Denote

$$V(G(I)) \equiv V_1(X, K) \cup V_2(X, K).$$

For $i = 1, ..., n$ and $j = 1, ..., r$ define the set $E_1(i, j)$ as follows:

$$E_1(i,j) \equiv$$

$$\begin{cases} \{(v_{11}(i,j), v_{12}(i,j)), (v_{12}(i,j), v_{22}(i,j)), (v_{21}(i,j), v_{22}(i,j)), \\ (v_{21}(i,j), u_{12}(i,j)), (v_{22}(i,j), u_{22}(i,j)), (u_{11}(i,j), u_{12}(i,j)), & \text{if} \quad D_j \in M_1(i,K); \\ (u_{21}(i,j), u_{22}(i,j))\}, \\ \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), (u_{12}(i,j), v_{11}(i,j)), \\ (u_{22}(i,j), v_{21}(i,j)), (v_{11}(i,j), v_{12}(i,j)), (v_{21}(i,j), v_{22}(i,j)), & \text{if} \quad D_j \in M_0(i,K); \\ (v_{12}(i,j), v_{22}(i,j))\}, & \text{if} \quad D_j \notin M(i,K). \\ \emptyset, \end{cases}$$

Assume:

$$E_1(X, K) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} E_1(i, j).$$

For $j = 1, ..., r$ define the set $E_2(j)$ as follows:

$$E_2(j) \equiv \{(w_1(j), w_2(j))\},$$

and let

$$E_2(X, K) \equiv \bigcup_{j=1}^{r} E_2(j).$$

For $i = 1, ..., n$ and $j = 1, ..., r$ define the set $E_3(i, j)$ as follows:

$$E_3(i,j) \equiv \begin{cases} \{(w_1(j), v_{12}(i,j))\}, & \text{if} \quad D_j \in M(i,K); \\ \emptyset, & \text{if} \quad D_j \notin M(i,K), \end{cases}$$

and let

$$E_3(X, K) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} E_3(i, j).$$

For $i = 1, ..., n$ define the set $E_4(i)$ as follows:

$$E_4(i) \equiv$$

$$\begin{cases} \{(v_{11}(i, m(1,i)), v_{21}(i, m(1,i)))\}, & \text{if} \quad s(i) = 1; \\ \{(v_{21}(i, m(1,i)), v_{11}(i, m(2,i))), (v_{21}(i, m(2,i)), v_{11}(i, m(3,i))), ..., \\ (v_{21}(i, m(s(i)-1,i)), v_{11}(i, m(s(i),i))), (v_{21}(i, m(s(i),i)), v_{11}(i, m(1,i)))\}, & \text{if} \quad s(i) > 1 \end{cases}$$

Assume:

$$E_4(X,K) \equiv \bigcup_{i=1}^{n} E_4(i).$$

Denote

$$E(G(I)) \equiv E_1(X,K) \cup E_2(X,K) \cup E_3(X,K) \cup E_4(X,K).$$

The definition of the graph $G(I)$ is complete. Clearly, $|V(G(I))| = 18r$, $|E(G(I))| = 19r$, $|\beta(G(I))| = 9r$, $\Delta(G(I)) = 3$.

The definition of the **PROBLEM2** and the construction of $G(I)$ imply that $G(I)$ is connected. It is not hard to see that **Lemma** implies $G(I)$ to be bipartite.

Consider an individual problem I' of the **PROBLEM3** where $G = G(I)$ and $k = 5r + l$. Clearly, I' is constructed from I in a time which is bounded by a polynom from $Length[I]$ [5].

