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New Bounds for E-capacities of Arbitrarily Varying Channel and Channel with Random Parameter

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Abstract

The channel with random parameter and the arbitrarily varying channel in the case, when states sequence is known to the sender, are considered. The upper and the lower bounds of E-capacity are constructed. When $E \to 0$ the random coding bounds coincide with known capacities of corresponding channels.

1 Introduction

Let $\mathcal{X}, \mathcal{Y}, \mathcal{S}$ be finite sets and the transition probabilitites of a discrete memoryless channel with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ depend on a parameter s with values in $\mathcal{S}$. In other words we have a set of conditional probabilities

$$W_s = \{W(y|x,s), x \in \mathcal{X}, y \in \mathcal{Y}\}, s \in \mathcal{S}.$$

Values of the parameter s can be changed by different rules, depending on which different definitions of channels can be considered.

The channel W_Q with random parameter is a family of discrete memoryless channels $W_s : \mathcal{X} \to \mathcal{Y}$, where s is the channel state, varying independently in each moment with the same probability distribution $Q(s)$ on $\mathcal{S}$.

Let $\mathcal{Q}(\mathcal{S})$ is the set of all probability distributions on $\mathcal{S}$ and $\mathcal{Q}$ is its some subset. We shall consider also the generalized channel with random parameter $W_{\mathcal{Q}}$, where the probability distribution of the random parameter s is fixed during the transmission of length N, but can be arbitrarily changed for the next transmission within the set $\mathcal{Q}$.

The discrete arbitrarily varying channel W is a channel, where s is varying arbitrarily.

The considered channels are memoryless and stationar, that is for N-length sequences: input word $\mathbf{x} = (x_1, x_2, ..., x_N) \in \mathcal{X}^N$, output word $\mathbf{y} = (y_1, y_2, ..., y_N) \in \mathcal{Y}^N$, and states sequence $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathcal{S}^N$, the transition probabilities are

$$W^N(\mathbf{y}|\mathbf{x},\mathbf{s}) = \prod_{n=1}^{N} W(y_n|x_n, s_n), \quad Q^N(\mathbf{s}) = \prod_{n=1}^{N} Q(s_n).$$

As in [1] and [2] we shall suppose that the choice of the channel states does not depend on the input or output signals; the states sequence $\mathbf{s}$ is known at the encoder and unknown at the decoder.

Let $\mathcal{M}$ be the message set and M be its cardinality.

The code for the channels is defined by encoding $f : \mathcal{M} \times \mathcal{S}^N \to \mathcal{X}^N$ and decoding $g : \mathcal{Y}^N \to \mathcal{M}$. The number $R = 1/N \log M$ is called code rate. Here and later we use the logarithmical and exponential functions to the base 2.

Denote

$$e(m, \mathbf{s}) = e(f, g, N, m, \mathbf{s}) \triangleq W^N(\mathcal{Y}^N - g^{-1}(m)|f(m, \mathbf{s}), \mathbf{s}). \tag{1}$$

Then the maximal e_Q and average $\overline{e}_Q$ error probabilities of the channel W_Q are

$$e_Q = e_Q(f, g, N, W_Q) \triangleq \max_{m\in\mathcal{M}} \sum_{\mathbf{s}\in\mathcal{S}^N} Q^N(\mathbf{s})e(m, \mathbf{s}), \tag{2}$$

$$\overline{e}_Q = \overline{e}_Q(f, g, N, W_Q) \triangleq \frac{1}{M} \sum_{m\in\mathcal{M}} \sum_{\mathbf{s}\in\mathcal{S}^N} Q^N(\mathbf{s})e(m, \mathbf{s}). \tag{3}$$

For the channel W the maximal and average error probabilities are, correspondingly,

$$e = e(f, g, N, W) \triangleq \max_{m\in\mathcal{M}} \max_{\mathbf{s}\in\mathcal{S}^N} e(m, \mathbf{s}),$$

$$\overline{e} = \overline{e}(f, g, N, W) \triangleq \frac{1}{M} \sum_{m\in\mathcal{M}} \max_{\mathbf{s}\in\mathcal{S}^N} e(m, \mathbf{s}).$$

We call E-capacity of the channel the function

$$C(E, W) \triangleq \overline{\lim_{N\to\infty}} \frac{1}{N} \log M(N, E), \tag{4}$$

where

$$M(N, E) \triangleq \sup_{f,g}\{M : e(f, g, N, W) \leq \exp(-NE)\}.$$

E-capacity of the channel W for average error probability we denote by $\overline{C}(E, W)$. Analogically, we define E-capacities of the channels W_Q and W_Q for average and maximal error probabilities and denote correspondingly by $\overline{C}(E, W_Q)$, $C(E, W_Q)$, $\overline{C}(E, W_Q)$ and $C(E, W_Q)$.

The channel with random parameter W_Q was investigated by Gelfand and Pinsker [1]. They found the capacity $\overline{C}(W_Q)$ of this channel for the average error probability in the situation, when during the choice of the codeword $\mathbf{x}$ the sequence $\mathbf{s}$ is known. Ahlswede [2] proved that the capacity $C(W_Q)$ of this channel for the maximal error probability is equal to $\overline{C}(W_Q)$. He also considered the generalized channel with random parameter W_Q and arbitrarily varying channel W with states sequence known at the encoder, he obtained the corresponding capacities $C(W_Q) = \overline{C}(W_Q), C(W) = \overline{C}(W)$ of this channels.

Some versions of upper and lower bounds for E-capacity were obtained in [3], [4], [5]. In 2000 professor Shlomo Shamai (Shitz) proposed a conterexample and professor Neri Merhav found the mistake in the proof of the upper bound in [5], about which the author informed by the letter [6].

In this paper new upper and improved lower bounds for all this channels are constructed. For $E \to 0$ the limits of random coding bounds coincide with the capacities of corresponding channels.

2 Formulation of results

Let U, S, X, Y be random variables with values in finite alphabets $\mathcal{U}$, $\mathcal{S}$, $\mathcal{X}$, $\mathcal{Y}$, respectively, with probability distributions $Q(s)$, $P(ux|s)$ and $V(y|x,s)$, $s \in \mathcal{S}, u \in \mathcal{U}, x \in \mathcal{X}, y \in \mathcal{Y}$.

For our notations of entropies, mutual informations and divergences as well as for notions of types, conditional types we refer to [7], [8], [9], [10]. We remind here some notations and relations, that we use in our proofs.

The subset of $\mathcal{Q}(\mathcal{S})$ consisting of the possible types of sequences $\mathbf{s} \in \mathcal{S}^N$ is denoted by $\mathcal{Q}_N(\mathcal{S})$. The set of vectors $\mathbf{s}$ of type Q is denoted by $\mathcal{T}_Q^N(S)$, the set of all sequences $\mathbf{x} \in \mathcal{X}^N$ of conditional type P for given $\mathbf{s} \in \mathcal{T}_Q^N(S)$ is denoted by $\mathcal{T}_{Q,P}^N(X|\mathbf{s})$. The following properties are now well known

$$|\mathcal{Q}_N(\mathcal{S})| \le (N+1)^{|\mathcal{S}|}, \tag{5}$$

$$|\mathcal{P}_N(\mathcal{X}, Q)| \le (N+1)^{|\mathcal{X}||\mathcal{S}|}, \tag{6}$$

$$|\mathcal{V}_N(\mathcal{Y}, Q, P)| \le (N+1)^{|\mathcal{U}||\mathcal{X}||\mathcal{S}||\mathcal{Y}|}. \tag{7}$$

$$(N+1)^{-|\mathcal{S}|} \exp\{N H_Q(S)\} \le |\mathcal{T}_Q^N(S)| \le \exp\{N H_Q(S)\}, \tag{8}$$

for any conditional type V and a pair of vectors $(\mathbf{x}, \mathbf{s})$ from $\mathcal{T}_{Q,P}^N(XS)$

$$(N+1)^{-|\mathcal{S}||\mathcal{X}||\mathcal{Y}|} \exp\{N H_{Q,P,V}(Y|X,S)\} \le |\mathcal{T}_{Q,P,V}^N(Y|\mathbf{x},\mathbf{s})| \le \exp\{N H_{Q,P,V}(Y|X,S)\}. \tag{9}$$

If $\mathbf{s} \in \mathcal{T}_{Q'}^N(S)$, then

$$Q^N(\mathbf{s}) = \exp\{-N(H_{Q'}(S) + D(Q'\|Q))\}, \tag{10}$$

if $\mathbf{s} \in \mathcal{T}_Q^N(S)$, $\mathbf{x} \in \mathcal{T}_{Q,P}^N(X|\mathbf{s})$, $\mathbf{y} \in \mathcal{T}_{Q,P,V}^N(Y|\mathbf{x},\mathbf{s})$, then

$$W^N(\mathbf{y}|\mathbf{x},\mathbf{s}) = \exp\{-N(H_{Q,P,V}(Y|X,S) + D(V\|W|Q,P))\}. \tag{11}$$

Becides

$$D(Q'PV\|QPW) = D(Q'\|Q) + D(V\|W|Q',P). \tag{12}$$

To formulate the outer and the inner bounds of the E-capacity let us denote

$$R_{sp}(Q,P,V) \triangleq I_{Q,P,V}(Y \wedge X|S),$$

$$R_r(Q,P,V) \triangleq I_{Q,P,V}(Y \wedge U) - I_{Q,P}(S \wedge U) =$$
$$= I_{Q,P,V}(Y \wedge X|S) - I_{Q,P,V}(S \wedge U|Y) - I_{Q,P,V}(Y \wedge X|U,S).$$

Consider the following functions:

$$R_{sp}(E, W_Q) \triangleq \min_{Q' \in \mathcal{Q}(\mathcal{S})} \max_P \min_{V: D(Q'PV\|QPW) \le E} R_{sp}(Q',P,V),$$

$$R_r(E, W_Q) \triangleq \min_{Q' \in \mathcal{Q}(\mathcal{S})} \max_P \min_{V: D(Q'PV\|QPW) \le E} |R_r(Q',P,V) + D(Q'PV\|QPW) - E|^+,$$

$$R_{sp}(E, W_{\mathcal{Q}}) \triangleq \inf_{Q \in \mathcal{Q}} R_{sp}(E, W_Q),$$

$$R_r(E, W_{\mathcal{Q}}) \triangleq \inf_{Q \in \mathcal{Q}} R_r(E, W_Q),$$

$$R_{sp}(E,W) \triangleq \min_{Q\in\mathcal{Q}(S)} \max_{P} \min_{V:D(V\|W|Q,P)\le E} R_{sp}(Q,P,V),$$

$$R_r(E,W) \triangleq \min_{Q\in\mathcal{Q}(S)} \max_{P} \min_{V:D(V\|W|Q,P)\le E} |R_r(Q,P,V)+D(V\|W|Q,P)-E|^{+}.$$

The following statements will be proved.

Theorem 1. For all $E>0$, for channel with random parameter with states sequence known to the sender the following inequalities are valid

$$R_r(E,W_Q)\le C(E,W_Q)\le \overline{C}(E,W_Q)\le R_{sp}(E,W_Q).$$

Corollary. For all $E>0$, for generalized channel with random parameter with states sequence known to the sender the following inequalities are valid

$$R_r(E,W_Q)\le C(E,W_Q)\le \overline{C}(E,W_Q)\le R_{sp}(E,W_Q).$$

Theorem 2. For all $E>0$, for arbitrarily varying channel with states sequence known to the sender the following inequalities are valid

$$R_r(E,W)\le C(E,W)\le \overline{C}(E,W)\le R_{sp}(E,W).$$

The upper bounds are proved by the combinatorial method proposed by Haroutunian [11] and the lower bounds are obtained by Shannon random coding method [12], some ideas of constructing the random matrix are adopted from [1]. It must be noted that the problem statement is closely related to the case of broadcast channels [8], [13], [14], [15].

Note that when $E\to 0$ we obtain the upper and the lower bounds for capacities of the corresponding channels W_Q and W:

$$R_{sp}(W_Q)\triangleq \max_P I_{Q,P,W}(Y\wedge X|S),$$

$$R_r(W_Q)\triangleq \max_P \{I_{Q,P,W}(Y\wedge U)-I_{Q,P}(S\wedge U)\},$$

$$R_{sp}(W)\triangleq \min_{Q\in\mathcal{Q}(S)} R_{sp}(W_Q),$$

$$R_r(W)\triangleq \min_{Q\in\mathcal{Q}(S)} R_r(W_Q).$$

$R_r(W_Q)$ coincides with the capacity $C(W_Q)$ of the channel with random parameter, obtained by Gelfand and Pinsker [1], who also stated that $|\mathcal{U}|\le|\mathcal{X}|+|\mathcal{S}|$. $R_r(W)$ coincides with the capacity $C(W)$ of the arbitrarily varying channel with states sequence known to the sender, obtained by Ahlswede [2]. The upper bounds $R_{sp}(W_Q)$ and $R_{sp}(W)$ are not tite, they coincide with the capacities of corresponding channels in the case, when states sequence is known also at the decoder. The introduction of auxiliary random variable U does not improve the obtained upper bounds.

3 Proof of the upper bounds

We begin with the proof of the upper bound in the theorem 1.

Let $\delta > 0$ and a code (f, g) is given, with rate $R = (1/N)\log M$ and error probability

$$\bar{e}_Q \leq \exp\{-N(E-\delta)\}, E - \delta > 0.$$

According to (1) and (3) it means that

$$\frac{1}{M}\sum_{m\in\mathcal{M}}\sum_{\mathbf{s}\in\mathcal{S}^N} Q^N(\mathbf{s})W^N\{\mathcal{Y}^N - g^{-1}(m)|\mathbf{x}(m,\mathbf{s}),\mathbf{s}\} \leq \exp\{-N(E-\delta)\}.$$

The left side of this inequality can only decrease if we take the sum by vectors $\mathbf{s}$ of some fixed type Q':

$$\sum_{\mathbf{s}\in\mathcal{T}^N_{Q'}(S)}\sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})} Q^N(\mathbf{s})W^N\{\mathcal{Y}^N - g^{-1}(m)|\mathbf{x}(m,\mathbf{s}),\mathbf{s}\} \leq M\exp\{-N(E-\delta)\},$$

where $f(\mathcal{M},\mathbf{s})$ is the set of all codewords used with the states vector $\mathbf{s}$.

Let us fix the conditional type P of the sequence $\mathbf{x}$ for given $\mathbf{s}$. For each $\mathbf{s}\in\mathcal{T}^N_{Q'}(S)$ we consider the vectors $\mathbf{x}(m,\mathbf{s})$ from $\mathcal{T}^N_{Q',P}(X|\mathbf{s})$. As for each $\mathbf{s}$

$$\sum_P |f(\mathcal{M},\mathbf{s})\bigcap\mathcal{T}^N_{Q',P}(X|\mathbf{s})| = M$$

and hence

$$M = \frac{1}{|\mathcal{T}^N_{Q'}(S)|}\sum_{\mathbf{s}\in\mathcal{T}^N_{Q'}(S)}\sum_P |f(\mathcal{M},\mathbf{s})\bigcap\mathcal{T}^N_{Q',P}(X|\mathbf{s})|,$$

then it follows from (6) that

$$M \leq (N+1)^{|\mathcal{S}||\mathcal{X}|}\frac{1}{|\mathcal{T}^N_{Q'}(S)|}\sum_{\mathbf{s}\in\mathcal{T}^N_{Q'}(S)}\max_P |f(\mathcal{M},\mathbf{s})\bigcap\mathcal{T}^N_{Q',P}(X|\mathbf{s})|,$$

from where we conclude that for each Q' there exists at least one type P' such that

$$M|\mathcal{T}^N_{Q'}(S)|(N+1)^{-|\mathcal{S}||\mathcal{X}|} \leq \sum_{\mathbf{s}\in\mathcal{T}^N_{Q'}(S)} |f(\mathcal{M},\mathbf{s})\bigcap\mathcal{T}^N_{Q',P'}(X|\mathbf{s})|. \qquad (13)$$

We shall remember that the choice of P' depends on Q'. Now for any conditional type V we have

$$\sum_{\mathbf{s}\in\mathcal{T}^N_{Q'}(S)}\sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})\bigcap\mathcal{T}^N_{Q',P'}(X|\mathbf{s})} Q^N(\mathbf{s})\times$$

$$\times W^N\{\mathcal{T}^N_{Q',P',V}(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s}) - g^{-1}(m)|\mathbf{x}(m,\mathbf{s}),\mathbf{s}\} \leq M\exp\{-N(E-\delta)\}.$$

Let then type Q' and conditional type V be such that

$$D(Q'P'V\|QP'W) \leq E. \qquad (14)$$

As the conditional probability $Q^N(\mathbf{s})W^N(\mathbf{y}|\mathbf{x},\mathbf{s})$ is constant for different $\mathbf{s} \in \mathcal{T}_{Q'}^N(S)$, $\mathbf{x} \in \mathcal{T}_{Q',P'}^N(X|\mathbf{s})$ and $\mathbf{y} \in \mathcal{T}_{Q',P',V}^N(Y|\mathbf{x},\mathbf{s})$, we can write

$$\sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} \sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})\bigcap \mathcal{T}_{Q',P'}^N(X|\mathbf{s})} Q^N(\mathbf{s})W^N(\mathbf{y}|\mathbf{x},\mathbf{s})\times$$

$$\times\{|\mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})| - |g^{-1}(m)\bigcap \mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})|\} \leq M\exp\{-N(E-\delta)\},$$

or

$$\sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} \sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})\bigcap \mathcal{T}_{Q',P'}^N(X|\mathbf{s})} \{|\mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})| - |g^{-1}(m)\bigcap \mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})|\} \leq$$

$$\leq \frac{M\exp\{-N(E-\delta)\}}{Q^N(\mathbf{s})W^N(\mathbf{y}|\mathbf{x},\mathbf{s})}.$$

From (10), (11) and (12) we obtain

$$\sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} \sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})\bigcap \mathcal{T}_{Q',P'}^N(X|\mathbf{s})} |\mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})| -$$

$$-\frac{M\exp\{-N(E-\delta)\}}{\exp\{-N(H_{Q'}(S)+H_{Q'P'V}(Y|X,S)+D(Q'P'V\|QP'W))\}} \leq$$

$$\leq \sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} \sum_{\mathbf{x}(m,\mathbf{s})\in f(\mathcal{M},\mathbf{s})\bigcap \mathcal{T}_{Q',P'}^N(X|\mathbf{s})} |g^{-1}(m)\bigcap \mathcal{T}_{Q',P',V}^N(Y|\mathbf{x}(m,\mathbf{s}),\mathbf{s})|.$$

The sets $g^{-1}(m)$, $m \in \mathcal{M}$, are disjoint so the right part of the last inequality is upper bounded by

$$\sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} |\mathcal{T}_{Q',P',V}^N(Y|\mathbf{s})|.$$

Taking into account (8), (9) we receive

$$\sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} |f(\mathcal{M},\mathbf{s})\bigcap \mathcal{T}_{Q',P'}^N(X|\mathbf{s})|(N+1)^{-|\mathcal{S}||\mathcal{X}||\mathcal{Y}|}\exp\{NH_{Q',P',V}(Y|X,S)\} -$$

$$-M\exp\{N(H_{Q'}(S)+H_{Q',P',V}(Y|X,S)+D(Q'P'V\|QP'W)-E+\delta)\} \leq$$

$$\leq \sum_{\mathbf{s}\in \mathcal{T}_{Q'}^N(S)} \exp\{NH_{Q',P',V}(Y|S)\}.$$

Now from (13) we obtain

$$M\exp\{N(H_{Q'}(S)+H_{Q',P',V}(Y|X,S))\}\times$$

$$\times[(N+1)^{-|\mathcal{S}||\mathcal{X}|(1+|\mathcal{Y}|)} - \exp\{N(D(Q'P'V\|QP'W)-E+\delta)\}] \leq$$

$$\leq \exp\{N(H_{Q'}(S)+H_{Q',P',V}(Y|S))\},$$

or

$$M \leq \frac{\exp\{N(H_{Q',P',V}(Y|S)-H_{Q',P',V}(Y|X,S))\}}{(N+1)^{-|\mathcal{S}||\mathcal{X}|(1+|\mathcal{Y}|)} - \exp\{N(D(Q'P'V\|QP'W)-E+\delta)\}}.$$

The right side of this inequality can be minimized by the choice of conditional type V, meeting the condition (14). Taking into account the continuity of all expressions we can minimize not only by types but also by any conditional probability distributions V. Then it can be maximized by conditional probability distribution P for given Q' and minimized by probability distribution Q'. It is left to note that

$$H_{Q,P,V}(Y|S) - H_{Q,P,V}(Y|X,S) = I_{Q,P,V}(Y \wedge X|S).$$

The upper bound of the theorem 1 is proved.

The upper bounds in the corollary and the theorem 2 follows from the upper bound for the channel with random parameter, because

$$\overline{C}(E, W_Q) \leq \inf_{Q \in \mathcal{Q}} \overline{C}(E, W_Q) \leq \inf_{Q \in \mathcal{Q}} R_{sp}(E, W_Q),$$

$$\overline{C}(E, W) \leq \min_{Q \in \mathcal{Q}(S)} \overline{C}(E, W_Q) \leq \min_{Q \in \mathcal{Q}(S)} R_{sp}(E, W_Q).$$

4 Proof of the lower bounds

The proofs of lower bounds are based on the random coding arguments.

Let us fix positive integers N, M, type Q, conditional type P, $\delta > 0$. For brevity we shall denote $\mathbf{u}(m,\mathbf{s})\mathbf{x}(m,\mathbf{s}) = \mathbf{ux}(m,\mathbf{s})$. Denote by $\mathcal{L}_M(\mathcal{T}_Q^N(S))$ the family of all matrices $\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in\mathcal{T}_Q^N(S)}$, such that the rows $\mathbf{L}(\mathbf{s}) = (\mathbf{ux}(1,\mathbf{s}), \mathbf{ux}(2,\mathbf{s}), \ldots, \mathbf{ux}(M,\mathbf{s}))$ are collections of not necessarily distinct vector pairs, majority of which are from $\mathcal{T}_{Q,P}^N(UX|\mathbf{s})$.

Let us denote by $\mathcal{A}_{Q,P}(m,\mathbf{s})$ for any $m \in \mathcal{M}$ and $\mathbf{s} \in \mathcal{T}_Q^N(S)$ the random event

$$\mathcal{A}_{Q,P}(m,\mathbf{s}) \triangleq \{\mathbf{ux}(m,\mathbf{s}) \in \mathcal{T}_{Q,P}^N(UX|\mathbf{s})\}.$$

Let us now consider the sets

$$\mathcal{S}(m,Q,P) \triangleq \{\mathbf{s} \in \mathcal{T}_Q^N(S) : \mathcal{A}_{Q,P}(m,\mathbf{s})\},\ m \in \mathcal{M},$$

$$\mathcal{M}(\mathbf{s},Q,P) \triangleq \{m \in \mathcal{M} : \mathcal{A}_{Q,P}(m,\mathbf{s})\},\ \mathbf{s} \in \mathcal{T}_Q^N(S),$$

$$(\mathcal{M},\mathcal{S})(Q,P) \triangleq \{(m,\mathbf{s}),\ m \in \mathcal{M},\ \mathbf{s} \in \mathcal{T}_Q^N(S) : \mathcal{A}_{Q,P}(m,\mathbf{s})\}.$$

We shall prove the following modification of the packing lemma from [9].

Lemma 1. For all $E > 2\delta \geq 0$, types Q, P, there exists a matrix $\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in\mathcal{T}_Q^N(S)}$ with

$$M = \exp\left\{N \min_{V: D(V\|W|Q,P) \leq E} |I_{Q,P,V}(Y \wedge U) - I_{Q,P}(U \wedge S) + D(V\|W|Q,P) - E - 2\delta|^+\right\}, \tag{15}$$

such that for each $\mathbf{s} \in \mathcal{T}_Q^N(S)$ vectors $\mathbf{ux}(m,\mathbf{s})$ for different m from $\mathcal{M}(\mathbf{s},Q,P)$ are distinct and

$$\Pr\{\overline{\mathcal{A}}_{Q,P}(m,\mathbf{s})\} \leq \exp\{-\exp\{N\delta/4\}\}, \tag{16}$$

and for any pair $(m, \mathbf{s}) \in (\mathcal{M}, \mathcal{S})(Q, P)$, conditional types $V : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ and $\hat{V} : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ for sufficiently large N the following inequality holds

$$\left| \mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m \neq m'} \bigcup_{\mathbf{s}' \in \mathcal{S}(m',Q,P)} \mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}') \right| \leq$$

$$\leq |\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})| \exp\{-N|E - D(\hat{V}\|W|Q,P)|^+\}. \quad (17)$$

The proof of lemma 1 is given in the appendix.

The next lemma follows from the first one.

Lemma 2. For all $E > 2\delta \geq 0$, there exists a matrix $\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}^{\mathbf{s} \in S^N}_{m=\overline{1,M}}$ with

$$M = \exp\left\{N \min_{Q \in \mathcal{P}(S)} \max_{P} \min_{V: D(V\|W|Q,P) \leq E} |I_{Q,P,V}(Y \wedge U) - \right. \quad (18)$$

$$\left. -I_{Q,P}(U \wedge S) + D(V\|W|Q,P) - E - 2\delta|^+ \right\} =$$

$$= \exp\left\{N \min_{Q \in \mathcal{P}(S)} \min_{V: D(V\|W|Q,P^*_Q) \leq E} \left|I_{Q,P^*_Q,V}(Y \wedge U) - \right.\right.$$

$$\left.\left. -I_{Q,P^*_Q}(U \wedge S) + D(V\|W|Q,P^*_Q) - E - 2\delta\right|^+ \right\},$$

such that for each Q and $\mathbf{s} \in \mathcal{T}^N_Q(S)$ vectors $\mathbf{ux}(m,\mathbf{s})$ for different $m \in \mathcal{M}(\mathbf{s}, Q, P^*_Q)$ are distinct and

$$\Pr\{\overline{\mathcal{A}}_{Q,P^*_Q}(m,\mathbf{s})\} \leq \exp\{-\exp\{N\delta/4\}\}, \quad (19)$$

and for any $(m,\mathbf{s}) \in (\mathcal{M}, \mathcal{S})(Q, P)$, $V_Q : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ and $\hat{V} : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$, types $\hat{Q}$ and $\hat{P}$ for sufficiently large N the following inequality holds

$$\left| \bigcap_{Q \in \mathcal{Q}_N(S)} \mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m \neq m'} \bigcup_{\mathbf{s}' \in \mathcal{S}(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}') \right| \leq$$

$$\leq \min_{Q \in \mathcal{Q}_N(S)} |\mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})| \exp\{-N|E - D(\hat{V}\|W|\hat{Q},\hat{P})|^+\}. \quad (20)$$

The next lemma can be proved similarly to the lemma 1.

Lemma 3. For all $E > 2\delta \geq 0$, type Q' such that $D(Q'\|Q) \leq E$, conditional type P, there exists a matrix $\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}^{\mathbf{s} \in \mathcal{T}^N_{Q'}(S)}_{m=\overline{1,M}}$ with

$$M = \exp\left\{N \min_{V: D(Q'PV\|QPW) \leq E} |I_{Q',P,V}(Y \wedge U) - \right. \quad (21)$$

$$\left. - I_{Q',P}(U \wedge S) + D(Q'PV\|QPW) - E - 2\delta|^+ \right\},$$

such that for each $\mathbf{s} \in \mathcal{T}^N_{Q'}(S)$ vectors $\mathbf{ux}(m,\mathbf{s})$ for different $m \in \mathcal{M}(\mathbf{s}, Q', P)$ are distinct and (16) holds and for any $(m,\mathbf{s}) \in (\mathcal{M}, \mathcal{S})(Q', P)$, $V : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ and $\hat{V} : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$, for sufficiently large N the following inequality holds

$$\left| \mathcal{T}^N_{Q',P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m \neq m'} \bigcup_{\mathbf{s}' \in \mathcal{S}(m',Q',P)} \mathcal{T}^N_{Q',P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}') \right| \leq$$

$$\leq |\mathcal{T}^N_{Q',P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})| \exp\{-N|E - D(Q'P\hat{V}\|QPW)|^+\}. \quad (22)$$

To formulate the next two lemmas, which follow from the lemma 3, let us denote

$$\mathcal{T}^E_Q(S) = \bigcup_{Q':D(Q'\|Q)\leq E} \mathcal{T}^N_{Q'}(S).$$

Lemma 4. For all $E > 2\delta \geq 0$, there exists a matrix $\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in\mathcal{T}^E_Q(S)}$ with

$$M = \exp\Big\{N \min_{Q'} \max_{P} \min_{V:D(Q'PV\|QPW)\leq E} |I_{Q',P,V}(Y\wedge U) - \quad (23)$$
$$-I_{Q',P}(U\wedge S) + D(Q'PV\|QPW) - E - 2\delta|^+\Big\} =$$
$$= \exp\Big\{N \min_{Q'} \min_{V:D(Q'P^*_QV\|QP^*_QW)\leq E} \Big|I_{Q',P^*_Q,V}(Y\wedge U) -$$
$$-I_{Q',P^*_Q}(U\wedge S) + D(Q'P^*_QV\|QP^*_QW) - E - 2\delta\Big|^+\Big\},$$

such that for each $Q' : D(Q'\|Q) \leq E$ and $\mathbf{s} \in \mathcal{T}^N_{Q'}(S)$ vectors $\mathbf{ux}(m,\mathbf{s})$, $m \in \mathcal{M}(\mathbf{s}, Q', P^*_{Q'})$ are distinct and (19) is true and for any $(m,\mathbf{s}) \in (\mathcal{M},\mathcal{S})(Q',P_{Q'})$, $V_{Q'} : \mathcal{X}\times\mathcal{S}\to\mathcal{Y}$ and $\hat{V} : \mathcal{X}\times\mathcal{S}\to\mathcal{Y}$, type $\hat{Q}$ such that $D(\hat{Q}\|Q) \leq E$, $\hat{P}$, for sufficiently large N the following inequality holds

$$\left|\bigcap_{Q':D(Q'\|Q)\leq E} \mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m\neq m'} \bigcup_{\mathbf{s}'\in S(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right| \leq$$
$$\leq \min_{Q':D(Q'\|Q)\leq E} |\mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})| \exp\{-N|E - D(\hat{Q}\hat{P}\hat{V}\|Q\hat{P}W)|^+\}. \quad (24)$$

Lemma 5. For all $E > 2\delta \geq 0$, $\mathcal{Q} \subset \mathcal{Q}(\mathcal{S})$, there exists a matrix

$$\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in \bigcup_{Q\in\mathcal{Q}} \mathcal{T}^E_Q(S)}$$

with

$$M = \exp\Big\{N \inf_{Q\in\mathcal{Q}} \min_{Q'} \max_{P} \min_{V:D(Q'PV\|QPW)\leq E} |I_{Q',P,V}(Y\wedge U) - \quad (25)$$
$$-I_{Q',P}(U\wedge S) + D(Q'PV\|QPW) - E - 2\delta|^+\Big\} =$$
$$= \exp\Big\{N \inf_{Q\in\mathcal{Q}} \min_{Q'} \min_{V:D(Q'P^*_QV\|QP^*_QW)\leq E} \Big|I_{Q',P^*_Q,V}(Y\wedge U) -$$
$$-I_{Q',P^*_Q}(U\wedge S) + D(Q'P^*_QV\|QP^*_QW) - E - 2\delta\Big|^+\Big\},$$

such that for each $\mathbf{s} \in \bigcup_{Q\in\mathcal{Q}} \mathcal{T}^E_Q(S)$ vectors $\mathbf{ux}(m,\mathbf{s})$ for $m \in \mathcal{M}(\mathbf{s}, Q', P^*_{Q'})$ are distinct and (19) takes place and for any $m \in \mathcal{M}$, $V_{Q'} : \mathcal{X}\times\mathcal{S}\to\mathcal{Y}$ and $\hat{V} : \mathcal{X}\times\mathcal{S}\to\mathcal{Y}$, types $Q \in \mathcal{Q}$, $\hat{Q}$ such that $D(\hat{Q}\|Q) \leq E$, $\hat{P}$, for sufficiently large N the following inequality holds

$$\left|\bigcap_{Q'\in\mathcal{Q}'} \mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m\neq m'} \bigcup_{\mathbf{s}'\in S(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right| \leq$$

$$\leq \min_{Q' \in \mathcal{Q}'} |\mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})| \exp\{-N|E - D(\hat{Q}\hat{P}\hat{V}\|Q\hat{P}W)|^+\}, \tag{26}$$

where $\mathcal{Q}' = \{Q' : D(Q'\|Q) \leq E, Q \in \mathcal{Q}\}$.

Proof of the lower bound of theorem 2. The existence of a matrix

$$\mathbf{L} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in\mathcal{S}^N}$$

satisfying (18), (19) and (20) is guaranteed by Lemma 2.

Let us denote by $\mathcal{P}$ and $\mathcal{V}$ the corresponding sets of conditional types P_Q and V_Q, where $Q \in \mathcal{Q}_N(S)$ and

$$\mathcal{S}^N(m,P) = \bigcup_{Q\in\mathcal{Q}_N(S)} \mathcal{S}(m,Q,P_Q),\ m \in \mathcal{M}.$$

Let us apply the following decoding rule for decoder g: each $\mathbf{y}$ is decoded to such m for which $\mathbf{y} \in \bigcap_{Q\in\mathcal{Q}_N(S)} \mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})$ and

$$\min_{Q\in\mathcal{Q}_N(S)} D(V_Q\|W|Q,P^*_Q)$$

is minimal.

The decoder g can make an error during the transmission of message m, if $\overline{\mathcal{A}}_{Q,P^*_Q}(m,\mathbf{s})$ takes place or if $\mathbf{s} \in \mathcal{S}^N(m,P^*)$, but there exist $m' \neq m$, types $\hat{Q},\hat{P},\hat{V}$, some $\mathbf{s}' \in \mathcal{S}(m',\hat{Q},\hat{P})$ such that

$$\mathbf{y} \in \bigcap_{Q\in\mathcal{Q}_N(S)} \mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')$$

and

$$D(\hat{V}\|W|\hat{Q},\hat{P}) \leq \min_{Q\in\mathcal{Q}_N(S)} D(V_Q\|W|Q,P^*_Q). \tag{27}$$

Denote

$$\mathcal{D} = \left\{\hat{Q},\hat{P},\mathcal{V},\hat{V} : (27)\text{ is valid}\right\}.$$

Then the error probability of the message m is upper bounded by the following way:

$$\max_{\mathbf{s}\in\mathcal{S}^N} e(m,\mathbf{s}) \leq \max_{Q\in\mathcal{Q}_N(S)} \Pr\{\overline{\mathcal{A}}_{Q,P^*_Q}(m,\mathbf{s})\} +$$

$$+ \max_{\mathbf{s}\in\mathcal{S}^N(m,P^*)} W^N \left\{ \bigcup_{\mathcal{D}} \bigcap_{Q\in\mathcal{Q}_N(S)} \mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \right.$$

$$\left. \bigcap \bigcup_{m\neq m'} \bigcup_{\mathbf{s}'\in\mathcal{S}(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')|\mathbf{x}(m,\mathbf{s}),\mathbf{s} \right\}.$$

Taking into account that the probability distribution $W^N(\mathbf{y}|\mathbf{x}(m,\mathbf{s}),\mathbf{s})$ is constant for fixed Q,P,V the second item can be upper bounded by

$$\sum_{\mathcal{D}} \left| \bigcap_{Q\in\mathcal{Q}_N(S)} \mathcal{T}^N_{Q,P^*_Q,V_Q}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m\neq m'} \bigcup_{\mathbf{s}'\in\mathcal{S}(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}') \right| \times \tag{28}$$

$$\times \max_{\mathbf{s}\in\mathcal{S}^N(m,P^*)} W^N(\mathbf{y}|\mathbf{x}(m,\mathbf{s}),\mathbf{s}).$$

From (11), (19) and (20) for sufficiently large N we obtain

$$\max_{s\in S^N} e(m,s) \le \exp\{-\exp\{N\delta/4\}\}+$$

$$+\sum_{D} \min_{Q\in\mathcal{Q}_N(S)} \exp\left\{NH_{Q,P_Q^*,V_Q}(Y|XS)\right\}\exp\left\{-N(E-D(\hat{V}\|W|\hat{Q},\hat{P}))\right\}\times$$

$$\times \max_{Q\in\mathcal{Q}_N(S)} \exp\left\{-N\left(H_{Q,P_Q^*,V_Q}(Y|XS)+D(V_Q\|W|Q,P_Q^*)\right)\right\} \le$$

$$\le \exp\{-\exp\{N\delta/4\}\}+$$

$$+(N+1)^{|S|(1+|\mathcal{U}||\mathcal{X}|+2|\mathcal{U}||\mathcal{X}||\mathcal{Y}|)} \exp\left\{N \min_{Q\in\mathcal{Q}_N(S)} H_{Q,P_Q^*,V_Q}(Y|XS)\right\}\times$$

$$\times \exp\left\{-N(E-\min_{Q\in\mathcal{Q}_N(S)} D(V_Q\|W|Q,P_Q^*)))\right\}\times$$

$$\times \exp\left\{-N\min_{Q\in\mathcal{Q}_N(S)}\left(H_{Q,P_Q^*,V_Q}(Y|XS)+D(V_Q\|W|Q,P_Q^*)\right)\right\}\le$$

$$\le \exp\{-N(E-\epsilon)\}.$$

The lower bound of theorem 2 is proved.

Proof of the lower bound in the theorem 1. We shall construct the code only for $s\in\mathcal{T}_Q^E(S)$, because for sufficiently large N $\Pr\{s\notin\mathcal{T}_Q^E(S)\}\le\exp\{-N(E-\epsilon_1)\}$. The existence of a matrix $L=\{ux(m,s)\}_{m=\overline{1,M}}^{s\in\mathcal{T}_Q^E(S)}$ satisfying (19), (23) and (24) is guaranteed by Lemma 4.

Now we denote by P and V the corresponding sets of conditional types $P_{Q'}$ and $V_{Q'}$, where Q' satisfies the condition $D(Q'\|Q)\le E$ and

$$S_Q^E(m,P)=\bigcup_{Q':D(Q'\|Q)\le E} S(m,Q',P_{Q'}),\ m\in\mathcal{M}.$$

Let us apply the following decoding rule for decoder g: each y is decoded to such m for which $y\in\bigcap_{Q':D(Q'\|Q)\le E}\mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|ux(m,s),s)$ and

$$\min_{Q':D(Q'\|Q)\le E} D(Q'P^*_{Q'}V_{Q'}\|QP^*_{Q'}W)$$

is minimal.

The decoder g can make an error when the message m is transmitted and $\overline{\mathcal{A}}_{Q',P^*_{Q'}}(m,s)$ takes place or $S_Q^E(m,P^*)$ takes place, but there exist $m'\ne m$, types $\hat{Q}$ (such that $D(\hat{Q}\|Q)\le E$), $\hat{P},\hat{V}$, vector $s'\in S(m',\hat{Q},\hat{P})$ such that

$$y\in\bigcap_{Q':D(Q'\|Q)\le E}\mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|ux(m,s),s)\bigcap\mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|ux(m',s'),s')$$

and

$$D(\hat{Q}\hat{P}\hat{V}\|Q\hat{P}W)\le\min_{Q':D(Q'\|Q)\le E} D(Q'P^*_{Q'}V_{Q'}\|QP^*_{Q'}W). \tag{29}$$

Denote

$$\mathcal{D}=\left\{\hat{Q},\hat{P},\mathcal{V},\hat{V}:(29)\text{ is valid}\right\}.$$

Then the error probability of the message m is upper bounded by the following way:

$$\sum_{\mathbf{s}\in\mathcal{T}_Q^E(S)} Q^N(\mathbf{s})e(m,\mathbf{s}) + \exp\{-N(E-\epsilon_1)\} \leq$$

$$\leq \sum_{\mathbf{s}\in\mathcal{T}_Q^E(S)-\mathcal{S}_Q^E(m,P^*)} Q^N(\mathbf{s})\Pr\{\overline{\mathcal{A}}_{Q',P^*_{Q'}}(m,\mathbf{s})\}+$$

$$+\sum_{\mathbf{s}\in\mathcal{S}_Q^E(m,P^*)} Q^N(\mathbf{s})W^N\left\{\bigcup_{\mathcal{D}}\bigcap_{Q':D(Q'\|Q)\leq E} \mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\bigcap\right.$$

$$\left.\bigcap\bigcup_{m\neq m'}\bigcup_{\mathbf{s}'\in\mathcal{S}(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')|\mathbf{x}(m,\mathbf{s}),\mathbf{s}\right\} + \exp\{-N(E-\epsilon_1)\} \leq$$

$$\leq \sum_{Q':D(Q'\|Q)\leq E}\sum_{\mathbf{s}\in\mathcal{T}_Q^{\prime N}(S)} Q^N(\mathbf{s})\exp\{-\exp\{N\delta/4\} + \exp\{-N(E-\epsilon_1)\}+$$

$$+\sum_{\mathcal{D}}\left|\bigcap_{Q':D(Q'\|Q)\leq E} \mathcal{T}^N_{Q',P^*_{Q'},V_{Q'}}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\bigcap\bigcup_{m\neq m'}\bigcup_{\mathbf{s}'\in\mathcal{S}(m',\hat{Q},\hat{P})} \mathcal{T}^N_{\hat{Q},\hat{P},\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right| \times$$

$$\times\sum_{\mathbf{s}\in\mathcal{T}_Q^E(S)} Q^N(\mathbf{s})W^N(\mathbf{y}|\mathbf{x},\mathbf{s}).$$

Taking into account (10), (11), (12) and (24) the error probability can be upper bounded by

$$(N+1)^{|S|}\exp\{-N(\min_{Q':D(Q'\|Q)\leq E} D(Q'\|Q)\}\exp\{-\exp\{N\delta/4\}\} + \exp\{-N(E-\epsilon_1)\}+$$

$$+\sum_{\mathcal{D}}\min_{Q':D(Q'\|Q)\leq E}\exp\{NH_{Q',P^*_{Q'},V_{Q'}}(Y|XS)\}\exp\{-N(E-D(\hat{Q}\hat{P}\hat{V}\|Q\hat{P}W))\}\times$$

$$\times\max_{Q':D(Q'\|Q)\leq E}[\exp\{-ND(Q'\|Q)\}\exp\{-N(H_{Q',P^*_{Q'},V_{Q'}}(Y|XS)+D(V_{Q'}\|W|Q',P^*_{Q'}))\}] =$$

$$= \exp\{-\exp\{N\delta/4\}+N\delta_1\} + \exp\{-N(E-\epsilon_1)\}+$$

$$+\sum_{\mathcal{D}}\exp\{N\min_{Q':D(Q'\|Q)\leq E} H_{Q',P^*_{Q'},V_{Q'}}(Y|XS)\}\exp\{-NE\}\times$$

$$\times\exp\left\{-N\left(\min_{Q':D(Q'\|Q)\leq E}[H_{Q',P^*_{Q'},V_{Q'}}(Y|XS)+\right.\right.$$

$$\left.\left.+D(Q',P^*_{Q'},V_{Q'}\|QP^*_{Q'}W)]-D(\hat{Q}\hat{P}\hat{V}\|Q\hat{P}W)\right)\right\} \leq$$

$$\leq \exp\{-\exp\{N\delta/4\}+N\delta_1\} + \exp\{-N(E-\epsilon_1)\} + (N+1)^{|S|(1+|\mathcal{U}||\mathcal{X}|+2|\mathcal{U}||\mathcal{X}||\mathcal{Y}|)}\exp\{-NE\} \leq$$

$$\leq \exp\{-N(E-\epsilon)\}.$$

The lower bound of theorem 1 is proved.

The lower bound in the Corollary can be proved similarly based on the Lemma 5.

Appendix

Proof of the lemma 1. Notice that if some matrix $\mathbf{L} \in \mathcal{L}_M(\mathcal{T}_Q^N(S))$ satisfies (17) for any V and $\hat{V}$, then $\mathbf{ux}(m,\mathbf{s}) \neq \mathbf{ux}(m',\mathbf{s})$ for $m \neq m'$ from $\mathcal{M}(\mathbf{s},Q,P)$, $\mathbf{s} \in \mathcal{T}_Q^N(S)$. To prove that, it is enough to choose $V = \hat{V}$ and $D(\hat{V}\|W|Q,P) < E$.

If $\hat{V}$ is such that $D(\hat{V}\|W|Q,P) \geq E$ then

$$\exp\left\{-N\left|E - D(\hat{V}\|W|Q,P)\right|^+\right\} = 1,$$

and (17) is valid for any M. It remains to prove inequality (17) for

$$\hat{\mathcal{V}}(Q,P,E) = \{\hat{V} : D(\hat{V}\|W|Q,P) < E\}.$$

For any matrix $\mathbf{L}$ denote

$$A_m(\mathbf{L}) = (N+1)^{|\mathcal{U}||\mathcal{S}||\mathcal{X}||\mathcal{Y}|}\sum_V \sum_{\hat{V}\in\hat{\mathcal{V}}(Q,P,E)} \exp\left\{N(E - D(\hat{V}\|W|Q,P) - H_{Q,P,V}(Y|XS))\right\} \times$$

$$\times \max_{\mathbf{s}\in\mathcal{S}(m,Q,P)} \left|\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s}) \bigcap \bigcup_{m\neq m'} \bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)} \mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right|.$$

It is clear that if $A_m(\mathbf{L}) \leq 1$ for all $m \in \mathcal{M}$, then $\mathbf{L}$ satisfies (17) for all $\mathbf{s} \in \mathcal{S}(m,Q,P)$, V and $\hat{V}$.

Notice that if for some $\mathbf{L} \in \mathcal{L}_M(\mathcal{T}_Q^N(S))$ the following inequality holds

$$\frac{1}{M}\sum_{m=1}^{M} A_m(\mathbf{L}) \leq \frac{1}{2}, \tag{30}$$

then $A_m(\mathbf{L}) \leq 1$ fot at least $M/2$ indices m. Further, if $\mathbf{L}'$ is subcollection of $\mathbf{L}$ with such indices then $A_m(\mathbf{L}') \leq A_m(\mathbf{L}) \leq 1$ for every such index m. Hence, the lemma will be proved if for an M with

$$2\exp\left\{N \min_{V:D(V\|W|Q,P)\leq E} |I_{Q,P,V}(Y\wedge U) - I_{Q,P}(U\wedge S) + D(V\|W|Q,P) - E - 2\delta|^+\right\} \leq$$

$$\leq M \leq \tag{31}$$

$$\leq \exp\left\{N \min_{V:D(V\|W|Q,P)\leq E} |I_{Q,P,V}(Y\wedge U) - I_{Q,P}(U\wedge S) + D(V\|W|Q,P) - E - \delta|^+\right\},$$

we find a matrix $\mathbf{L} \in \mathcal{L}_M(\mathcal{T}_Q^N(S))$ satisfying (30) and (16).

We shall construct a random matrix $\widetilde{\mathbf{L}} = \{\mathbf{ux}(m,\mathbf{s})\}_{m=\overline{1,M}}^{\mathbf{s}\in\mathcal{T}_Q^N(S)}$ in the following way. We choose at random from $\mathcal{T}^N_{Q,P}(U)$, according to uniform distribution, M collections $\mathcal{J}(m)$ each of

$$J = \exp\{N(I_{Q,P}(U\wedge S) + \delta/2)\}$$

vectors $\mathbf{u}_j(m)$, $j = \overline{1,J}, m = \overline{1,M}$.

For each $m = \overline{1,M}$ and $\mathbf{s} \in \mathcal{T}_Q^N(S)$ we choose such a $\mathbf{u}_j(m)$ from $\mathcal{J}(m)$ that $\mathbf{u}_j(m) \in \mathcal{T}^N_{Q,P}(U|\mathbf{s})$. We denote this vector by $\mathbf{u}(m,\mathbf{s})$. If there is no such vector, let $\mathbf{u}(m,\mathbf{s}) = \mathbf{u}_J(m)$.

Next, for each m and s we choose at random a vector $\mathbf{x}(m,\mathbf{s})$ from $\mathcal{T}^N_{Q,P}(X|\mathbf{u}(m,\mathbf{s}),\mathbf{s})$ if $\mathbf{u}(m,\mathbf{s})\in\mathcal{T}^N_{Q,P}(U|\mathbf{s})$ and from $\mathcal{T}^N_{Q,P}(X|\mathbf{s})$ if $\mathbf{u}(m,\mathbf{s})\notin\mathcal{T}^N_{Q,P}(U|\mathbf{s})$.

First we shall show that for N large enough and any m and s (16) takes plase. Really,

$$\Pr\{\overline{\mathcal{A}}_{Q,P}(m,\mathbf{s})\}=\Pr\left\{\bigcap_{j=1}^{J}\mathbf{u}_j(m)\notin\mathcal{T}^N_{Q,P}(U|\mathbf{s})\right\}\le$$

$$\le\prod_{j=1}^{J}\left[1-\Pr\left\{\mathbf{u}_j(m)\in\mathcal{T}^N_{Q,P}(U|\mathbf{s})\right\}\right]\le\left[1-\frac{|\mathcal{T}^N_{Q,P}(U|\mathbf{s})|}{|\mathcal{T}^N_{Q,P}(U)|}\right]^J\le$$

$$\le[1-\exp\{-N(I_{Q,P}(U\wedge S)+\delta/4)\}]^{\exp\{N(I_{Q,P}(U\wedge S)+\delta/2)\}}.$$

Using the inequality $(1-t)^a\le\exp\{-at\}$, which is true for any a and $t\in(0,1)$, we can see that

$$\Pr\{\overline{\mathcal{A}}_{Q,P}(m,\mathbf{s})\}\le\exp\{-\exp\{N\delta/4\}\}.$$

To prove that (30) holds for some L it sufficies to show that

$$\mathbf{E}A_m(\tilde{\mathbf{L}})\le 1/2,\ \ m=\overline{1,M}. \tag{32}$$

To this end we observe that

$$\mathbf{E}\left|\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\bigcap\bigcup_{m\neq m'}\ \bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right|\le$$

$$\le\sum_{m\neq m'}\sum_{\mathbf{y}\in\mathcal{Y}^N}\Pr\left\{\mathbf{y}\in\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\right\}\times\Pr\{\mathbf{y}\in\bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\},$$

because the events in the brackets are independent.

Let us note that the first probability is different from zero if and only if $\mathbf{y}\in\mathcal{T}^N_{Q,P,V}(Y|\mathbf{s})$, in this case we have for N large enough

$$\Pr\left\{\mathbf{y}\in\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\right\}=\frac{|\mathcal{T}^N_{Q,P,V}(UX|\mathbf{y},\mathbf{s})|}{|\mathcal{T}^N_{Q,P}(UX|\mathbf{s})|}\le$$

$$\le(N+1)^{|\mathcal{U}||\mathcal{X}||\mathcal{S}|}\exp\{-NI_{Q,P,V}(Y\wedge UX|S)\}=$$
$$=(N+1)^{|\mathcal{U}||\mathcal{X}||\mathcal{S}|}\exp\{-NI_{Q,P,V}(Y\wedge X|S)\}.$$

The second probability can be estimated in the following way

$$\Pr\left\{\mathbf{y}\in\bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right\}\le$$

$$\le\Pr\left\{\mathbf{y}\in\bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{u}(m',\mathbf{s}'),\mathbf{s}')\right\}\le$$

$$\le\Pr\left\{\mathbf{y}\in\bigcup_{\mathbf{u}_j(m')\in\mathcal{J}(m')}\ \bigcup_{\mathbf{s}'\in\mathcal{T}^N_{Q,P}(S|\mathbf{u}_j(m'))}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{u}_j(m'),\mathbf{s}')\right\}\le$$

$$\leq \sum_{\mathbf{u}_j(m')\in\mathcal{J}(m')} \Pr\left\{\mathbf{y}\in\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{u}_j(m'))\right\} \leq J\frac{|\mathcal{T}^N_{Q,P,\hat{V}}(U|\mathbf{y})|}{|\mathcal{T}^N_{Q,P}(U)|} \leq$$

$$\leq (N+1)^{|\mathcal{U}|}\exp\left\{-N(I_{Q,P,\hat{V}}(Y\wedge U)-I_{Q,P}(U\wedge S)-\delta/2)\right\}.$$

At last we obtain

$$\mathbf{E}\left|\mathcal{T}^N_{Q,P,V}(Y|\mathbf{ux}(m,\mathbf{s}),\mathbf{s})\bigcap\bigcup_{m\neq m'}\bigcup_{\mathbf{s}'\in\mathcal{S}(m',Q,P)}\mathcal{T}^N_{Q,P,\hat{V}}(Y|\mathbf{ux}(m',\mathbf{s}'),\mathbf{s}')\right|\leq$$

$$\leq (N+1)^{|\mathcal{U}|(|\mathcal{X}||\mathcal{S}|+1)}(M-1)|\mathcal{T}^N_{Q,P,V}(Y|\mathbf{s})|\times$$

$$\times\exp\{-N(I_{Q,P,V}(Y\wedge X|S)+I_{Q,P,\hat{V}}(Y\wedge U)-I_{Q,P}(U\wedge S)-\delta/2)\}.$$

From (31) it follows that for any $\hat{V}\in\hat{\mathcal{V}}(Q,P,E)$

$$M-1\leq\exp\{N(I_{Q,P,\hat{V}}(Y\wedge U)-I_{Q,P}(U\wedge S)+D(\hat{V}\|W|Q,P)-E-\delta\},$$

and we obtain

$$\mathbf{E}A_m(\tilde{\mathbf{L}})\leq(N+1)^{|\mathcal{U}|(|\mathcal{X}||\mathcal{S}||\mathcal{Y}|+|\mathcal{X}||\mathcal{S}|+1)}\times$$

$$\times\sum_V\sum_{\hat{V}\in\hat{\mathcal{V}}(Q,P,E)}\exp\{N(I_{Q,P,\hat{V}}(Y\wedge U)-I_{Q,P}(U\wedge S)+D(\hat{V}\|W|Q,P)-E-\delta\}\times$$

$$\times\exp\{NH_{Q,P,V}(Y|S)\}\times\exp\{-N(I_{Q,P,V}(Y\wedge X|S)+I_{Q,P,\hat{V}}(Y\wedge U)-I_{Q,P}(U\wedge S)-\delta)\}\times$$

$$\times\exp\{N(E-D(\hat{V}\|W|Q,P)-H_{Q,P,V}(Y|XS))\}\leq$$

$$\leq(N+1)^{|\mathcal{U}|(|\mathcal{X}||\mathcal{S}||\mathcal{Y}|+|\mathcal{X}||\mathcal{S}|+1)}\sum_{V,\hat{V}}\exp\{-N\delta/2\}.$$

Taking into account (7) we obtain

$$\mathbf{E}A_m(\tilde{\mathbf{L}})\leq(N+1)^{|\mathcal{U}|(3|\mathcal{X}||\mathcal{S}||\mathcal{Y}|+|\mathcal{X}||\mathcal{S}|+1)}\exp\{-N\delta/2\},$$

which for N large enough proves (32) and hence lemma.

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Կամայականորեն փոփոխվող և պատահական պարամետրով կապուղիների E-ունակության նոր գնահատականներ

Մ. Ե. Հարությունյան

Ամփոփում

Դիտարկված են պատահական պարամետրով և կամայականորեն փոփոխվող կապուղիները այն դեպքում, երբ վիճակների հաջորդականությունը հայտնի է առաքողին: Կառուցված են E-ունակության վերին և ստորին գնահատականները: Երբ $E \to 0$, պատահական կոդավորման գնահատականները և համապատասխան կապուղիների հայտնի ունակությունները համընկնում են: