Mathematical Problems of Computer Science 21, 2000, 164–180.

Algebraic Multilevel/Substructuring Preconditioner in Finite Element Method with Piecewise Quadratic Approximation*

Yuri R. Hakopian

Faculty of Informatics and Applied Mathematics, Yerevan State University

Abstract

A multilevel preconditioner for matrices arising in second–order finite element approximation of elliptic boundary value problems is proposed. Multilevel subdivision of the hierarchical triangular grids into substructures and using the inner Chebyshev iterations form the basis of the approach. The multilevel preconditioner constructed is proved to be spectrally equivalent to the initial stiffness matrix and its arithmetic cost is proportional to the dimension of the finest–grid algebraic problem.

Key words: multilevel preconditioning, finite element method, hierarchical grids, condition number.

1 Introduction

A considerable progress has recently been achieved in constructing optimal preconditioners for matrices arising in finite element approximation of elliptic boundary value problems. Most interesting and comprehensive results have been obtained by making use of multilevel procedures (see [5,6,10,12–14]).

Algebraic preconditioning methods occupy an important place among multilevel methods. In [5,6] an approach to constructing algebraic multilevel preconditioners of optimal order of computational complexity has been proposed. It is based on a nested sequence of grids and a special two–level ordering the nodes on each level. The preconditioner for correspondingly partitioned stiffness matrix is constructed by replacing the inverse of the Schur complement by certain matrix polynomial involving the inverse of the preconditioner on the previous (coarser) level and stiffness matrix on the current level. An another approach, closely related to the mentioned one, has been presented in [12–14]. Its main idea consists in partitioning the set of sides of the grid cells into substructures and using the inner Chebyshev iterations.

In [5,6,12–14] the recursive method of constructing the preconditioners is based upon the nested sequence of finite element spaces with nodal piecewise linear basis functions. Using the hierarchical basis functions in two–level methods has been considered in [1,3,8,9,15].

As is well–known, for second–order linear elliptic equations with solutions belonging to the Sobolev space H^2, finite element method on the base of piecewise linear approximation

*The work was supported by INTAS Association (93-377ex).

provides the optimal order of accuracy (see [16], for instance). However, using the piecewise linear approximation does not allow to obtain the greater rate of convergence for the problems with more smooth solutions. Increasing the rate of convergence can be achieved by using on triangles the second–order polynomial approximation (see [11,16,20]).

Constructing two–level preconditioners for second and third order finite element approximations has been considered in [3]. In present paper a multilevel preconditioner for the matrices which arise in finite element approximation of the self-adjoint elliptic boundary value problems on the base of piecewise quadratic basis functions is proposed. At first, using the technique of partitioning the domain into small substructures (see [12–14]), a two–stage preconditioner for the initial stiffness matrix is constructed. Then, as it turned out, the Schur complement of the two–stage preconditioner only by a numerical factor differs from a stiffness matrix corresponding to the piecewise linear basis functions at the vertices of triangles of the finest triangulation. This circumstance allows to construct multilevel preconditioners for the initial stiffness matrix using multilevel preconditioners for the "linear" case, i.e. for the linear finite element approximation. Note, that the similar idea but in another approach to constructing the preconditioners has been discussed in [4].

The remainder of the paper is organized as follows. In Section 2 the variational formulation of a second–order elliptic boundary value problem is posed and the matrix, spectrally equivalent to the initial finite element matrix, is constructed. In Section 3 a two–stage linerizing preconditioner is presented. The multilevel preconditioner and the associated condition number are analyzed in Section 4.

2 Preliminaries

2.1 Setting the problem

Let Ω be a simply–connected polygonal domain with the boundary $\partial\Omega$ in the plane of variables $x = (x_1, x_2)$ which is a union of some number $l \geq 1$ of triangles $\Delta_m, m = 1, 2, \ldots, l$. Any two triangles are supposed either not to intersect or to have only one common vertex or side. Define Γ_0 as a closed subset of $\partial\Omega$ consisting of the edges of triangles Δ_m. Denote by $H_0^1(\Omega)$ the subspace of the Sobolev space $H^1(\Omega)$ that consists of the functions vanishing on Γ_0.

Consider the variational formulation of model second–order elliptic boundary value problem: for a given function $f \in L_2(\Omega)$ find the function $u \in H_0^1(\Omega)$ such that

$$b(u,v) = (f,v)_{0,\Omega} \qquad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$b(u,v) \equiv \int_\Omega a \nabla u \nabla v \, dx \tag{2.2}$$

is a bilinear functional and $(\cdot,\cdot)_{0,\Omega}$ is the inner product in $L_2(\Omega)$. As regards the coefficient a, we suppose that it is a positive function constant in each triangle Δ_m. Namely,

$$a(x) \equiv a_m, \quad x \in \Delta_m \quad (m = 1, 2, \ldots, l).$$

2.2 Hierarchical sequence of triangular grids

Since the domain Ω is composed of triangles Δ_m, we have a well–defined coarsest triangulation τ_0 of the domain. By a refining procedure we obtain a sequence of triangulations

τ_k, $k = 0, 1, \ldots, p$. The refinement is performed by subdividing the triangles of the previou triangulation into four congruent ones. Let us agree that triangulation τ_k corresponds t the kth *level* of the refinement. With any triangulation τ_k we associate the grid ω_k, whos nodes are the vertices of triangles (*triangular elements*) which form the triangulation.

For all values $k = 0, 1, \ldots, p$ we introduce the following notation:

N_k is the set of nodes of the grid ω_k that belong to $\bar{\Omega} \setminus \Gamma_0$;

n_k is the number of nodes in the set N_k;

G_k is the space of grid functions defined on the set N_k;

V_k is the space of functions continuous in Ω, linear in each triangle of the triangulatio τ_k and vanishing on Γ_0.

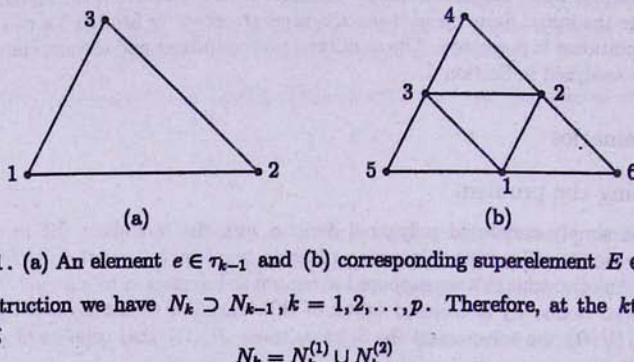
From now on we shall consider the operation of *prolongation* of grid functions in differen function spaces. Let us denote this operation by symbol prol having the following structure

$$\text{prol}\,(< a >:< b >),$$

where

$< a >$ is a prolonged grid function,

$< b >$ is a function space the prolongation belongs to.



(a) (b)

Fig.1. (a) An element $e \in \tau_{k-1}$ and (b) corresponding superelement $E \in T_k$.

By construction we have $N_k \supset N_{k-1}, k = 1, 2, \ldots, p$. Therefore, at the kth level the partitioning

$$N_k = N_k^{(1)} \cup N_k^{(2)} \tag{2.3}$$

of the nodes in N_k can be used, where

$$N_k^{(1)} = N_k \setminus N_{k-1} \quad , \quad N_k^{(2)} = N_{k-1} \,.$$

If the number of nodes in the set $N_k^{(i)}$ is denoted by $n_k^{(i)}\,(i = 1, 2)$, then $n_k^{(1)} = n_k - n_{k-1}$, $n_k^{(2)} = n_{k-1}$. The following ordering of the nodes will be used: *the nodes from* $N_k^{(1)}$ *are numbered first in some order, then the nodes from* $N_k^{(2)}$.

Let $G_k^{(i)}$ be the space of grid functions defined on the set $N_k^{(i)}\,(i = 1, 2)$.

Consider a triangular element $e \in \tau_{k-1}\,(1 \le k \le p)$. At the next level of refining the grid the element e is subdivided into four elements of the kth level. As a result, the element e turns into a superelement E (Fig.1). For all values $k = 1, 2, \ldots, p$ let T_k be the set of superelements at the kth level.

As has been said in Section 1, for solving the elliptic boundary value problem (2.1), (2.2) we intend to apply the second-order piecewise polynomial approximation (see [17,19]). To this end let us insert additional nodes at the midpoints of the edges of the triangular cells of triangulation τ_p. Thereby, the *linear* triangular elements of pth level turn into the *quadratic* triangular elements (see Fig.2). The triangulation τ_p, correspondingly, turns into triangulation τ which is formed by quadratic triangular elements. In consequence of inserting the additional nodes we obtain the new grid ω.

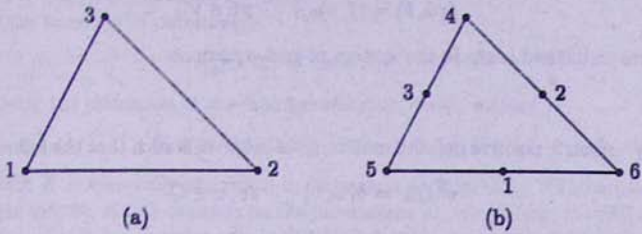


Fig.2. Inserting the additional nodes: (a) a linear triangular element of pth level and (b) corresponding quadratic triangular element.

Let us introduce the following notation:

N is the set of nodes of the grid ω which belong to $\bar{\Omega} \setminus \Gamma_0$;

n is the number of nodes in the set N;

G is the space of grid functions defined on the set N;

V is the space of functions continuous in Ω and vanishing on Γ_0, which are second-order polynomials of two variables on each quadratic triangular element.

The partitioning

$$N = N^{(1)} \cup N^{(2)} \tag{2.4}$$

of the nodes in N can be used, where

$$N^{(1)} = N \setminus N_p \quad , \quad N^{(2)} = N_p .$$

Let $n^{(i)}$ be the number of nodes in the set $N^{(i)}\,(i = 1,2)$. We have $n^{(1)} = n - n_p$, $n^{(2)} = n_p$. The following ordering of the nodes will be used: *the nodes from $N^{(1)}$ are numbered first in some order, then the nodes from $N^{(2)}$*.

Let $G^{(i)}$ be the space of grid functions defined on the set $N^{(i)}\,(i = 1,2)$.

Consider a triangle $\Delta_m, 1 \le m \le l$. Due to the rule of generating the sequence of triangulations, each triangulation τ_k (or τ) of the domain Ω defines the triangulation τ_k^m (or τ^m) of the triangle Δ_m. In other words, τ_k^m (or τ^m) is the *restriction* of the triangulation τ_k (or τ) onto triangle Δ_m. Relative to that let us introduce the following notation:

ω_k^m (or ω^m) is the restriction of the grid ω_k (or ω) onto triangle Δ_m (the nodes of the grid ω_k^m (or ω^m) are those of the grid ω_k (or ω) belonging to $\bar{\Delta}_m$);

N_k^m (or N^m) is the restriction of the set N_k (or N) onto triangle Δ_m;

n_k^m (or n^m) is the number of nodes in the set N_k^m (or N^m);

G_k^m (or G^m) is the space of grid functions defined on the set N_k^m (or N^m);

V_k^m (or V^m) is the space of restrictions of functions from V_k (or V) onto triangle Δ_m;

T_k^m is the restriction of the set of superelements T_k onto triangle Δ_m.

2.3 Spectrally equivalent matrix

Let us formulate the finite element problem corresponding to the problem (2.1), (2.2) : find the function $\bar{u} \in V$ such that

$$b(\bar{u}, \bar{v}) = (f, \bar{v})_{0,\Omega} \qquad \forall \bar{v} \in V\,. \tag{2.5}$$

The problem so defined leads to the system of grid equations

$$Qu = g\,, \tag{2.6}$$

where the symmetric positive definite matrix Q of order n is such that the following relation

$$v^T Q w = b(\bar{w}, \bar{v}) \qquad \forall v, w \in G \tag{2.7}$$

holds and the grid function $g \in G$ is determined by the relation

$$v^T g = (f, \bar{v})_{0,\Omega} \qquad \forall v \in G \tag{2.8}$$

(in (2.7) and (2.8) we have $\bar{v} = \mathrm{prol}(v : V), \bar{w} = \mathrm{prol}(w : V)$).

As is generally known, an arbitrary finite element matrix for some domain can be obtained by using the operation of *assembling* the matrices for subdomains (see [2,19], for instance). From now on let us denote this operation by symbol assem.

Define for each triangle $\Delta_m, m = 1, 2, \ldots, l$ a matrix Q_m of order n^m with the help of the relation

$$v^T Q_m w = a_m \int_{\Delta_m} \nabla\bar{w} \nabla\bar{v}\, dx \qquad \forall v, w \in G^m \tag{2.9}$$

$(\bar{v} = \mathrm{prol}(v : V^m), \bar{w} = \mathrm{prol}(w : V^m))$. Then

$$Q = \mathrm{assem}\{Q_m : m = 1, 2, \ldots, l\}\,. \tag{2.10}$$

For constructing a matrix spectrally equivalent to the stiffness matrix Q and subsequent constructing the multilevel preconditioner we shall use an approach already applied in [12,13].

Let us take a triangle Δ_m and perform an isoparametric linear mapping $\mathcal{L}_m$ which transforms it into an equilateral unit triangle Δ in the plane of variables $\xi = (\xi_1, \xi_2)$. Under this mapping the triangulation τ_k^m (or τ^m) is transformed into a uniform triangulation $\tau_k^m(\Delta)$ (or $\tau^m(\Delta)$) of the triangle Δ. The elements forming the triangulation $\tau_k^m(\Delta)$ (or $\tau^m(\Delta)$) are equilateral triangles with edge of length

$$h_k = 2^{-k}\,, \qquad k = 0, 1, \ldots, p \qquad (h_p \equiv h)\,. \tag{2.11}$$

The grid ω_k^m (or ω^m) is transformed into a grid $\omega_k^m(\Delta)$ (or $\omega^m(\Delta)$). Further, the linear mapping $\mathcal{L}_m$ transfers the set of nodes N_k^m (or N^m) of the grid ω_k^m (or ω^m) into the set of nodes $N_k^m(\Delta)$ (or $N^m(\Delta)$) of the grid $\omega_k^m(\Delta)$ (or $\omega^m(\Delta)$). Note, that the numbering of nodes remains unchanged under the mapping, i.e. the nodes of the set $N_k^m(\Delta)$ (or $N^m(\Delta)$) have the same numbers as their preimages from N_k^m (or N^m). Therefore, the space of grid functions G_k^m (or G^m) defined on the set N_k^m (or N^m) can be considered as the one defined on the set $N_k^m(\Delta)$ (or $N^m(\Delta)$). Finally, let $V_k^m(\Delta)$ (or $V^m(\Delta)$) be the space of piecewise linear (or piecewise quadratic) functions vanishing on the image of $\partial\Delta_m \cap \Gamma_0$ under the mapping $\mathcal{L}_m$.

For all values $m = 1, 2, \ldots, l$ define the matrices K_m of order n^m with the help of the relations

$$v^T K_m w = a_m \int_\triangle \nabla\bar{\bar{w}} \nabla\bar{\bar{v}}\, d\xi \qquad \forall v, w \in G^m \tag{2.12}$$

$(\bar{\bar{v}} = \mathrm{prol}(v : V^m(\triangle)), \bar{\bar{w}} = \mathrm{prol}(w : V^m(\triangle)))$.

Let us take some set of parameters

$$\kappa_m > 0, \qquad m = 1, 2, \ldots, l \tag{2.13}$$

and then, using the operation of assembling construct $n \times n$ matrix

$$K = \mathrm{assem}\{\kappa_m K_m : m = 1, 2, \ldots, l\}. \tag{2.14}$$

The matrix K is spectrally equivalent to the matrix Q from (2.6). The spectral condition number of the matrix $K^{-1}Q$ depends on the parameters κ_m involved in the definition (2.14) of the matrix K. It has been shown in [12,13] how this parameters should be chosen to minimize the estimate of $\mathrm{cond}(K^{-1}Q)$. Namely, for each triangle Δ_m there exist positive constants $\delta_m^{(1)}$ and $\delta_m^{(2)}$ such that the equivalence relation

$$\delta_m^{(1)} v^T K_m v \le v^T Q_m v \le \delta_m^{(2)} v^T K_m v$$

holds for all $v \in G^m$. The constants $\delta_m^{(1)}$ and $\delta_m^{(2)}$ depend only on geometrical parameters of the triangle Δ_m but do not depend on the number of refinement levels. For example, the constants can be taken as follows:

$$\delta_m^{(1)} = \frac{2\sqrt{3}\, s_m}{5 l_{m,max}^2} \quad , \quad \delta_m^{(2)} = \frac{\sqrt{3}(9 l_{m,max}^2 - 2 l_{m,min}^2)}{20 s_m}$$

(here s_m is the area of triangle Δ_m, $l_{m,min}$ and $l_{m,max}$ are the lengths of its shortest and longest edges, respectively). Then the condition number of the matrix $K^{-1}Q$ is estimated as follows:

$$\mathrm{cond}(K^{-1}Q) \le \max_{1 \le m \le l} \frac{\delta_m^{(2)}}{\kappa_m} / \min_{1 \le m \le l} \frac{\delta_m^{(1)}}{\kappa_m}.$$

The right-hand side of the last inequality takes its least value if we choose the parameters κ_m in the following way

$$\kappa_m = \sqrt{\delta_m^{(1)} \delta_m^{(2)}}, \qquad m = 1, 2, \ldots, l. \tag{2.15}$$

Under this choice

$$\mathrm{cond}(K^{-1}Q) \le \max_{1 \le m \le l} \frac{\delta_m^{(2)}}{\delta_m^{(1)}}. \tag{2.16}$$

Further we shall construct a multilevel preconditioner for the matrix K which will then be used for the initial stiffness matrix Q.

3 Two-stage linearizing preconditioner

3.1 Two-level preconditioners on the sequence of grids

For an arbitrary triangle e with vertices numbered 1,2 and 3 (see Fig.1a) define the bilinear functional

$$\varphi_e(u, v) \equiv (u_2 - u_1)(v_2 - v_1) + (u_3 - u_2)(v_3 - v_2) + (u_1 - u_3)(v_1 - v_3), \tag{3.1}$$

where u_i and v_i are the values of the functions u and v, respectively, at the ith vertex.

Consider some refinement level $k, 0 \le k \le p$. For all values $m = 1, 2, \ldots, l$ let us define matrices $L_m^{(k)}$ of order n_k^m with the help of relations

$$v^T L_m^{(k)} w = a_m \int_\Delta \nabla\hat{w}\, \nabla\hat{v}\, d\xi \qquad \forall v, w \in G_k^m \tag{3.2}$$

$(\hat{v} = \mathsf{prol}(v : V_k^m(\Delta)), \hat{w} = \mathsf{prol}(w : V_k^m(\Delta)))$. In [12,13] it has been shown that matrices $L_m^{(k)}$ so defined also satisfy the following relations

$$v^T L_m^{(k)} w = \frac{\sqrt{3}}{6} a_m \sum_{e' \in \tau_k^m(\Delta)} \varphi_{e'}(\hat{w}, \hat{v}) \qquad \forall v, w \in G_k^m .$$

It is obvious that

$$\varphi_{e'}(\hat{\hat{w}}, \hat{\hat{v}}) = \varphi_e(\hat{w}, \hat{v}),$$

where $e \in \tau_k^m$ is the preimage of the equilateral element $e' \in \tau_k^m(\Delta)$ under the mapping $\mathcal{L}_m$ and $\hat{v} = \mathsf{prol}(v : V_k^m)$, $\hat{w} = \mathsf{prol}(w : V_k^m)$. It can be readily seen from the definition (3.1) of the functional. Thereby, the matrices from (3.2) satisfy the relations

$$v^T L_m^{(k)} w = \frac{\sqrt{3}}{6} a_m \sum_{e \in \tau_k^m} \varphi_e(\hat{w}, \hat{v}) \qquad \forall v, w \in G_k^m . \tag{3.3}$$

Then, using the operation of assembling, let us construct $n_k \times n_k$ matrix

$$L^{(k)} = \mathsf{assem}\{\kappa_m L_m^{(k)} : m = 1, 2, \ldots, l\}, \tag{3.4}$$

where the constants κ_m are those of (2.14). As follows from definition (3.4) and relations (3.3), the matrix $L^{(k)}$ satisfies the relation

$$v^T L^{(k)} w = \frac{\sqrt{3}}{6} \sum_{m=1}^{l} \kappa_m a_m \sum_{e \in \tau_k^m} \varphi_e(\hat{w}, \hat{v}) \qquad \forall v, w \in G_k \tag{3.5}$$

$(\hat{v} = \mathsf{prol}(v : V_k), \hat{w} = \mathsf{prol}(w : V_k))$. In accordance with the rule of ordering the nodes, the matrix $L^{(k)}$ for $k \ge 1$ can be partitioned in two by two block form

$$L^{(k)} = \begin{bmatrix} L_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix} \tag{3.6}$$

with $n_k^{(i)} \times n_k^{(j)}$ submatrices $L_{ij}^{(k)}\ (i, j = 1, 2)$.

Further, with each superelement $E \in T_k$ (Fig.1b) we shall associate the bilinear functional

$$\begin{aligned} \Phi_E(u, v) \equiv\ & (u_1 - u_5)(v_1 - v_5) + (u_1 - u_6)(v_1 - v_6) + (u_2 - u_6)(v_2 - v_6) + \\ & (u_2 - u_4)(v_2 - v_4) + (u_3 - u_4)(v_3 - v_4) + (u_3 - u_5)(v_3 - v_5), \end{aligned} \tag{3.7}$$

where u_i and v_i are the values of the functions u and v, respectively, at the ith node.

Let us return to the relation (3.5) for $1 \le k \le p$. If we group the elements of the kth level to form superelements, then the matrix $L^{(k)}$ will satisfy the relation

$$v^T L^{(k)} w = \frac{\sqrt{3}}{6} \sum_{m=1}^{l} \kappa_m a_m \sum_{E \in T_k^m} [\Phi_E \hat{w}, \hat{v}) + 2\varphi_{e_E}(\hat{w}, \hat{v})] \qquad \forall v, w \in G_k, \tag{3.8}$$

where $e_E \in \tau_k^m$ is the triangular element whose vertices are the midpoints of the edges of superelement E (Fig.1b).

Define a matrix $B^{(k)}$ of order n_k by means of the relation

$$v^T B^{(k)} w = \frac{\sqrt{3}}{6} \sum_{m=1}^{l} \kappa_m a_m \sum_{E \in T_k^m} \Phi_E(\hat{w}, \hat{v}) \qquad \forall v, w \in G_k . \tag{3.9}$$

The matrix $B^{(k)}$ has the following block representation

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix}, \tag{3.10}$$

where $B_{11}^{(k)}$ is a diagonal matrix and blocks $L_{12}^{(k)}, L_{21}^{(k)}, L_{22}^{(k)}$ are identical with those of block representation (3.6) of the matrix $L^{(k)}$.

From now on let us denote by $\mathsf{sp}(A)$ the spectrum of a matrix A.

The following statement takes place (see [12,13]).

Theorem 1 *For all values $k = 1, 2, \ldots, p$, regardless of the values of the coefficient a in subdomains $\triangle_m$,*

$$\mathsf{sp}(B^{(k)^{-1}} L^{(k)}) \subset [1, 5] . \tag{3.11}$$

We shall consider the matrix $B^{(k)}$ as a *preconditioner* for the matrix $L^{(k)}$. As regards the Schur complement

$$S_{22}^{(k)} = L_{22}^{(k)} - L_{21}^{(k)} B_{11}^{(k)^{-1}} L_{12}^{(k)} \tag{3.12}$$

of the matrix $B^{(k)}$, then

$$S_{22}^{(k)} = \frac{1}{2} L^{(k-1)} \tag{3.13}$$

(see [12,13]). For this reason the matrix $B^{(k)}$ is called a *two-level preconditioner.* The block representation (3.10) of the matrix $B^{(k)}$ can be rewritten in the following form:

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & \frac{1}{2} L^{(k-1)} + L_{21}^{(k)} B_{11}^{(k)^{-1}} L_{12}^{(k)} \end{bmatrix} . \tag{3.14}$$

3.2 Two-stage preconditioner for the matrix K

The matrix K spectrally equivalent to the matrix Q has been constructed in Section 2 (see (2.12)-(2.14)). According to the rule of ordering the nodes of set N, the matrix K admits the block representation

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{3.15}$$

with $n^{(i)} \times n^{(j)}$ submatrices $K_{ij}\,(i, j = 1, 2)$.

Consider a quadratic triangular element e whose nodes are numbered as shown in Fig.2b. Let us insert the bilinear functionals

$$\begin{aligned} \Psi_e(u, v) \;\equiv\; & 4[(u_1 - u_5)(v_1 - v_5) + (u_1 - u_6)(v_1 - v_6) + (u_2 - u_4)(v_2 - v_4) + \\ & (u_2 - u_6)(v_2 - v_6) + (u_3 - u_4)(v_3 - v_4) + (u_3 - u_5)(v_3 - v_5)] - \\ & [(u_5 - u_4)(v_5 - v_4) + (u_6 - u_5)(v_6 - v_5) + (u_4 - u_6)(v_4 - v_6)] \end{aligned} \tag{3.16}$$

and

$$\psi_e(u,v) \equiv 8[(u_2-u_1)(v_2-v_1)+(u_3-u_2)(v_3-v_2)+(u_1-u_3)(v_1-v_3)] \tag{3.17}$$

into our consideration, where u_i and v_i are the values of the functions u and v, respectively, at the ith node. It may easily make sure that

$$\Psi_e(u,u) \geq 0 \quad , \quad \psi_e(u,u) \geq 0 \tag{3.18}$$

for any function u defined at the nodes of the element e.

Let us formulate a statement which form the base of the further considerations. It may be proved by direct calculation.

Lemma 2 *If e is an equilateral quadratic triangular element, then*

$$\int_e \nabla u \nabla v \, dx = \frac{\sqrt{3}}{18}[\Psi_e(u,v) + \psi_e(u,v)] \tag{3.19}$$

for any functions u and v which are second–order polynomials of two variables in e.

Proceeding from the relation (2.12) and using Lemma 3.1, we obtain that for matrix K_m the following equality

$$v^T K_m w = \frac{\sqrt{3}}{18} a_m \sum_{e' \in \tau^m(\Delta)} [\Psi_{e'}(\bar{\bar{w}},\bar{\bar{v}}) + \psi_{e'}(\bar{\bar{w}},\bar{\bar{v}})] \qquad \forall v,w \in G^m$$

holds. It is obvious that

$$\Psi_{e'}(\bar{\bar{w}},\bar{\bar{v}}) = \Psi_e(\bar{w},\bar{v}) \quad , \quad \psi_{e'}(\bar{\bar{w}},\bar{\bar{v}}) = \psi_e(\bar{w},\bar{v}) \, ,$$

where $e \in \tau^m$ is the preimage of the equilateral quadratic element $e' \in \tau^m(\Delta)$ under the mapping $\mathcal{L}_m$ and $\bar{v} = \mathrm{prol}(v : V^m), \bar{w} = \mathrm{prol}(w : V^m)$. Thus, the matrix K_m satisfies the relation

$$v^T K_m w = \frac{\sqrt{3}}{18} a_m \sum_{e \in \tau^m} [\Psi_e(\bar{w},\bar{v}) + \psi_e(\bar{w},\bar{v})] \qquad \forall v,w \in G^m . \tag{3.20}$$

Then, as follows from (2.14) and (3.20), the matrix K satisfies the relation

$$v^T K w = \frac{\sqrt{3}}{18} \sum_{m=1}^{l} \kappa_m a_m \sum_{e \in \tau^m} [\Psi_e(\bar{w},\bar{v}) + \psi_e(\bar{w},\bar{v})] \qquad \forall v,w \in G . \tag{3.21}$$

Define a matrix B of order n by the relation

$$v^T B w = \frac{\sqrt{3}}{18} \sum_{m=1}^{l} \kappa_m a_m \sum_{e \in \tau^m} \Psi_e(\bar{w},\bar{v}) \qquad \forall v,w \in G . \tag{3.22}$$

The matrix B can be represented in the block form

$$B = \begin{bmatrix} B_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \tag{3.23}$$

where B_{11} is a diagonal matrix and blocks K_{12}, K_{21}, K_{22} are identical with those of the block representation (3.15) of the matrix K.

The matrix B constructed will be considered as a preconditioner for the matrix K. Let us now determine the bounds of spectrum of the matrix $B^{-1}K$. Consider the generalized eigenvalue problem

$$Ku = \lambda Bu. \tag{3.24}$$

The smallest and the largest eigenvalues of the problem (3.24) are denoted by λ_{min} and λ_{max}, respectively.

The inequality

$$u^T K u \geq u^T B u$$

holds for all $u \in G$. It follows directly from relations (3.21), (3.22) and the second of the inequalities (3.18). This implies that for the eigenvalues of the problem (3.24) the inequality $\lambda \geq 1$ holds. Moreover, as it follows from block representations (3.15) and (3.23), $\lambda = 1$ is an eigenvalue of the problem (3.24). Hence,

$$\lambda_{min} = 1. \tag{3.25}$$

The largest eigenvalue λ_{max} can be evaluated by means of passing onto the element level. Let K^e and B^e be the restrictions of the matrices K and B, respectively, on the element $e \in \tau$ in the following sense:

$$K = \mathsf{assem}\{K^e : e \in \tau\},\ B = \mathsf{assem}\{B^e : e \in \tau\}.$$

Further, let w_e be the restriction of an arbitrary grid function $w \in G$ onto an element $e \in \tau$.

For a non-zero grid function $w \in G$ we have

$$\frac{w^T K w}{w^T B w} = \frac{\sum_{e\in\tau} w_e^T K^e w_e}{\sum_{e\in\tau} w_e^T B^e w_e} \leq \max_{\substack{e\in\tau,\\ w_e \notin \mathsf{ker} B^e}} \frac{w_e^T K^e w_e}{w_e^T B^e w_e} \leq \max_{e\in\tau} \max_{v \notin \mathsf{ker} B^e} \frac{v^T K^e v}{v^T B^e v}$$

(here we take into account that $\mathsf{ker} K^e = \mathsf{ker} B^e$ for all $e \in \tau$). It is readily seen that the maximum over $e \in \tau$ in the right-hand side of the last inequality is achieved on any element of which all the nodes belong to the set N. Let $e \in \tau$ be such an element. So, we arrive at a conclusion that

$$\lambda_{max} \leq \lambda_e, \tag{3.26}$$

where λ_e is the largest eigenvalue of the problem

$$K^e v = \lambda B^e v, \quad v \notin \mathsf{ker} B^e, \quad \lambda \neq 1. \tag{3.27}$$

Let us represent the matrices K^e and B^e in the block form

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}, \quad B^e = \begin{bmatrix} B_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

according to the rule of ordering the nodes. The problem (3.27) is equivalent to the eigenvalue problem

$$S_K^e v_1 = \lambda S_B^e v_1, \quad v_1 \notin \mathsf{ker} S_B^e, \tag{3.28}$$

where S_K^e and S_B^e are the Schur complements of the matrices K^e and B^e, respectively:

$$S_K^e = K_{11}^e - K_{12}^e K_{22}^{e^{-1}} K_{21}^e\,, \quad S_B^e = B_{11}^e - K_{12}^e K_{22}^{e^{-1}} K_{21}^e\,. \tag{3.29}$$

Taking advantage of the relations (3.21) and (3.22), let us write down the matrices K^e and B^e using the node numbering given in Fig.2b:

$$K^e = \frac{\sqrt{3}}{18}\kappa_m a_m \left[\begin{array}{ccc|ccc} 24 & -8 & -8 & 0 & -4 & -4 \\ -8 & 24 & -8 & -4 & 0 & -4 \\ -8 & -8 & 24 & -4 & -4 & 0 \\ \hline 0 & -4 & -4 & 6 & 1 & 1 \\ -4 & 0 & -4 & 1 & 6 & 1 \\ -4 & -4 & 0 & 1 & 1 & 6 \end{array}\right],$$

$$B^e = \frac{\sqrt{3}}{18}\kappa_m a_m \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 0 & -4 & -4 \\ 0 & 8 & 0 & -4 & 0 & -4 \\ 0 & 0 & 8 & -4 & -4 & 0 \\ \hline 0 & -4 & -4 & 6 & 1 & 1 \\ -4 & 0 & -4 & 1 & 6 & 1 \\ -4 & -4 & 0 & 1 & 1 & 6 \end{array}\right].$$

Then, the matrices S_K^e and S_B^e from (3.29) are

$$S_K^e = \frac{8\sqrt{3}}{15}\kappa_m a_m \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad S_B^e = \frac{4\sqrt{3}}{45}\kappa_m a_m \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

From here we find that the unique eigenvalue of the problem (3.28) is $\lambda = 6$. Hence, taking into account (3.26) we obtain the estimate

$$\lambda_{max} \le 6\,. \tag{3.30}$$

Having at our disposal (3.25) and (3.30), we arrive at the following statement.

Theorem 3 *Regardless of the values of the coefficient a in subdomains Δ_m,*

$$\mathrm{sp}(B^{-1}K) \subset [1\,,6]\,. \tag{3.31}$$

Consider the Schur complement of the matrix B represented in the block form (3.23):

$$S_{22} = K_{22} - K_{21}B_{11}^{-1}K_{12}\,. \tag{3.32}$$

For matrix S_{22} the following statement takes place. It can be proved by direct calculation.

Theorem 4 *The equality*

$$S_{22} = \frac{1}{3}L^{(p)}, \tag{3.33}$$

where $L^{(p)}$ is the matrix defined in (3.4), holds.

The matrix B will be referred to as *two-stage linearizing preconditioner* for the matrix K. Using (3.32) and (3.33), the block representation (3.23) of the preconditioner B can be written in the following form:

$$B = \begin{bmatrix} B_{11} & K_{12} \\ K_{21} & \frac{1}{3}L^{(p)} + K_{21}B_{11}^{-1}K_{12} \end{bmatrix}. \tag{3.34}$$

4 Multilevel preconditioner

4.1 Constructing the preconditioner

In Section 2 and Section 3 we have constructed the sequence of finite element matrices

$$L^{(0)}, L^{(1)}, \dots, L^{(p)}, K \tag{4.1}$$

and corresponding sequence of preconditioners

$$B^{(1)}, \dots, B^{(p)}, B. \tag{4.2}$$

Recall that the matrices $L^{(k)}$ and $B^{(k)}$ have been computed by the piecewise linear basis functions while the matrices K and B by the piecewise quadratic ones. Let us now turn to the construction of a multilevel preconditioner for the matrix K (and, thereby, for the matrix Q) using the inner Chebyshev iterations.

At first, let us describe briefly the construction of the multilevel preconditioner for the matrix $L^{(p)}$, proposed in [12,13]. For all values $k = 1, 2, \dots, p$ successively define the matrices

$$M^{(k)} = \begin{bmatrix} B_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & \frac{1}{2}R^{(k-1)} + L_{21}^{(k)} {B_{11}^{(k)}}^{-1} L_{12}^{(k)} \end{bmatrix}, \tag{4.3}$$

where

$$R^{(0)} = L^{(0)}, \tag{4.4}$$

$$R^{(k-1)} = L^{(k-1)} \left[I - \prod_{j=1}^{3} (I - \theta_j^{(k-1)} {M^{(k-1)}}^{-1} L^{(k-1)}) \right]^{-1}, \; k = 2, 3, \dots, p \tag{4.5}$$

(here I is the identity matrix). The parameters $\theta_j^{(k-1)}$ are chosen as follows

$$\theta_j^{(k-1)} = \frac{2}{\beta_{k-1} + \alpha_{k-1} + (\beta_{k-1} - \alpha_{k-1}) z_j^{(3)}}, \quad j = 1, 2, 3,$$

where $z_j^{(3)}$ are the roots of the Chebyshev polynomial of the first kind of degree 3 and $[\alpha_{k-1}; \beta_{k-1}]$ is the interval containing $\mathrm{sp}({M^{(k-1)}}^{-1} L^{(k-1)})$. The bounds $[\alpha_k; \beta_k]$ of spectra of the matrices ${M^{(k)}}^{-1} L^{(k)}$ $(k = 1, 2, \dots, p)$ are determined by the following formulae:

$$\begin{aligned}
&\text{for} \quad k = 1 \; : \; \alpha_1 = 1, \beta_1 = 5; \\
&\text{for} \quad 2 \le k \le p \; : \; \alpha_k = 1 - \gamma_{k-1}, \beta_k = 5(1 + \gamma_{k-1}), \\
&\qquad \gamma_{k-1} = \frac{2q_{k-1}^3}{1 + q_{k-1}^6}, q_{k-1} = \frac{\sqrt{c_{k-1}} - 1}{\sqrt{c_{k-1}} + 1}, c_{k-1} = \frac{\beta_{k-1}}{\alpha_{k-1}}.
\end{aligned}$$

In [12,13] the matrix $M^{(p)}$ has been considered as a multilevel preconditioner for the matrix $L^{(p)}$. It is also proved there that regardless of the number of levels

$$c_p = \frac{\beta_p}{\alpha_p} \le c_* \equiv 3 + 2\sqrt{5}. \tag{4.6}$$

Note, that the estimate does not depend on the values of coefficient a in subdomains Δ_m.

Let us compare the block representations (3.14) and (4.3) of the matrices $B^{(k)}$ and $M^{(k)}$, respectively. We see, that the matrix $L^{(k-1)}$ in (3.14) has been replaced by specially chosen matrix $R^{(k-1)}$ (see (4.4),(4.5)). The similar replacement is performed in the block representation (3.34) of matrix B to obtain the multilevel preconditioner for the matrix K.

Let us choose an integer $\nu \geq 1$ and define a matrix

$$M = \begin{bmatrix} B_{11} & K_{12} \\ K_{21} & \frac{1}{3}R^{(p)} + K_{21}B_{11}^{-1}K_{12} \end{bmatrix}, \tag{4.7}$$

where

$$R^{(p)} = L^{(p)}\left[I - \prod_{j=1}^{\nu}(I - \theta_j^{(p)} M^{(p)^{-1}} L^{(p)})\right]^{-1}. \tag{4.8}$$

Here

$$\theta_j^{(p)} = \frac{2}{\beta_p + \alpha_p + (\beta_p - \alpha_p) z_j^{(\nu)}}, \quad j = 1, 2, \cdots, \nu,$$

where $z_j^{(\nu)}$ are the roots of the Chebyshev polynomial of the first kind of degree ν.

We have the block representations (3.34) and (4.7) of the matrices B and M, respectively. As follows from the theory of Chebyshev methods (see [18], for instance), due to definition (4.8) of the matrix $R^{(p)}$ we obtain

$$\mathrm{sp}(M^{-1}B) \subset [1-\gamma, 1+\gamma], \tag{4.9}$$

where

$$\gamma = \frac{2q^{\nu}}{1+q^{2\nu}} \quad , \quad q = \frac{\sqrt{c_p}-1}{\sqrt{c_p}+1}. \tag{4.10}$$

Then, proceeding from the equality

$$M^{-1}K = (M^{-1}B)(B^{-1}K)$$

and using (3.31) and (4.9), we find the bounds of spectrum of the matrix $M^{-1}K$:

$$\mathrm{sp}(M^{-1}K) \subset [\alpha, \beta], \tag{4.11}$$

where

$$\alpha = 1-\gamma, \beta = 6(1+\gamma). \tag{4.12}$$

Thus, from (4.10)-(4.12) we obtain the inequality

$$\mathrm{cond}(M^{-1}K) \leq \frac{\beta}{\alpha} = 6\left[\frac{(\sqrt{c_p}+1)^{\nu} + (\sqrt{c_p}-1)^{\nu}}{(\sqrt{c_p}+1)^{\nu} - (\sqrt{c_p}-1)^{\nu}}\right]^2.$$

From here, taking into account estimate (4.6), we arrive at the following statement.

Theorem 5 *Regardless of the values of the coefficient a in subdomains Δ_m, the estimate*

$$\mathrm{cond}(M^{-1}K) \leq \hat{c}, \tag{4.13}$$

where

$$\hat{c} = 6\left[\frac{(\sqrt{c_*}+1)^{\nu} + (\sqrt{c_*}-1)^{\nu}}{(\sqrt{c_*}+1)^{\nu} - (\sqrt{c_*}-1)^{\nu}}\right]^2, \quad c_* = 3 + 2\sqrt{5}, \tag{4.14}$$

holds.

Below we give in Table 1 the values of the quantity $\hat{c}$ for some values of parameter ν.

Table 1.

$\nu =$	$\hat{c} =$
3	$6(3+2\sqrt{5})/5 \simeq 8.97$
4	$54(2\sqrt{5}-3)/11 \simeq 7.23$
5	$3(5\sqrt{5}-9) \simeq 6.55$
6	$3(3651+3127\sqrt{5})/5120 \simeq 6.24$

Finally, let us obtain the estimate of the spectral condition number of the matrix $M^{-1}Q$. We have

$$\mathrm{cond}(M^{-1}Q) \leq \mathrm{cond}(M^{-1}K)\,\mathrm{cond}(K^{-1}Q).$$

The condition numbers entering the right-hand side of the last inequality have already been estimated in (2.16) and (4.13). Thus, we obtain the following statement.

Theorem 6 *Regardless of the values of the coefficient a in subdomains Δ_m, the estimate*

$$\mathrm{cond}(M^{-1}Q) \leq \hat{c} \max_{1\leq m\leq l} \frac{\delta_m^{(2)}}{\delta_m^{(1)}}, \tag{4.15}$$

where the quantity $\hat{c}$ is calculated from the formula (4.14), holds.

4.2 Implementational details

In an iterative method with matrix M as a multilevel preconditioner we need to solve linear systems with matrices M and $M^{(k)}$ $(k=1,2,\ldots,p)$.

At first let us describe the process of solving a system with matrix M. For certainty, we consider the system

$$Mv = g, \tag{4.16}$$

where

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \quad v_i, g_i \in G^{(i)}, i = 1,2.$$

Algorithm (M)

- the grid function

$$z_2 = 3\,(g_2 - K_{21}B_{11}^{-1}g_1) \tag{4.17}$$

is calculated;

- the system

$$R^{(p)}v_2 = z_2 \tag{4.18}$$

is solved; finding the solution of system (4.18) is equivalent to performing ν steps of the Chebyshev iterative process

$$M^{(p)}\frac{v_2^{(j)} - v_2^{(j-1)}}{\theta_j^{(p)}} = -L^{(p)}v_2^{(j-1)} + z_2, \quad j = 1,2,\ldots,\nu, \quad v_2^{(0)} = 0; \tag{4.19}$$

$$v_2 = v_2^{(\nu)};$$

• the grid function

$$v_1 = B_{11}^{-1}\,(g_1 - K_{12}v_2) \tag{4.20}$$

is calculated;

end

Further, consider a system with matrix $M^{(k)}\,(1 \le k \le p)$:

$$M^{(k)}v = g\,, \tag{4.21}$$

where

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\,,\quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\quad;\quad v_i\,,\, g_i \in G_k^{(i)}\,,\, i = 1,2\,.$$

Algorithm $(\mathbf{M^{(k)}})$

• the grid function

$$z_2 = 2\,(g_2 - L_{21}^{(k)}{B_{11}^{(k)}}^{-1}\,g_1) \tag{4.22}$$

is calculated;

• the system

$$R^{(k-1)}v_2 = z_2 \tag{4.23}$$

is solved;

in the case $2 \le k \le p$: finding the solution of system (4.23) is equivalent to performing 3 steps of the Chebyshev iterative process

$$M^{(k-1)}\,\frac{v_2^{(j)} - v_2^{(j-1)}}{\theta_j^{(k-1)}} = -L^{(k-1)}v_2^{(j-1)} + z_2\,,\quad j = 1,2,3,\quad v_2^{(0)} = 0\,; \tag{4.24}$$

$$v_2 = v_2^{(3)}\,;$$

in the case $k = 1$: the system

$$L^{(0)}v_2 = z_2 \tag{4.25}$$

is solved;

• the grid function

$$v_1 = {B_{11}^{(k)}}^{-1}\,(g_1 - L_{12}^{(k)}v_2) \tag{4.26}$$

is calculated;

end

Let us give some comments to **Algorithm (M)** and **Algorithm** $(\mathbf{M^{(k)}})$. The matrices B_{11} and $B_{11}^{(k)}\,(1 \le k \le p)$ are diagonal. Thereby, calculations by formulae (4.17), (4.20) and (4.22), (4.26) present no difficulties. Then, the system of grid equations (4.25) at the coarsest level is assumed to be solved by a direct method spending $O(1)$ arithmetic operations.

In conclusion, a few words about the computational complexity of a preconditioning step. Let A_{ops} be the number of arithmetic operations required for solving a system with matrix M. By means of direct calculation we find

$$A_{ops} \le 4\,(\nu + 1)\,n + 16\,\nu\,[n_p + 3n_{p-1} + \cdots + 3^{p-1}n_1] + \nu\,3^{p-1}A_{ops}^{(0)}\,,$$

where $A^{(0)}_{ops}$ is the number of arithmetic operations required for solving a system with matrix $L^{(0)}$. Then, using the inequalities from [7], we obtain the arithmetic cost of a preconditioning step

$$A_{ops} \leq C \nu n \quad ,$$

where C is a positive constant independent of n and ν.

Thus, the preconditioner M constructed may be considered to belong to the class of optimal preconditioners, since it is spectrally equivalent to the original stiffness matrix Q and its arithmetic cost is proportional to the dimension of the finest-grid problem.

References

[1] O. Axelsson, On multigrid methods of the two-level type.– In: *Multigrid Methods*, Proceedings, Köln-Porz, 1981 (W.Hackbush and U.Trottenberg, eds.), Lecture Notes in Math., 960, Springer-Verlag, 1982, 352–367.

[2] O. Axelsson and V. A. Barker, Finite Element Solution of Boundary Value Problems: Theory and Computation.– Academic Press, Orlando, Fla, 1984.

[3] O. Axelsson and I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximation.– *Math. Comput.*, v.40, No.161, 1983, 219–242.

[4] O. Axelsson and L. Kolotilina, Diagonally Compensated Reduction and Related Preconditioning Methods.– *J. Numer. Linear Algebra with Appl.*, v.1(2), 1994, 155–177.

[5] O. Axelsson and P. S. Vassilevski. Algebraic multilevel preconditioning methods. I.– *Numer. Math.*, 56, 1989, 157–177.

[6] O. Axelsson and P. S. Vassilevski, Algebraic multilevel preconditioning methods. II.– *SIAM J. Numer. Anal.*, v.27, No.6, 1990, 1569–1590.

[7] O. Axelsson and P. S. Vassilevski, Asymptotic work estimates for AMLI methods.– *Applied Numer. Math.*, 7, 1991, 437–451.

[8] R. Bank and T. Dupont, Analysis of a two-level scheme for solving finite element equations.– *Report CNA-159*, Centre for Numerical Analysis, The University of Texas, Austin, TX, 1980.

[9] D. Braess, The contraction number of a multigrid method for solving the Poisson equation.– *Numer. Math.*, 37, 1981, 387–404.

[10] J. H. Bramble, J. E. Pasciak and J. Xu, Parallel multilevel preconditioners.– *Math. Comput.*, v.55, No.191, 1990, 1–22.

[11] J. H. Bramble and M. Zlamal, Triangular elements in the finite element method.– *Math. Comput.*, v.24, No.112, 1970, 809–820.

[12] Yu. R. Hakopian and Yu. A. Kuznetsov, Algebraic multigrid method for solving finite element equations on hierarchical triangular grids.– *Report No.250*, Dept. Numer. Math. USSR Acad. Sci., Moscow, 1990 (in Russian).

[13] YU. R. HAKOPIAN AND YU. A. KUZNETSOV, Algebraic multigrid/substructuring preconditioners on triangular grids.– *Sov. J. Numer. Anal. Math. Modelling*, v.6, No.6, 1991, 453–483.

[14] YU. A. KUZNETSOV, Algebraic multigrid domain decomposition methods.– *Sov. J. Numer. Anal. Math. Modelling*, v.4, No.5, 1989, 351–379.

[15] J. F. MAITRE AND F. MUSY, The contraction number of a class of two–level methods: an exact evaluation for some finite element subspaces and model problems.– In: *Multigrid Methods*, Proceedings, Köln–Porz, 1981 (W. Hackbush and U. Trottenberg, eds.), Lecture Notes in Math., 960, Springer–Verlag, 1982, 535–544.

[16] L. A. OGANESYAN AND L. A. RUCHOVETS, Variation–difference methods for solving the elliptic equations.– Acad. Sci. of Armenia Publ., Yerevan, 1979 (in Russian).

[17] G. STRANG AND G. J. FIX, An Analysis of the Finite Element Method.– Prentice–Hall, Englewood Cliffs, NJ, 1973.

[18] R. S. VARGA, Matrix Iterative Analysis.– Prentice–Hall, Englewood Cliffs, NJ, 1962.

[19] O. C. ZENKIEWICZ AND K. MORGAN, Finite Elements and Approximation.– A Wiley–Interscience Publ., John Wiley&Sons, 1983.

[20] M. ZLAMAL, On the finite element method.– *Numer. Math.*, 12, 1968, 394–409.