Let us show that I has a positive answer if and only if I' has a positive answer. Suppose I has a positive answer and the sequence $\tilde{\varepsilon} = (\varepsilon_1, ..., \varepsilon_n)$ where $\varepsilon_i \in \{0,1\}$ for $i = 1, ..., n$ satisfies the inequality $|Sat(K, \tilde{\varepsilon})| \geq l$. Let us show that there is a maximum matching $F_0(G(I))$ of the graph $G(I)$ such that $\beta(G \backslash F_0(G(I))) \geq 5r + l$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $F_{0,1}(G(I), i, j)$ of the set $E(G(I))$ in the following way:

$$F_{0,1}(G(I), i, j) \equiv$$
$$\begin{cases} \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), \\ (v_{21}(i,j), v_{22}(i,j)), (v_{11}(i,j), v_{12}(i,j))\}, & \text{if } D_j \in M(i,K) \text{ and } \varepsilon_i = 0; \\ \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), \\ (v_{12}(i,j), v_{22}(i,j))\}, & \text{if } D_j \in M(i,K) \text{ and } \varepsilon_i = 1; \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$F_{0,1}(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} F_{0,1}(G(I), i, j).$$

For $j = 1, ..., r$ define a subset $F_{0,2}(G(I), j)$ of the set $E(G(I))$ in the following way:

$$F_{0,2}(G(I), j) \equiv E_2(j),$$

and let

$$F_{0,2}(G(I)) \equiv \bigcup_{j=1}^{r} F_{0,2}(G(I), j).$$

For $i = 1, ..., n$ define a subset $F_{0,3}(G(I), i)$ of the set $E(G(I))$ in the following way:

$$F_{0,3}(G(I), i) \equiv \begin{cases} E_4(i), & \text{if } \varepsilon_i = 1; \\ \emptyset, & \text{if } \varepsilon_i = 0, \end{cases}$$

and let

$$F_{0,3}(G(I)) \equiv \bigcup_{i=1}^{n} F_{0,3}(G(I),i).$$

Assume:

$$F_0(G(I)) \equiv F_{0,1}(G(I)) \cup F_{0,2}(G(I)) \cup F_{0,3}(G(I)).$$

It is easy to see that $F_0(G(I))$ is a perfect matching of the graph $G(I)$. Let us show that $\beta(G(I)\backslash F_0(G(I))) \geq k$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $F_{1,1}(G(I), i, j)$ of the set $E(G(I))$ in the following way:

$$F_{1,1}(G(I),i,j) \equiv \begin{cases} \{(u_{12}(i,j), v_{21}(i,j)), (u_{22}(i,j), v_{22}(i,j))\}, & \text{if } D_j \in M_1(i,K); \\ \{(u_{12}(i,j), v_{11}(i,j)), (u_{22}(i,j), v_{21}(i,j))\}, & \text{if } D_j \in M_0(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$F_{1,1}(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} F_{1,1}(G(I),i,j).$$

Clearly $|F_{1,1}(G(I))| = 4r$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $U_1(i, j)$ of the set $V(G(I))$ as follows:

$$U_1(i,j) \equiv \begin{cases} \{v_{11}(i,j), v_{12}(i,j)\}, & \text{if } D_j \in M_1(i,K); \\ \{v_{12}(i,j), v_{22}(i,j)\}, & \text{if } D_j \in M_0(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

For $j = 1, ..., r$ assume:

$$U_1(j) \equiv (\bigcup_{i=1}^{n} U_1(i,j)) \cup \{w_1(j)\}.$$

Clearly $|U_1(j)| = 5$, $j = 1, ..., r$. Let G'_j denote the subgraph of the graph $G(I)\backslash F_0(G(I))$ induced by the set of vertices $U_1(j)$, $j = 1, ..., r$. Clearly $1 \leq \beta(G'_j) \leq 2$, $j = 1, ..., r$.

As for $i = 1, ..., n$ and $p = 1, ..., s(i)$ the vertices $u_{12}(i, p)$ and $u_{22}(i, p)$ are endpoints [1] of the graph $G(I)\backslash F_0(G(I))$, there is a maximum matching F of the graph $G(I)\backslash F_0(G(I))$ such that $F_{1,1}(G(I)) \subset F$. Note that for $j = 1, ..., r$ the set $F \cap E(G'_j)$ is a maximum matching of the graph G'_j, thus

$$\beta(G(I)\backslash F_0(G(I))) = 4r + \sum_{j=1}^{r} \beta(G'_j).$$

It is easy to see that for $j = 1, ..., r$ the equality $D_j(\varepsilon_1, ..., \varepsilon_n) = 1$ implies $\beta(G'_j) = 2$, and therefore

$$\beta(G(I)\backslash F_0(G(I))) \geq 4r + 2\,|Sat(K,\bar{\varepsilon})| + (r - |Sat(K,\bar{\varepsilon})|) = 5r + |Sat(K,\bar{\varepsilon})| \geq 5r + l = k.$$

Now suppose that the answer of the individual problem I' is positive. Let us show that the answer of the problem I is positive.

Let T_1 and T_2 be matchings of the graphs $G(I)$ and $G(I)\backslash T_1$, respectively, satisfying the conditions :

$$|T_1| = \beta(G(I)),\ |T_2| = \beta(G(I)\backslash T_1) \geq 5r + l = k.$$

Let us show that there is a sequence $\tilde{\nu} = (\nu_1, ..., \nu_n)$ with $\nu_i \in \{0,1\}$ for $i = 1, ..., n$, such that $|Sat(K, \tilde{\nu})| \geq l$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $E_5(i,j)$ of the set $E(G(I))$ in the following way:

$$E_5(i,j) \equiv \begin{cases} \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j))\}, & \text{if } D_j \in M(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$E_5(G(I)) \equiv \bigcup_{j=1}^{r}((\bigcup_{i=1}^{n} E_5(i,j)) \cup E_2(j)).$$

Clearly $|E_5(G(I))| = 5r$. As the graph $G(I)$ contains a perfect matching, the following holds: $E_5(G(I)) \subset T_1$, therefore, without loss of generality, we may assume that $F_{1,1}(G(I)) \subset T_2$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $V_1(i,j)$ of the set $V(G(I))$ as follows:

$$V_1(i,j) \equiv \begin{cases} \{v_{11}(i,j), v_{12}(i,j), v_{21}(i,j), v_{22}(i,j)\}, & \text{if } D_j \in M(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

For $i = 1, ..., n$ assume:

$$V_1(i) \equiv \bigcup_{j=1}^{r} V_1(i,j),$$

and let

$$V_1(G(I)) \equiv \bigcup_{i=1}^{n} V_1(i).$$

For $i = 1, ..., n$ let $G(i)$ denote the subgraph of the graph $G(I)$ induced by the set of vertices $V_1(i)$, and let G_1 be the subgraph of the graph $G(I)$ induced by the set of vertices $V_1(G(I))$. It is easy to see that

$$|V_1(G(I))| = |E(G_1)| = \sum_{i=1}^{n} |V_1(i)| = 8r.$$

Clearly, $G(1), ..., G(n)$ are connected components of G_1, and for $i = 1, ..., n$ the subgraph $G(i)$ is an even simple cycle containing two edge disjoint perfect matchings $F'(i)$ and $F''(i)$. Without loss of generality we may assume that for $i = 1, ..., n$

$$F'(i) \cap E_4(i) \neq \emptyset,\ F''(i) \cap E_4(i) = \emptyset.$$

Evidently, G_1 has a perfect matching containing $\beta(G_1) = 4r$ edges. As $(T_1 \backslash E_5(G(I))) \subset E(G_1)$ and $|T_1 \backslash E_5(G(I))| = 4r$, the set $T_1 \backslash E_5(G(I))$ is a perfect matching of the graph G_1, therefore $(T_1 \backslash E_5(G(I))) \cap E(G(i))$ is a perfect matching in $G(i)$, $i = 1, ..., n$. For $i = 1, ..., n$ set

$$\nu_i \equiv \begin{cases} 1, & \text{if } (T_1 \backslash E_5(G(I))) \cap E(G(i)) = F'(i); \\ 0, & \text{if } (T_1 \backslash E_5(G(I))) \cap E(G(i)) = F''(i), \end{cases}$$

and

$$\tilde{\nu} \equiv (\nu_1, ..., \nu_n).$$

Let us show that $|Sat(K, \tilde{\nu})| \geq l$. For $j = 1, ..., r$ denote by G''_j the subgraph of the graph $G(I) \backslash T_1$ induced by the set of vertices $U_1(j)$. As $F_{1,1}(G(I)) \subset T_2$, we have

$$5r + l \leq |T_2| = |F_{1,1}(G(I))| + \sum_{j=1}^{r} \beta(G''_j) = 4r + \sum_{j=1}^{r} \beta(G''_j),$$

and therefore

$$r + l \leq \sum_{j=1}^{r} \beta(G''_j).$$

It is not hard to see that for $j = 1, ..., r$ $1 \leq \beta(G''_j) \leq 2$. Set

$$J \equiv \{D_j | 1 \leq j \leq r, \beta(G''_j) = 2\}.$$

Clearly $|J| \geq l$. Let us show that $J \subseteq Sat(K, \tilde{\nu})$. It is not hard to check that if $D_j \notin Sat(K, \tilde{\nu})$, $1 \leq j \leq r$, then $\beta(G''_j) = 1$, and therefore $D_j \notin J$. Thus $|Sat(K, \tilde{\nu})| \geq |J| \geq l$. The proof of the **Theorem3** is complete.

Theorem4. The **PROBLEM4** is NP-complete for connected bipartite graphs G with $\Delta(G) = 3$.

Proof. Evidently [7,8], the **PROBLEM4** belongs to NP. Let us describe a polynomial algorithm which reduces the **PROBLEM2** to the **PROBLEM4** restricted to connected bipartite graphs G with $\Delta(G) = 3$.

Consider an individual problem I of the **PROBLEM2**, in which $X = \{x_1, ..., x_n\}$ is the set of variables, $K(x_1, ..., x_n) = D_1 \& ... \& D_r$ is the conjunctive normal form and l is the positive integer.

For $i = 1, ..., n$ and $j = 1, ..., r$ define the set $V(i, j)$ as follows:

$$V(i,j) \equiv$$
$$\begin{cases} \{v_{11}(i,j), v_{12}(i,j), v_{21}(i,j), v_{22}(i,j), u_{11}(i,j), u_{12}(i,j), u_{21}(i,j), u_{22}(i,j)\}, & \text{if } D_j \in M(i,K) \\ \emptyset, & \text{if } D_j \notin M(i,K) \end{cases}$$

Assume:

$$V(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} V(i,j).$$

For $i = 1, ..., n$ and $j = 1, ..., r$ define the set $E_1(i, j)$ as follows:

$$E_1(i,j) \equiv$$
$$\begin{cases} \{(v_{11}(i,j), v_{12}(i,j)), (v_{12}(i,j), v_{22}(i,j)), (v_{21}(i,j), v_{22}(i,j)), \\ (v_{21}(i,j), u_{12}(i,j)), (v_{22}(i,j), u_{22}(i,j)), (u_{11}(i,j), u_{12}(i,j)), & \text{if } D_j \in M_1(i,K); \\ (u_{21}(i,j), u_{22}(i,j))\}, \\ \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), (u_{12}(i,j), v_{11}(i,j)), \\ (u_{22}(i,j), v_{21}(i,j)), (v_{11}(i,j), v_{12}(i,j)), (v_{21}(i,j), v_{22}(i,j)), \\ (v_{12}(i,j), v_{22}(i,j))\}, & \text{if } D_j \in M_0(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

Assume:

$$E_1(X,K) \equiv \bigcup_{i=1}^{n}\bigcup_{j=1}^{r} E_1(i,j).$$

For $j=1,...,r$ define the set $E_2(j)$ as follows:

$$E_2(j) \equiv \{(v_{12}(i_1,j), v_{12}(i_2,j))\},$$

where $\tau(D_j)=\{i_1,i_2\}$.
Assume:

$$E_2(X,K) \equiv \bigcup_{j=1}^{r} E_2(j).$$

For $i=1,...,n$ define the set $E_3(i)$ as follows:

$$E_3(i) \equiv \begin{cases} \{(v_{11}(i,m(1,i)), v_{21}(i,m(1,i)))\}, & \text{if } s(i)=1; \\ \{(v_{21}(i,m(1,i)), v_{11}(i,m(2,i))), (v_{21}(i,m(2,i)), v_{11}(i,m(3,i))), ..., \\ (v_{21}(i,m(s(i)-1,i)), v_{11}(i,m(s(i),i))), (v_{21}(i,m(s(i),i)), v_{11}(i,m(1,i)))\}, & \text{if } s(i)>1. \end{cases}$$

Assume:

$$E_3(X,K) \equiv \bigcup_{i=1}^{n} E_3(i).$$

Denote

$$E(G(I)) \equiv E_1(X,K) \cup E_2(X,K) \cup E_3(X,K).$$

The definition of the graph $G(I)$ is complete. Clearly, $|V(G(I))| = 16r$, $|E(G(I))| = 17r$, $|\beta(G(I))| = 8r$, $\Delta(G(I)) = 3$.

The definition of the **PROBLEM2** and the construction of $G(I)$ imply that $G(I)$ is connected. It is not hard to see that **Lemma** implies $G(I)$ to be bipartite.

Consider an individual problem I' of the **PROBLEM4** where $G=G(I)$ and $k=6r-l$. Clearly, I' is constructed from I in a time which is bounded by a polynom from $Length[I]$.

Let us show that I has a positive answer if and only if I' has a positive answer. Suppose I has a positive answer and the sequence $\tilde{\varepsilon}=(\varepsilon_1,...,\varepsilon_n)$ where $\varepsilon_i \in \{0,1\}$ for $i=1,...,n$ satisfies the inequality $|Sat(K,\tilde{\varepsilon})| \geq l$. Let us show that there is a maximum matching $F_0(G(I))$ of the graph $G(I)$ such that $\beta(G \backslash F_0(G(I))) \leq 6r-l$.

For $i=1,...,n$ and $j=1,...,r$ define a subset $F_{0,1}(G(I),i,j)$ of the set $E(G(I))$ in the following way:

$$F_{0,1}(G(I),i,j) \equiv \begin{cases} \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), \\ (v_{21}(i,j), v_{22}(i,j)), (v_{11}(i,j), v_{12}(i,j))\}, & \text{if } D_j \in M(i,K) \text{ and } \varepsilon_i=1; \\ \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j)), \\ (v_{12}(i,j), v_{22}(i,j))\}, & \text{if } D_j \in M(i,K) \text{ and } \varepsilon_i=0; \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$F_{0,1}(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} F_{0,1}(G(I),i,j).$$

For $i = 1, ..., n$ define a subset $F_{0,2}(G(I),i)$ of the set $E(G(I))$ in the following way:

$$F_{0,2}(G(I),i) \equiv \begin{cases} E_3(i), & \text{if } \varepsilon_i = 0; \\ \emptyset, & \text{if } \varepsilon_i = 1, \end{cases}$$

and let

$$F_{0,2}(G(I)) \equiv \bigcup_{i=1}^{n} F_{0,2}(G(I),i).$$

Assume:

$$F_0(G(I)) \equiv F_{0,1}(G(I)) \cup F_{0,2}(G(I)).$$

It is easy to see that $F_0(G(I))$ is a perfect matching of the graph $G(I)$. Let us show that $\beta(G(I)\backslash F_0(G(I))) \leq k$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $F_{1,1}(G(I),i,j)$ of the set $E(G(I))$ in the following way:

$$F_{1,1}(G(I),i,j) \equiv \begin{cases} \{(u_{12}(i,j), v_{21}(i,j)), (u_{22}(i,j), v_{22}(i,j))\}, & \text{if } D_j \in M_1(i,K); \\ \{(u_{12}(i,j), v_{11}(i,j)), (u_{22}(i,j), v_{21}(i,j))\}, & \text{if } D_j \in M_0(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$F_{1,1}(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} F_{1,1}(G(I),i,j).$$

Clearly $|F_{1,1}(G(I))| = 4r$.

For $i = 1, ..., n$ and $j = 1, ..., r$ define a subset $U_1(i,j)$ of the set $V(G(I))$ as follows:

$$U_1(i,j) \equiv \begin{cases} \{v_{11}(i,j), v_{12}(i,j)\}, & \text{if } D_j \in M_1(i,K); \\ \{v_{12}(i,j), v_{22}(i,j)\}, & \text{if } D_j \in M_0(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

For $j = 1, ..., r$ assume:

$$U_1(j) \equiv \bigcup_{i=1}^{n} U_1(i,j).$$

Clearly $|U_1(j)| = 4$, $j = 1, ..., r$. Let G'_j denote the subgraph of the graph $G(I)\backslash F_0(G(I))$ induced by the set of vertices $U_1(j)$, $j = 1, ..., r$. Clearly $1 \leq \beta(G'_j) \leq 2$, $j = 1, ..., r$.

As for $i = 1, ..., n$ and $p = 1, ..., s(i)$ the vertices $u_{12}(i,p)$ and $u_{22}(i,p)$ are endpoints of the graph $G(I)\backslash F_0(G(I))$, there is a maximum matching F of the graph $G(I)\backslash F_0(G(I))$ such that $F_{1,1}(G(I)) \subset F$. Note that for $j = 1, ..., r$ the set $F \cap E(G'_j)$ is a maximum matching of the graph G'_j, thus

$$\beta(G(I)\backslash F_0(G(I))) = 4r + \sum_{j=1}^{r} \beta(G'_j).$$

It is easy to see that for $j = 1, \dots, r$ the equality $D_j(\varepsilon_1, \dots, \varepsilon_n) = 1$ implies $\beta(G'_j) = 1$, and therefore

$$\beta(G(I)\backslash F_0(G(I))) \leq 4r + |Sat(K, \bar{\varepsilon})| + 2(r - |Sat(K, \bar{\varepsilon})|) = 6r - |Sat(K, \bar{\varepsilon})| \leq 6r - l = k.$$

Now suppose that the answer of the individual problem I' is positive. Let us show that the answer of the problem I is positive.

Let T_1 and T_2 be matchings of the graphs $G(I)$ and $G(I)\backslash T_1$, respectively, satisfying the conditions :

$$|T_1| = \beta(G(I)),\ |T_2| = \beta(G(I)\backslash T_1) \leq k = 6r - l.$$

Let us show that there is a sequence $\bar{\nu} = (\nu_1, \dots, \nu_n)$ with $\nu_i \in \{0, 1\}$ for $i = 1, \dots, n$, such that $|Sat(K, \bar{\nu})| \geq l$.

For $i = 1, \dots, n$ and $j = 1, \dots, r$ define a subset $E_4(i, j)$ of the set $E(G(I))$ in the following way:

$$E_4(i,j) \equiv \begin{cases} \{(u_{11}(i,j), u_{12}(i,j)), (u_{21}(i,j), u_{22}(i,j))\}, & \text{if } D_j \in M(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K), \end{cases}$$

and let

$$E_4(G(I)) \equiv \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} E_4(i,j).$$

Clearly $|E_4(G(I))| = 4r$. As the graph $G(I)$ contains a perfect matching, the following holds: $E_4(G(I)) \subset T_1$, therefore:

1) without loss of generality, we may assume that $F_{1,1}(G(I)) \subset T_2$,

2)$T_1 \cap E_2(X, K) = \emptyset$.

For $i = 1, \dots, n$ and $j = 1, \dots, r$ define a subset $V_1(i, j)$ of the set $V(G(I))$ as follows:

$$V_1(i,j) \equiv \begin{cases} \{v_{11}(i,j), v_{12}(i,j), v_{21}(i,j), v_{22}(i,j)\}, & \text{if } D_j \in M(i,K); \\ \emptyset, & \text{if } D_j \notin M(i,K). \end{cases}$$

For $i = 1, \dots, n$ assume:

$$V_1(i) \equiv \bigcup_{j=1}^{r} V_1(i,j),$$

and let

$$V_1(G(I)) \equiv \bigcup_{i=1}^{n} V_1(i).$$

For $i = 1, \dots, n$ let $G(i)$ denote the subgraph of the graph $G(I)$ induced by the set of vertices $V_1(i)$, and let G_1 be the subgraph of the graph $G(I)$ induced by the set of vertices $V_1(G(I))$. Set $G_2 \equiv G_1 \backslash E_2(X, K)$. It is easy to see that

$$|E(G_2)| = |V_1(G(I))| = \sum_{i=1}^{n} |V_1(i)| = 8r.$$

Clearly, $G(1), ..., G(n)$ are connected components of G_2, and for $i = 1, ..., n$ the subgraph $G(i)$ is an even simple cycle containing two edge disjoint perfect matchings $F'(i)$ and $F''(i)$. Without loss of generality we may assume that for $i = 1, ..., n$

$$F'(i) \cap E_3(i) \neq \emptyset,\ F''(i) \cap E_3(i) = \emptyset.$$

Evidently, G_2 has a perfect matching containing $\beta(G_2) = 4r$ edges. As $(T_1 \backslash E_4(G(I))) \subset E(G_2)$ and $|T_1 \backslash E_4(G(I))| = 4r$, the set $T_1 \backslash E_4(G(I))$ is a perfect matching of the graph G_2, therefore $(T_1 \backslash E_4(G(I))) \cap E(G(i))$ is a perfect matching of the graph $G(i)$, $i = 1, ..., n$. For $i = 1, ..., n$ set

$$\nu_i \equiv \begin{cases} 0, & \text{if } (T_1 \backslash E_5(G(I))) \cap E(G(i)) = F'(i); \\ 1, & \text{if } (T_1 \backslash E_5(G(I))) \cap E(G(i)) = F''(i), \end{cases}$$

and

$$\tilde{\nu} \equiv (\nu_1, ..., \nu_n).$$

Let us show that $|Sat(K, \tilde{\nu})| \geq l$. For $j = 1, ..., r$ denote by G''_j the subgraph of the graph $G(I) \backslash T_1$ induced by the set of vertices $U_1(j)$. As $F_{1,1}(G(I)) \subset T_2$, we have

$$6r - l \geq |T_2| = |F_{1,1}(G(I))| + \sum_{j=1}^{r} \beta(G''_j) = 4r + \sum_{j=1}^{r} \beta(G''_j),$$

and therefore

$$\sum_{j=1}^{r} \beta(G''_j) \leq 2r - l.$$

It is not hard to see that for $j = 1, ..., r$ $1 \leq \beta(G''_j) \leq 2$. Set

$$J \equiv \{D_j | 1 \leq j \leq r, \beta(G''_j) = 1\}.$$

Clearly $|J| \geq l$. Let us show that $J \subseteq Sat(K, \tilde{\nu})$. It is not hard to check that if $D_j \notin Sat(K, \tilde{\nu})$, $1 \leq j \leq r$, then $\beta(G''_j) = 2$, and therefore $D_j \notin J$. Thus $|Sat(K, \tilde{\nu})| \geq |J| \geq l$. The proof of the **Theorem4** is complete.

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Հատուկ տիպի մաքսիմալ զուգակցումների կառուցման մասին

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Ամփոփում

Երկկողմանի գրաֆների համար ցույց է տրվել այնպիսի մաքսիմալ զուգակցման կառուցման խնդրի NP-լրիվությունը, որի հեռացումից ստացված գրաֆի մաքսիմալ զուգակցման հզորությունը բավարարում է նախապես տրված վերին (ստորին) գնահատականին: