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Cycle-Extensions and Long Cycles in k-connected Graphs

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Abstract

Let G be a k–connected ($k \geq 2$) graph with minimum degree δ and let C be a longest cycle in G. If $G - C$ has a cycle of length h with $h \geq k$ then $|C| \geq \frac{(h+1)k}{h+k+1}(\delta+2)$.

1 Introduction

In this paper we present a lower bound for the length c of a longest cycle C in k–connected ($k \geq 2$) graph G in terms of the minimum degree δ, vertex-connectivity k and the length h of any cycle in $G - C$.

We consider only finite undirected graphs without loops and multiple edges. For unexplained terminology see [1]. The vertex set of a graph G is denoted by $V(G)$ or just V; the set of edges by $E(G)$ or just E. We use $|G|$ as a symbol of the cardinality $|V|$. For a subset S of V, $G - S$ denotes the subgraph $<V - S>$ induced by $V - S$. If H is a subgraph of G, we also use the symbol $G - H$ for $G - V(H)$.

Paths and cycles in a graph G are considered as subgraphs of G, they are connected and have maximum degree 0, 1 or 2. The length of path P is $|P| - 1$ and the length of cycle Q is $|Q|$.

We need the following extension of a notion for cycles: every edge (respectively, vertex) will be interpreted as a cycle of length 2 (respectively, 1). For Q a cycle in G the following equalities $|Q| = 0$ and $V(Q) = \emptyset$ are equivalent. A graph is said to be hamiltonian if its longest cycle passes through all of its vertices.

By the definition, G is hamiltonian iff $h = 0$. If $h = 1$ then $V - V(C)$ is an independent set of vertices or, in other words, C is a dominating cycle.

Let c (the circumference) denote the length of a longest cycle in G.

In view of the main purpose the following results can be considered as starting points:

(A) $k \geq 2 \Rightarrow c \geq 2\delta$ or $h = 0$ (1952, Dirac [3]),

(B) $k \geq 3 \Rightarrow c \geq 3(\delta - 1)$ or $h \leq 1$ (1977, Voss [12]).

Since 1952, we know more about the impact of 2 - connectivity and 3 - connectivity on circumference and cycle structures in graphs. But a very little was known for k–connected graphs in general. The following three results give some understanding how k–connectivity, minimum degree and independence number α effect long cycles in graphs:

(C) $k \geq \alpha \Rightarrow h = 0$ (1972, Chvatal and Erdos [2]),

(D) $k \geq 3 \Rightarrow c \geq 3\delta - k$ or $h = 0$ (1982, Nikoghosyan [8])

(E) $k \geq 4 \Rightarrow c \geq 4\delta - 2k$ or $h \leq 1$ (1985, Nikoghosyan [9]).

In 1998 the notion of path-extensions was introduced [10], which allows to use path and cycle structures of $G-C$ much more effectively. For q the length of a longest path in $G-C$ the following lower bound for the circumference is obtained:

(F) $c \geq (q+2)(\delta - q)$ (1998, Nikoghosyan [10]).

Some progress on circumference has been made for τ-tough graphs. A graph G is t-tough if $|S| \geq t\omega(G-S)$ for every subset $S \subseteq V(G)$ with $\omega(G-S) > 1$, where $\omega(G)$ denote the number of components of G. The toughness of G, denoted $\tau(G)$, is the minimum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Using ruther strong properties of t-tough graphs, Jung and Wittmann were able to prove:

(G) $k \geq 2 \Rightarrow c \geq (\tau+1)(\delta+1) - 1$ or $h = 0$ (1999, Jung and Wittmann [5]).

Returning to the vertex connectivity k, we find in [6] some bounds of the type $c \geq k(\delta - k + 2)$ for k-connected graphs with $k \leq 6$. In [4] a similar result was obtained for k-connected graphs with a rather strong condition with respect to $G-C$ (for any two distinct vertices x, y in some component of $G-C$, there is a path of length at least $k-2$ with endvertices x and y).

Applying so called "cycle-extensions technique" for k-connected graphs, the following result was obtained in 1999:

(H) $h \geq k \Rightarrow c \geq \frac{(h+1)k}{h+k+1}\left(\delta + 1 + \frac{2}{h}\right)$. (1999, Nikoghosyan [11]).

In this paper, improving (H) we have obtained its complete and final verssion:

Theorem. Let G be a k–connected ($k \geq 2$) graph with minimum degree δ and let C be a longest cycle in G. If $G-C$ has a cycle of length h with $h \geq k$ then $|C| \geq \frac{(h+1)k}{h+k+1}(\delta+2)$.

For h the length of longest cycle in $G-C$, the result is sharp, as can be seen from the following family of graphs. Take $k+1$ disjoint copies of the complete graph $K_{\delta-k+1}$ and join each vertex in their union to every vertex of a disjoint complete graph K_k. This graph $(k+1)K_{\delta-k+1} + K_k$ is clearly not hamiltonian. Moreover, $c = k(\delta - k + 2)$ and $h = \delta - k + 1$, implying that $c = \frac{(h+1)k}{h+k+1}(\delta+2)$.

The next section is devoted to standard terminology. In section 3 we introduce some special definitions and convenient notations, where the notion of HC–extensions plays a central role in the sequel. In section 4 we investigate the main properties of HC–extensions and in the last section we prove our main result.

2 Terminology

An (x,y)-path is a path with endvertices x and y. Given an (x,y)-path L of G we denote by $\overrightarrow{L}$ the path L with an orientation from x to y. If $u,v \in V(L)$ then $u\overrightarrow{L}v$ denotes the consequtive vertices on L from u to v in the direction specified by $\overrightarrow{L}$. The same vertices, in reverse order, are given by $v\overleftarrow{L}u$. For $\overrightarrow{L} = x\overrightarrow{L}y$ and $u \in V(L)$, let $u^+(\overrightarrow{L})$ (or just u^+) denotes the successor of u ($u \neq y$) on $\overrightarrow{L}$ and u^- denotes its predecessor ($u \neq x$). If $A \subseteq V(L)$ then $A^+ = \{v^+ \mid v \in A - y\}$ and $A^- = \{v^- \mid v \in A - x\}$. If Q is a cycle in G and $A \subseteq V(Q)$ then $\overrightarrow{Q}$, A^+ and A^- are analogously defined. For $v \in V(Q)$, $v\overrightarrow{Q}v$ will be interpreted as a vertex v.

For $v \in V$, put $N(v) = \{u \in V \mid uv \in E\}$, $d(v) = |N(v)|$ and
$\delta = min\{d(u) \mid u \in V\}$.

3 Special Definitions

We begin introducing some special definitions and convenient notations. For the remainder of this section let a longest cycle C in graph G and a longest cycle $H = u_1 \ldots u_h u_1$ in $G - C$ be fixed.

Definition 3.1. T is an HC−extension; $T(u_i)$; $\mathring{u}$; $\hat{u}$.

Let $T(u_1), \ldots, T(u_h)$ are vertex-disjoint $(u_i, \hat{u}_i)$-paths in $G - C$ for $i = 1, \ldots, h$ respectively. The union $T = \bigcup_{i=1}^{h} T(u_i)$ is called HC−extension if $N(\hat{u}_i) \subseteq V(T) \cup V(C)$ for each $i \in \overline{1,h}$. An HC-extension T is called maximal if it is chosen so as to maximize $|\{u \in V(H) \,|u \neq \hat{u}\}|$. If $u \neq \hat{u}$ for some $u \in V(H)$ then we use $\mathring{u}$ to denote $u^+(\overrightarrow{T}(u))$.

Definition 3.2. (A, B)−path.

Let $A, B \subset V$ and $A \bigcap B = \emptyset$. A path E of G with all its inner vertices in $V - (A \bigcup B)$ is called (A, B)-path if E starts at any vertex in A and terminates at any vertex in B. For subgraphs H_1 and H_2 of G, an (H_1, H_2)-path is analogously defined.

Definition 3.3.$\Theta(\overrightarrow{P}, V_{neut}, V_{fin}) = (P_0, \ldots, P_\pi)$; $P_i = y_i \overrightarrow{P_i} z_i$ $(i = 0, \ldots, \pi)$.

Let $V' \subset V$. A path with endvertices in $V - V'$ and all internal vertices in V' is called a V'−path. Let $\overrightarrow{P} = v_0 v_1 \ldots v_n$ be a path in G of length $n \geq 1$ and let V_{neut}, V_{fin} be vertex-disjoint subsets in $V - V(\overrightarrow{P})$. We define $\Theta(\overrightarrow{P}, V_{neut}, V_{fin})$ as a sequence of paths $P_0, \ldots, P_\pi$ as follows: For $i = 0$, put $P_0 = \overrightarrow{y_0 z_0}$ and $X = V(v_0 \overrightarrow{P} z_0)$, where $y_0 = v_0$ and $z_0 = v_1$. Now let $P_{i-1} = y_{i-1} \overrightarrow{P}_{i-1} z_{i-1}$ and X_{i-1} are defined for some integer $i \geq 1$. In order to define P_i and X_i we distinguish three cases.

(i) If every V_{neut}−path, starting in $X_{i-1} - z_{i-1}$, terminates in X_{i-1} then $X_i = \emptyset$ and $P_\pi = P_{i-1}$ (so P_i is undefined).

(ii) If there is a V_{neut}−path $P' = v' \overrightarrow{P'} v''$ with $v' \in X_{i-1} - z_{i-1}$ and $v'' \in V_{fin}$ then $X_i = \emptyset$ and $P_\pi = P_i = y_i \overrightarrow{P'} z_i$ where $y_i = v'$ and $z_i = v''$.

(iii) There is a V_{neut}−path $P'' = w' \overrightarrow{P''} w''$ with $w' \in X_{i-1} - z_{i-1}$ and $w'' \in V(z_{i-1}^+ \overrightarrow{P} v_n)$ but there is no V_{neut}−path satisfying (ii).

Choose P'' so as to maximize $|\, v_0 \overrightarrow{P} w'' \,|$. Then putting

$$P_i = y_i \overrightarrow{P''} z_i, \; X_i = V(v_0 \overrightarrow{P} z_i),$$

where $y_i = w'$ and $z_i = w''$, we complete the definition of P_i and X_i. Since $X_o \subset X_1 \subset \cdots$, there must be some integer j $(j \geq 1)$ with $P_j = P_\pi$, which, in fact, completes the definition of $\Theta(\overrightarrow{P}, V_{neut}, V_{fin})$.

Definition 3.4. $\Phi_u; \varphi_u; \Psi_u; \psi_u$.

Let T be a maximal HC-extension. For each $u \in V(H)$, put

$$\Phi_u = N(\hat{u}) \bigcap V(T), \quad \varphi_u = |\Phi_u|,$$

$$\Psi_u = N(\hat{u}) \bigcap V(C), \quad \psi_u = |\Psi_u|.$$

Definition 3.5. $U_0; \overline{U}_0; U_1; U_2; U_*$.

For T a maximal HC−extension, put

$$U_0 = \{u \in V(H) \,|u = \hat{u}\}, \quad \overline{U}_0 = V(H) - U_0, \quad U_1 = \{u \in \overline{U}_0 \,|\Phi_u \not\subseteq V(T(u))\}.$$

Let $u \in V(H) - (U_0 \bigcup U_1)$ and let $\Theta(\overleftarrow{T}(u), V_{neut}, V_{fin}) = (P_0, \ldots, P_\pi)$, where

$$V_{neut} = V - (V(T) \bigcup V(C)), \quad V_{fin} = V(T) - V(T(u)).$$

A vertex u is called to be special if P_π starts and terminates in $V(T(u))$. The set of all nonspecial vertices in $V(H)-(U_0\bigcup U_1)$ is denoted by U_2 and the set of all special vertices by U_*.

Definition 3.6. $B_u; B_u^*; b_u; b_u^*$.

Let T be a maximal HC-extension. For each $u\in V(H)$, put $B_u=\{v\in U_0 \mid v\,\mathring{u}\in E\}$. Clearly $B_u=\emptyset$ if $u\in U_0$. Furthermore, for each $u\in U_0$, put $B_u^*=\{v\in V(H)\mid u\,\mathring{v}\in E$. Clearly $B_u^*\subseteq\overline{U}_0$. Let $b_u=|B_u|$ and $b_u^*=|B_u^*|$.

Definition 3.7. $A_u(v);\rho_u(v);\overline{\rho}_u(v);\Lambda_u;\Lambda_u(v,w)$.

Let T be a maximal HC-extension. For each $u,v\in V(H)$, put

$$A_u(v)=(\Phi_u\bigcup B_u)\bigcap V(T(v)).$$

Let $\rho_u(v)$ denote the vertex in $A_u(v)$ maximizing $|\,v\overrightarrow{T}(v)\rho_u(v)\,|$. In particular, $\rho_u(u)=\hat{u}^-$. Put $\overline{\rho}_u(v)=\hat{u}$ if $\rho_u(v)\in\Phi_u$ and $\overline{\rho}_u(v)=\mathring{u}$ if $\rho_u(v)\in B_u$. Clearly $\overline{\rho}_u(u)=\hat{u}$. Put $\Lambda_u=\{v\in V(H)\,|A_u(v)\neq\emptyset\}$. For each $v,w\in\Lambda_u\quad(v\neq w)$, put

$$\Lambda_u(v,w)=vT(v)\rho_u(v)\overline{\rho}_u(v)T(u)\overline{\rho}_u(w)\rho_u(w)T(w)w.$$

Definition 3.8. $\varphi_u';\gamma_u;\beta_u;\mu(T)$.

For T a maximal HC–extension, put

$$\varphi_u'=\begin{cases}\varphi_u & \text{if}\quad u\in V(H)-U_*,\\ 0 & \text{if}\quad u\in U_*,\end{cases}\qquad \gamma_u=\begin{cases}\varphi_u'+b_u & \text{if}\quad u\in\overline{U}_0,\\ \varphi_u'-b_u^* & \text{if}\quad u\in U_0,\end{cases}$$

$$\beta_u=\frac{(\gamma_u+\gamma_{u^+})}{2}\quad(u\in V(H)),\qquad \mu(T)=\frac{1}{h}\sum_{u\in V(H)}\beta_u.$$

Definition 3.9. T–transformation; $T_{tr}(E_1,...,E_n);T_{tr}(v_1,...,v_n)$.

Let T be a maximal HC–extension and let $E_1,...,E_n$ are vertex-disjoint (H,C)–paths with $E_i=v_i\overrightarrow{E}_iw_i\quad(i=1,...,n)$. Assume that the union of $E_1,...,E_n$ intersect $T(z)$ for some $z\in V(H)-\{v_1,...,v_n\}$. Clearly $z\in\overline{U}_0$. walking along $T(z)$ from z to $\hat{z}$ we stop at the first vertex $w\in\bigcup_{i=1}^n V(E_i)$. Assume w.l.o.g. that $w\in V(E_1)$. Replacing the segment v_1E_1w of a path E_1 by $zT(z)w$ we get a new path E_1^0 instead of E_1. If the union of $E_1^0,E_2,...,E_n$ intersect $T(z')$ for some $z'\in V(H)-\{z,v_2,...,v_n\}$ then continue this procedure. In a finite number of steps we obtain

$$|\,\{v\in V(H)\mid(\bigcup_{i=1}^n V(E_i'))\bigcap V(T(v))\neq\emptyset\}\,|=n$$

for some vertex-disjoint (H,C)-paths $E_1',...,E_n'$. Let $E_i'=v_i'E_i'w_i\quad(i=1,...,n)$. Writing

$$T_{tr}\left(E_1,...,E_n\right)=\left(E_1',...,E_n'\right),\quad T_{tr}\left(v_1,...,v_n\right)=\left(v_1',...,v_n'\right),$$

we say that $E_1',...,E_n'$ is a T– transformation of $E_1,...,E_n$. By the definition,

$$v_i'\in\{v_i\}\bigcup\overline{U}_0\quad(i=1,...,n),\quad T_{tr}\left(w_1,...,w_n\right)=\left(w_1,...,w_n\right).$$

Definition 3.10. $O(x,y);O_x(x,y);O(y,\mathring{x});O(x,\mathring{x});O_y(x,y);O(x,\mathring{y});O(y,\mathring{y})$.

Let T be a maximal HC–extension. For each pair of distinct vertices $x,y\in V(H)$, put

$$V_1=\bigcup_{v\notin\{x,y\}}V\left(T(v)\right)\bigcup\{x,y\},\quad V_2=V_1\bigcup\{\mathring{x}\}.$$

Let $O(x,y)$ (resp. $O_z(x,y), O(y,\mathring{z}), O(x,\mathring{z})$) be the longest (x,y)−path (resp. (x,y)-path, $(y,\mathring{z})$−path, $(x,\mathring{z})$−path) in $<V_1>$ (resp. $<V_2>,<V_2>,<V_2>$). The paths $O_y(x,y)$, $O(x,\mathring{y})$, and $O(y,\mathring{y})$ are analogously defined.

Definition 3.11. $\Omega(x,y);\Omega(x,y,E,F);\Omega(v,w,x,y,E,F)$.

Let T be a maximal HC−extension and let E,F be a pair of vertex disjoint T−transformed (H,C)−paths with $E=xEv$ and $F=yFw$. If $|T(x)|-1\neq 1$ then we denote $\Omega_x(x,y,E,F)=O(x,y)$. Otherwise,

$$\Omega_x(x,y,E,F)=\begin{cases} O_z(x,y) & \text{if } \mathring{z}\notin V(E)\cup V(F),\\ O(\mathring{z},y) & \text{if } \mathring{z}\in V(E),\\ O(\mathring{z},x) & \text{if } \mathring{z}\in V(F).\end{cases}$$

Defining $\Omega_y(x,y,E,F)$ analogously, we denote by $\Omega(x,y,E,F)$ the longest path among $O(x,y),\Omega_x(x,y,E,F)$ and $\Omega_y(x,y,E,F)$. Let $\Omega(x,y)$ be the shortest path $\Omega(x,y,E,F)$ for fixed x,y and all possible E,F. By definition 3.9, $vE\mu\Omega(x,y,E,F)\nu Fw$ is a simple path for appropriate $\mu,\nu\in\{x,y,\mathring{x},\mathring{y}\}$ and will be denoted by $\Omega(v,w,x,y,E,F)$.

Definition 3.12. $(v,L)\in\Delta$.

Let L be a path of G with $L=v_1...v_{2t-1}$ $(t\geq 1)$ and let $v\in V-V(L)$. We will write $(v,L)\in\Delta$ if $vv_{2i-1}\in E$ $(i=1,...,t)$. If $w\in V(L)$ then we will write $(w,L)\in\Delta$ if $wu\in E$ for each $u\in V(L)-w$.

Remarks. If no ambiguity can arise, any notation of the type R_{u_i} in definitions 2.4 and 2.6-2.8, having index u_i (say Φ_{u_i}), we abbreviate $R_{u_i}=R_i$.

4 Preliminaries

Throughout in this section we let C be a longest cycle of a graph G and $H=u_1...u_hu_1$ a longest cycle of $G-C$ with a maximal HC−extension T.

Lemma 1. Let G be a graph.

(a1) Let E,F be a pair of vertex-disjoint (H,C)−paths with $E=xEv$ and $F=yFw$. If $T_{tr}(E,F)=(E',F')$ and $T_{tr}(x,y)=(x',y')$ then

$$|v\overrightarrow{C}w|-1\geq|\Omega(v,w,x',y',E',F')|-1\geq a(x')+a(y')+|\Omega(x',y')|-1,$$

where $a(z)=1$ if $z\notin U_*$ and $a(z)=\varphi_z+1$ if $z\in U_*$ for each $z\in\{x',y'\}$.

(a2) Let $u\in V(H)$ and $\Theta(\overleftarrow{T}(u),V_{neut},V_{fin})=(P_0,...,P_\pi)$, where $P_i=y_i\overrightarrow{P}_iz_i$ $(i=0,...,\pi)$ and

$$V_{neut}=V-(V(T)\cup V(C)),\quad V_{fin}=V(T)-V(T(u)).$$

If $u\in U_2$ then there is an (u,z_π)−path L of length at least φ_u+1 with $V(L)\subseteq V(T(u))\cup V^*$, where $V^*=\bigcup_{i=0}^{\pi}V(P_i)$. If $u\in U_*$ then for each vertex

$$z\in(V(\hat{u}\overleftarrow{T}(u)z_\pi)\cup V^*)-z_\pi$$

there is an (u,z)−path L of length at least φ_u+1 with $V(L)\subseteq V(T(u))\cup V^*$.

Lemma 2. For each $u\in V(H)$,

(b1) $u\in\overline{U}_0,\hat{u}\neq\mathring{u}\Longrightarrow\Phi_u\cap B_u=\emptyset$.

(b2) $\sum_{u\notin U_0}b_u=\sum_{u\in U_0}b_u^*,\ \sum_{u\in V(H)}\gamma_u=\sum_{u\in V(H)}\varphi_u',|\Phi_u\cup B_u|=\sum_{v\in V(H)}|A_u(v)|$.

Lemma 3. Let C be a longest cycle of a graph G, Q be a path in $G - C$ and $P_i = v_i\overrightarrow{P_i}w_i$ $(i = 0, ..., q)$ are vertex-disjoint paths in $G - C$ having only $v_0, ..., v_q$ in common with Q. Then

$$c \geq \sum_{i=0}^{q} |Z_i| + |\bigcup_{i=0}^{q} Z_i|,$$

where $Z_i = N(w_i) \cap V(C)$ $(i = 0, ..., q)$.

Lemma 4. For each $u \in V(H)$,

(d1) $|T(u)| - 1 \geq 2 \implies h \geq 2\gamma_u$.

(d2) $|T(u)| - 1 = 1 \implies h \geq 2\varphi'_u \geq \gamma_u + 1$.

(d3) $h \geq \gamma_u + 1$.

Lemma 5. Let $\Lambda_u \subseteq V(x\overrightarrow{H}y)$ for some $u, x, y \in V(H)$.

(e1) $|T(u)| - 1 \geq 2 \implies |x\overrightarrow{H}y| - 1 \geq \gamma_u$.

(e2) $|T(u)| - 1 = 1 \implies |x\overrightarrow{H}y| - 1 \geq \gamma_u - 1$.

(e3) $|T(u)| - 1 = 1, |x\overrightarrow{H}y| - 1 = \gamma_u - 1 \implies (\hat{u}, x\overrightarrow{H}y) \in \Delta, B_u = \Lambda_u - u \subseteq U_0, \gamma_u - 1 = 2(\varphi_u - 1)$.

Lemma 6. For each $u \in U_1 \cup U_2$ let $x_1\overrightarrow{H}y_1$ and $x_2\overrightarrow{H}y_2$ be vertex-disjoint segments in H with $\{x_1, x_2, y_1, y_2\} \subseteq \Lambda_u \subseteq V(x_1\overrightarrow{H}y_1) \cup V(x_2\overrightarrow{H}y_2)$ and let $v \in \{x_2, y_2\}$.

(f1) If $B_u \cup \{u\} \subseteq V(x_1\overrightarrow{H}y_1)$ and $\Lambda_u - (B_u \cup \{u\}) \subseteq V(x_2\overrightarrow{H}y_2)$ then

$$|x_1\overrightarrow{H}y_1| - 1 + |x_2\overrightarrow{H}y_2| - 1 + |A_u(u)| + |A_u(v)| \geq \gamma_u - 1.$$

Otherwise,

$$|x_1\overrightarrow{H}y_1| - 1 + |x_2\overrightarrow{H}y_2| - 1 + |A_u(u)| + |A_u(v)| \geq \gamma_u - 2 + |A_u(u)| \geq \gamma_u - 1.$$

(f2) If $|x_1\overrightarrow{H}y_1| - 1 + |x_2\overrightarrow{H}y_2| - 1 + |A_u(u)| + |A_u(v)| = \gamma_u - 1$ then

$$(\hat{u}, x_i\overrightarrow{H}y_i) \in \Delta \ (i = 1, 2), \quad B_u = \Lambda_u - u \subseteq U_0, \quad \gamma_u - 1 = 2(\varphi_u - 1).$$

Lemma 7. Let x, y be a pair of distinct vertices of H. For each $u \in V(H)$,

(g1) $u \in U_* \implies |O(x, y)| - 1 \geq \gamma_u + 1$.

(g2) $|T(u)| - 1 \geq 2 \implies |O(x, y)| - 1 \geq \gamma_u$.

(g3) $|T(u)| - 1 = 1 \implies |O_u(x, y)| - 1 \geq \gamma_u - 1$.

(g4) Let $|T(u)| - 1 = 1$ and $|O_u(x, y)| - 1 = \gamma_u - 1$. If either $\Lambda_u \subseteq V(x\overrightarrow{H}y)$ and $(\mathring{u}, H) \notin \Delta$ or $\Lambda_u \subseteq V(y\overrightarrow{H}x)$ and $(\mathring{u}, H) \notin \Delta$ (say $\Lambda_u \subseteq V(x\overrightarrow{H}y)$ and $(\mathring{u}, H) \notin \Delta$) then

(g4.1) $(\mathring{u}, x\overrightarrow{H}y) \in \Delta$,

(g4.2) $B_u = \Lambda_u - u \subseteq U_0, |O_u(x, y)| - 1 = |x\overrightarrow{H}y - 1| = \gamma_u - 1 = 2(\varphi_u - 1)$,

(g4.3) $z \in V(x\overrightarrow{H}y) - \Lambda_u \implies$ either $z \in U_*$ or $z \in U_0, \Lambda_z \subseteq \Lambda_u \cup \{z\}$, $\gamma_z \leq \varphi_u = (\gamma_u + 1)/2$,

(g4.4) $z \in V(x\overrightarrow{H}y) - \{x, y\} \implies \Lambda_z \subseteq V(x\overrightarrow{H}y)$.

Otherwise,

(g4.5) $(\mathring{u}, H) \in \Delta$,

(g4.6) $B_u = \Lambda_u - u \subseteq U_0, |O_u(x, y)| - 1 = h - 2 = \gamma_u - 1 = 2(\varphi_u - 1)$,

(g4.7) $z \in V(H) - \Lambda_u \implies$ either $z \in U_*$ or $z \in U_0, \Lambda_z \subseteq \Lambda_u \cup \{z\}$, $\gamma_z \leq \varphi_u = (\gamma_u + 1)/2 = h/2$.

(g5) $|T(x)|-1=1 \Longrightarrow \min(|O(\mathring{x},x)|-1,|O(\mathring{x},y)|-1)\geq \gamma_x$.

(g6) $u\in\{x^+,x^-,y^+,y^-\}\Longrightarrow |O(x,y)|-1\geq \gamma_u$.

(g7) If $u\in\{x^+,x^-,y^+,y^-\}$ (say $u=x^+$) and $|O(x,y)|-1=\gamma_u$ then $|T(u)|-1\leq 1$ and $(\hat{u},v\overrightarrow{H}y)\in\Delta$ for some $v\in\Lambda_u$ with $\Lambda_u\subseteq V(v\overrightarrow{H}y)$.

(g8) If $|T(x)|-1=1$ and $|O_x(x,y)|-1=|O_x(x,w)|-1=\gamma_x-1$ for some $w\in V(H)-\{x,y\}$ then for each $z\in\{x^+,x^-\}$,

$$\min(|O_z(x,y)|-1,|O_z(x,w)|-1)\geq \gamma_z+1.$$

Lemma 8. Let x,y be a pair of distinct vertices of H and let

$$a=\min(|O_x(x,y)|,|O(\mathring{x},y)|,|O(\mathring{x},x)|)-1,$$
$$b=\min(|O_y(x,y)|,|O(\mathring{y},x)|,|O(\mathring{y},y)|)-1.$$

Then $|\Omega(x,y)|-1\geq \max(|O(x,y)|-1,a,b)$.

Lemma 9. Let x,y be a pair of distinct vertices of H.

(i1) $\{u_i,u_{i+1}\}\cap\{x,y\}=\emptyset\ (i\in\overline{1,h})\Longrightarrow |\Omega(x,y)|-1\geq(\gamma_i+\gamma_{i+1})/2=\beta_i$.

(i2) $|T(x)|-1\geq 2,\ z\in\{x^+,x^-\}\Longrightarrow |\Omega(x,y)|-1\geq(\gamma_x+\gamma_z)/2$.

(i3) $x\in U_*, z\in\{x^+,x^-\}\Longrightarrow |\Omega(x,y)|-1\geq(\gamma_x+\gamma_z+1)/2$.

(i4) If $|T(x)|-1=1$ then for each $w\in V(H)-\{x,y\}$ and $z\in\{x^+,x^-\}$,

$$\max(|\Omega(x,y)|-1,|\Omega(x,w)|-1)\geq(\gamma_x+\gamma_z)/2.$$

(i5) $z\in\{x^+,x^-\},w\in V(H)-z\Longrightarrow \max(|\Omega(x,y)|-1,\ |\Omega(z,w)|-1)\geq(\gamma_x+\gamma_z)/2$.

(i6) If $x\in\overline{U}_0$ and $h\neq 4$ then $|\Omega(x,y)|-1\geq(\gamma_x+\gamma_z)/2$ for some $z\in\{x^+,x^-\}$.

(i7) $x,y\in\overline{U}_0\Longrightarrow |\Omega(x,y)|-1\geq\max_i\beta_i$.

(i8) $|x\overrightarrow{H}y|-1=1\Longrightarrow |\Omega(x,y)|-1\geq\max_i\beta_i$.

(i9) $|x\overrightarrow{H}y|-1=2,\ h\neq 4\Longrightarrow |\Omega(x,y)|-1\geq(\gamma_x+\gamma_{x^+})/2$.

5 Proofs

Proof of lemma 1. (a1) Following definition 3.11, we distinguish three cases.

Case 1. $x,y\notin U_*$.

Clearly,

$$|v\overrightarrow{C}w|-1\geq|\Omega(v,w,x,y,E,F)|-1\geq 2+|\Omega(x,y,E,F)|-1\geq|\Omega(x,y)|+1.$$

Case 2. $x,y\in U_*$.

If $|T(x)|-1=1$ then $\mathring{x}\notin V(\Omega(x,y))$, since otherwise the segment of $\Omega(x,y)$ between $\mathring{x}$ and y, contradict the fact that $x\in U_*$. Therefore, $\Omega(x,y,E,F)=O(x,y)$. On the other hand, $\Omega_x(x,y,E,F)=O(x,y)$ if $|T(x)|-1\geq 2$. Also by the symmetric arguments, $\Omega_y(x,y,E,F)=O(x,y)$. Thus $\Omega(x,y,E,F)=\Omega(x,y)$ and

$$|v\overrightarrow{C}w|-1\geq|\Omega(v,w,x,y,E,F)|-1$$
$$\geq(|E|-1)+(|F|-1)+|\Omega(x,y)|-1\geq\varphi_x+\varphi_y+|\Omega(x,y)|+1.$$

Case 3. Either $x\notin U_*, y\in U_*$ or $x\in U_*, y\notin U_*$.

Apply the arguments in case 1 and case 2.□

(a2) Suppose first that $u \in U_2$. By definition 3.3, $z_1 \in V(T(u))$ and $z_\pi \in V_{fin}$. Let $z_\pi \in V(T(w))$ for some $w \in V(H) - u$. Choose $z_{01} \in V(u\overrightarrow{T}(u)y_2^-)$ such that $z_{01}\hat{u} \in E$ and $| z_{01}\overrightarrow{T}(u)y_2 |$ is minimum. Then we get the desired result putting together the following paths

$$P_2, ..., P_\pi, \hat{u}z_{01}, \hat{u}\overleftarrow{T}(u)y_2, z_{01}\overleftarrow{T}(u)y_3, z_{\pi-1}\overleftarrow{T}(u)u, z_i\overleftarrow{T}(u)y_{i+2} \ (i = 2, ..., \pi - 2).$$

A similar proof holds for $u \in U_*$.□

Proof of lemma 2. (b1) Case 1. $u \in U_1$.

Suppose, to the contrary, that $\Phi_u \cap B_u \neq \emptyset$ and let $z \in \Phi_u \cap B_u$. Then, by definitions 3.4 and 3.1, the collection

$$\{T(u_1), ..., T(u_h), u\mathring{u}, z\hat{u}\} - \{T(u), T(z)\}$$

generates another $HC-$extension, contradicting the maximality of T.

Case 2. $u \in U_2 \cup U_*$.

By definition 3.5, $\Phi_u \subseteq V(T(u))$ and the result follows.□

(b2) Immediately from definitions 3.6-3.8.□

Proof of lemma 3. Assume first that $v_i = w_i \ (i = 0, ..., q)$. The result is immediate if $\bigcup_{i=0}^q Z_i = \emptyset$. Let $\bigcup_{i=0}^q Z_i \neq \emptyset$ and let $\xi_1, ..., \xi_m \ (m \geq 1)$ be the elements of $\bigcup_{i=0}^q Z_i$ occuring on $\overrightarrow{C}$ in consequtive order. Set

$$F_i = N(\xi_i) \cap \{w_0, ..., w_q\} \quad (i = 1, ..., m).$$

Suppose that $m = 1$. If $|F_1| = 1$ then $q = 0$ and $Z_0 = Z_q = \{\xi_1\}$ implying that

$$c \geq 2 = \sum_{i=0}^q |Z_i| + |\bigcup_{i=0}^q Z_i|.$$

If $|F_1| \geq 2$ then choosing $u, v \in F_1 \ (u \neq v)$ such that $| u\overrightarrow{Q}v |$ is maximum,

$$c \geq | \xi_1 u\overrightarrow{Q}v\xi_1 | \geq \sum_{i=0}^q |Z_i| + 1 = \sum_{i=0}^q |Z_i| + |\bigcup_{i=0}^q Z_i|.$$

Thus, we may assume $m \geq 2$. It means, in particular, that $c \geq 3$. For $i = 1, ..., m$, put $f(\xi_i) = | \xi_i\overrightarrow{C}\xi_{i+1} | -1$ (indices mod m). It is easy to see that

$$c = \sum_{i=1}^m f(\xi_i), \quad \sum_{i=1}^m |F_i| = \sum_{i=0}^q |Z_i|, \quad m = |\bigcup_{i=0}^q Z_i|. \qquad (1)$$

For every $i \in \overline{1, m}$ choose $x_i, y_i \in F_i \cup F_{i+1}$ such that $| x_i\overrightarrow{Q}y_i |$ is maximum (indices mod m).

Claim 3.1 $f(\xi_i) \geq (|F_i| + |F_{i+1}| + 2)/2 \quad (i = 1, ..., m).$

Proof of claim 3.1. Case 1. Either $x_i \in F_i$, $y_i \in F_{i+1}$ or $x_i \in F_{i+1}$, $y_i \in F_i$.

If $x_i \in F_i$, $y_i \in F_{i+1}$ then $f(\xi_i) \geq | \xi_i x_i\overrightarrow{Q}y_i\xi_{i+1} | -1$ and hence

$$f(\xi_i) \geq \max(|F_i|, |F_{i+1}|) + 1 \geq (|F_i| + |F_{i+1}| + 2)/2.$$

Otherwise, the result holds from $f(\xi_i) \geq |\xi_i y_i \overleftarrow{Q} x_i \xi_{i+1}| - 1$ in the same way.

Case 2. Either $x_i,\ y_i \in F_i$ or $x_i,\ y_i \in F_{i+1}$.

First suppose $x_i,\ y_i \in F_i$. We can assume also $x_i,\ y_i \notin F_{i+1}$, since otherwise we could argue as in case 1. Choose $x_i',\ y_i' \in F_{i+1}$ such that $|x_i' \overrightarrow{Q} y_i'|$ is maximum. If $|x_i \overrightarrow{Q} x_i'| - 1 \geq (|F_i| - |F_{i+1}|)/2$ then

$$f(\xi_i) \geq |\xi_i x_i \overrightarrow{Q} y_i' \xi_{i+1}| - 1 \geq (|F_i| - |F_{i+1}|)/2 + |F_{i+1}| + 1 \geq (|F_i| + |F_{i+1}| + 2)/2.$$

Otherwise,

$$\begin{aligned} f(\xi_i) &\geq |\xi_i y_i \overleftarrow{Q} x_i' \xi_{i+1}| - 1 = |x_i' \overrightarrow{Q} y_i| + 1 = |x_i \overrightarrow{Q} y_i| - |x_i \overrightarrow{Q} x_i'| + 2 \geq \\ &\geq |F_i| - (|F_i| - |F_{i+1}| - 1)/2 + 2 \geq (|F_i| + |F_{i+1}| + 2)/2. \end{aligned}$$

By symmetry, the case $x_i,\ y_i \in F_{i+1}$ requires the same arguments.□

By claim 3.1,

$$\sum_{i=1}^{m} f(\xi_i) \geq \sum_{i=1}^{m} (|F_i| + |F_{i+1}| + 2)/2 = \sum_{i=1}^{m} |F_i| + m,$$

which by (1) gives the desired result. Finally, if $v_i \neq w_i$ for some $i \in \overline{0,q}$ then we could argue exactly as in case $v_i = w_i \quad (i = 0, ..., q)$.□

Proof of lemma 4. **(d1)** **Case1.** $u \in U_1$.

Let $\Lambda_u = \{\xi_1, ..., \xi_f\}$. Assume w.l.o.g that $u = \xi_1$ and $\xi_1, ..., \xi_f$ occurs on H in consequtive order. For each integer i $(1 \leq i \leq f)$ let

$$M_i = \xi_i \overrightarrow{H} \xi_{i+1}, \ \ \omega_i = |A_u(\xi_i)| + |A_u(\xi_{i+1})| \quad \text{(indices mod } f).$$

Since H is extreme,

$$|M_i| \geq |\Lambda_u(\xi_i, \xi_{i+1})| \quad (i = 1, ..., f). \tag{2}$$

Let $\xi_r \overrightarrow{H} \xi_s$ be the longest segment on H with

$$\xi_1 \in V(\xi_r \overrightarrow{H} \xi_s), \ \ \{\xi_r, \xi_{r+1}, ..., \xi_s\} \subseteq B_u \cup \{u\}.$$

Put

$$\begin{aligned} \Omega^+ &= \{M_i \in \{M_2, ..., M_{f-1}\} \,|\, \overline{p}_u(\xi_i) \neq \overline{p}_u(\xi_{i+1})\}, \\ \Omega^- &= \{M_i \in \{M_1, M_f\} \,|\, \overline{p}_u(\xi_i) \neq \overline{p}_u(\xi_{i+1})\}, \\ \Omega^0 &= \{M_1, ..., M_f\} - (\Omega^+ \cup \Omega^-). \end{aligned}$$

Observe that $|\Omega^-| \leq 2$ and $|M_i| - 1 \geq |\Lambda_u(\xi_i, \xi_{i+1})| - 1$ for each $i \in \overline{1, f}$. Then clearly

$$M_i \in \Omega^+ \implies |M_i| - 1 \geq \omega_i + |A_u(u)| - 1, \tag{3}$$

$$M_i \in \Omega^- \implies |M_i| - 1 \geq \omega_i - |A_u(u)| + 1, \tag{4}$$

$$M_i \in \Omega^0 \implies |M_i| - 1 \geq \omega_i. \tag{5}$$

Claim 4.1. If $|\Omega^-| = 0$ then $|M_i| - 1 \geq \omega_i \ (i = 1, ..., f)$.

Proof of claim 4.1. Immediate from (3), (4) and (5).□

Claim 4.2. **(k1)** If $|\Omega^-| = 1$, say $\Omega^- = \{M_1\}$, then $M_s \in \Omega^+$.

(k2) If $\Omega^- = \{M_1\}$ and $\Omega^+ = \{M_s\}$ then

$$B_u \cup \{u\} \subseteq V(\xi_r \overrightarrow{H} \xi_s), \ \ \Lambda_u - (B_u \cup \{u\}) \subseteq V(\xi_{s+1} \overrightarrow{H} \xi_{r-1}).$$

Proof of claim 4.2. (k1) Let $\Omega^- = \{M_1\}$. By the definition, $\{\xi_2, ..., \xi_s\} \subseteq B_u$ and $\xi_{s+1} \in \Lambda_u - (B_u \cup \{u\})$, implying that $M_s \in \Omega^+$.□

(k2) If follows that $\{\xi_{s+1}, ..., \xi_f\} \subseteq \Lambda_u - (B_u \cup \{u\})$. On the other hand (by the definition), $\{\xi_1, ..., \xi_s\} \subseteq B_u \cup \{u\}$ and the proof is complete.□

Claim 4.3. (l1) If $|\Omega^-| = 2$, i.e.$\Omega^- = \{M_1, M_f\}$, then $M_s, M_{r-1} \in \Omega^+$.

(l2) If $\Omega^- = \{M_1, M_f\}$ and $\Omega^+ = \{M_s, M_{r-1}\}$ then ξ_1, ξ_r, ξ_s are pairwise different and

$$B_u \cup \{u\} \subseteq V(\xi_r \overrightarrow{H} \xi_s), \;\; \Lambda_u - (B_u \cup \{u\}) \subseteq V(\xi_{s+1} \overrightarrow{H} \xi_{r-1}).$$

Proof of claim 4.3. (l1) By the definition, $\{\xi_2, \xi_f, \xi_s, \xi_r\} \subseteq B_u$ and $\xi_{s+1}, \xi_{r-1} \in \Lambda_u - (B_u \cup \{u\})$, which implies $M_s, M_{r-1} \in \Omega^+$.□

(l2) It follows that $\{M_{s+1}, ..., M_{r-2}\} \cap \Omega^+ = \emptyset$ and hence

$$\{\xi_{s+1}, ..., \xi_{r-1}\} \subseteq \Lambda_u - (B_u \cup \{u\}).$$

On the other hand (by the definition) $\{\xi_r, ..., \xi_s\} \subseteq B_u \cup \{u\}$, which completes the proof of claim 4.3.□

The following three results can be obtained easely from (3), (4), (5) and claims 4.1, 4.2 and 4.3.

Claim 4.4. $\sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i$.

Claim 4.5. $t \in \{1, ..., f\} \Longrightarrow \sum_{i \neq t} (|M_i| - 1) \geq \sum_{i \neq t} \omega_i - |A_u(u)| + 1$.

Claim 4.6. $g, t \in \{1, ..., f\}$ $(g \neq t) \Longrightarrow \sum_{i \notin \{g,t\}} (|M_i| - 1) \geq \sum_{i \notin \{g,t\}} \omega_i - 2|A_u(u)| + 2$.

Using ($b2$) and claim 4.4,

$$h = \sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i = \sum_{i=1}^{f} (|A_u(\xi_i)| + |A_u(\xi_{i+1})|)$$
$$= 2 \sum_{i=1}^{f} |A_u(\xi_i)| = 2|\Phi_u \cup B_u|.$$

By ($b1$), $|\Phi_u \cup B_u| = \varphi_u + b_u = \gamma_u$, implying that $h \geq 2\gamma_u$.

Case 2. $u \in U_2$.

Let $\Theta(\overleftarrow{T}(u), V_{neut}, V_{fin}) = (P_0, ..., P_\pi)$, where $P_i = y_i \overrightarrow{P_i} z_i$ $(i = 0, ..., \pi)$. By ($a2$), there is an (u, z_π)-path L of length at least $\varphi_u + 1$ with $V(L) \subseteq V(T(u)) \cup V^*$. Let $z_\pi \in V(T(w))$ for some $w \in V(H)$. Denoting $B_u \cup \{u, w\} = \{\xi_1, ..., \xi_f\}$ we can argue exactly as in case 1.

Case 3. $u \in U_*$.

Clearly $h \geq 2(b_u + 1) = 2(\varphi'_u + b_u + 1) > 2\gamma_u$.□

(d2) Since $|T(u)| - 1 = 1$, we have $u \in U_1 \cup U_*$.If $u \in U_*$ then $b_u = 0$ and $h \geq 2 = 2(\varphi'_u + b_u + 1) = 2(\gamma_u + 1) \geq \gamma_u + 1$. Let $u \in U_1$. Define

$$\xi_i, \omega_i, M_i, \xi_r \overrightarrow{H} \xi_s, \Omega^+, \Omega^-, \Omega^0 \tag{6}$$

as in proof of ($d1$). It is easy to see that $\Omega^+ = \Omega^- = \emptyset$. By claim 4.1, $\sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i$ and as in proof of ($d1$), $h \geq 2|\Phi_u \cup B_u| = 2\varphi_u$. Noting that $\varphi_u \geq b_u + |\{u\}| = b_u + 1$, we obtain $h \geq \varphi_u + b_u + 1 = \gamma_u + 1$.□

(d3) It is easely checked that $h \geq \gamma_u + 1$ if $u \in U_0$. If $u \in \overline{U}_0$ then by ($d1$) and ($d2$),$h \geq \min(2\gamma_u, \gamma_u + 1) \geq \gamma_u + 1$.□

Proof of lemma 5. Assume w.l.o.g. that $x, y \in \Lambda_u$.

(e1) **Case 1** $u \in U_1$.

Following (6) we let, in addition, $y\overrightarrow{H}x = M_t$ for some t $(1 \le t \le f)$. By claim 4.5 we can distingush the following two cases:

Case 1.1. $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$.

By $(b1)$, $|\Phi_u \cup B_u| = |\Phi_u| + |B_u| = \gamma_u$, and using $(b2)$,

$$|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i = \sum_{i=1}^{f}\omega_i - \omega_t = 2\sum_{i=1}^{f}|A_u(\xi_i)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})|$$
$$= \sum_{i=1}^{f}|A_u(\xi_i)| + \sum_{i\notin\{t,t+1\}}|A_u(\xi_i)| \ge \sum_{i=1}^{f}|A_u(\xi_i)| = |\Phi_u \cup B_u| = \gamma_u.$$

Case 1.2. $\sum_{i\ne t}\omega_i - |A_u(u)| + 1 \le \left|x\overrightarrow{H}y - 1\right| < \sum_{i\ne t}\omega_i$.

If $\Omega^- = \emptyset$ then by claim 4.1., $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$, a contradition. Let $\Omega^- \ne \emptyset$.

Case 1.2.1. $|\Omega^-| = 1$.

Assume w.l.o.g $\Omega^- = \{M_1\}$. By claim 4.2, $M_s \in \Omega^+$. If $|\Omega^+| \ge 2$ then by (3), (4) and (5), $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$,a contradiction. Thus we can assume $\Omega^+ = \{M_s\}$. If $M_t \ne M_s$ then again $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$, a contradiction. Finally, if $M_t = M_s$ then $A_u(\xi_t)$, $A_u(\xi_{t+1})$ and $A_u(u)$ are pairwise different and hence

$$|x\overrightarrow{H}y| - 1 \ge 2\sum_{i=1}^{f}|A_u(\xi_i)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| - |A_u(u)| + 1 =$$
$$\sum_{i=1}^{f}|A_u(\xi_i)| + \sum_{i\notin\{1,t,t+1\}}|A_u(\xi_i)| + 1 \ge \sum_{i=1}^{f}|A_u(\xi_i)| + (f-3) + 1$$
$$\ge \sum_{i=1}^{f}|A_u(\xi_i)| = |\Phi_u \cup B_u| = \gamma_u.$$

Case 1.2.2. $|\Omega^-| = 2$.

By claim 4.3, $M_s, M_{r-1} \in \Omega^+$. If $|\Omega^+| \ge 3$ then by (3), (4) and (5), $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$, a contradiction. Let $\Omega^+ = \{M_s, M_{r-1}\}$. If $M_t \notin \Omega^+$ then again $|x\overrightarrow{H}y| - 1 \ge \sum_{i\ne t}\omega_i$, a contradiction. Finally, if $M_t \in \Omega^+$, say $M_t = M_s$, then $A_u(\xi_t)$, $A_u(\xi_{t+1})$, $A_u(u)$ are pairwise different and we could argue exactly as in case 1.2.1.

Case 2. $u \in U_2 \cup U_*$.

Apply the arguments used in the proof of $(d1)$ (case 2 and case 3).

(e2) Clearly $u \in U_1$. Following (6) we see that $\Omega^+ = \Omega^- = \emptyset$. By claim 4.1, $|M_i| - 1 \ge \omega_i$ $(i = 1, ..., f)$. Recalling that $f \ge b_u + 1$,

$$|x\overrightarrow{H}y| - 1 = \sum_{i\ne t}(|M_i| - 1) \ge \sum_{i\ne t}\omega_i = \sum_{i=1}^{f}|A_u(\xi_i)| + \sum_{i\notin\{t,t+1\}}|A_u(\xi_i)|$$
$$\ge |\Phi_u \cup B_u| + f - 2 = \varphi_u + f - 2 = \varphi_u + b_u - 1 = \gamma_u - 1.$$

(e3) It was shown in $(e2)$ that $|x\overrightarrow{H}y| - 1 \ge \varphi_u + f - 2 \ge \gamma_u - 1$. Since $|x\overrightarrow{H}y| - 1 = \gamma_u - 1$, we have $|x\overrightarrow{H}y| - 1 = \varphi_u + f - 2 = \gamma_u - 1$. This implies $|B_u| = b_u = f - 1$ and therefore $B_u = \Lambda_u - u \subseteq U_0$. But then $\varphi_u = b_u + 1$ and $|x\overrightarrow{H}y| - 1 = \gamma_u - 1 = 2\varphi_u - 2$ implying that $(\hat{u}, x\overrightarrow{H}y) \in \Delta$.□

Proof of lemma 6. (f1) **Case 1.** $u \in U_1$.

By symmetry, we can assume $v = x_2$. Following (6) we let, in addition, $M_g = y_1\overrightarrow{H}x_2$ and $M_t = y_2\overrightarrow{H}x_1$ for some integers $g, t \in \{1, ..., f\}$. This means that

$$y_1 = \xi_g,\ x_2 = \xi_{g+1},\ y_2 = \xi_t,\ x_1 = \xi_{t+1},\ v = x_2 = \xi_{g+1},\ A_u(\xi_{g+1}) = A_u(v).$$

Putting $\beta = | x_1\overrightarrow{H}y_1 | -1 + | x_2\overrightarrow{H}y_2 | -1$ and using claim 4.6, we can distingush the following four cases.

Case 1.1. $\beta \geq \sum_{i\notin\{g,t\}} \omega_i + | A_u(u) | -1$.

Clearly

$$\beta \geq 2\sum_{i=1}^{f} |A_u(\xi_i)| - |A_u(\xi_g)| - |A_u(\xi_{g+1})| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| + |A_u(u)| - 1. \tag{7}$$

Observe that $|A_u(u)| \geq 1$ and $f \geq b_u + 1$. If $x_1 \neq y_1$ then $A_u(\xi_g)$, $A_u(\xi_t)$, $A_u(\xi_{t+1})$ are pairwise different and by (7),

$$\beta + |A_u(v)| = \beta + |A_u(\xi_{g+1})| \geq \sum_{i=1}^{f} |A_u(\xi_i)| + \sum_{i\notin\{g,t,t+1\}} |A_u(\xi_i)| \geq \sum_{i=1}^{f} |A_u(\xi_i)| + f - 3 \geq |\Phi_u \cup B_u| + f - 3 \geq |\Phi_u| + f - 3 \geq \varphi_u + b_u - 2 \geq \gamma_u - 1 - |A_u(u)|.$$

Otherwise $(x_1 = y_1)$, $A_u(\xi_{t+1}) = A_u(u)$, and by (7),

$$\beta + |A_u(v)| = \beta + |A_u(\xi_{g+1})| \geq 2\sum_{i=1}^{f} |A_u(\xi_i)| - |A_u(\xi_g)| - |A_u(\xi_t)| - 1 \geq \sum_{i=1}^{f} |A_u(\xi_i)| + \sum_{i\notin\{g,t\}} |A_u(\xi_i)| - 1 \geq |\Phi_u \cup B_u| + f - 3 \geq \varphi_u + b_u - 2 \geq \gamma_u - 1 - |A_u(u)|.$$

Case 1.2. $\sum_{i\notin\{g,t\}} \omega_i \leq \beta < \sum_{i\notin\{g,t\}} \omega_i + |A_u(u)| - 1$.

Clearly

$$\beta + |A_u(v)| = \beta + |A_u(\xi_{g+1})| \geq 2\sum_{i=1}^{f} |A_u(\xi_i)| - |A_u(\xi_g)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})|.$$

If $x_1 \neq y_1$ then we obtain the desired result as in case 1.1. Let $x_1 = y_1$, i.e. $M_g = M_1$, $M_t = M_f$ and $\xi_{t+1} = \xi_g = \xi_1 = u$. If $\Omega^+ \neq \emptyset$ then $\beta \geq \sum_{i\notin\{1,t\}} \omega_i + |A_u(u)| - 1$, a contradiction. Let $\Omega^+ = \emptyset$. This implies $M_s = M_1$ and $M_{r-1} = M_f$ and we deduce that

$$B_u \cup \{u\} = \{u\} = V(x_1\overrightarrow{H}y_1), \quad \Lambda_u - u \subseteq V(x_2\overrightarrow{H}y_2).$$

Recalling that $f \geq b_u + 1$,

$$\beta + |A_u(u)| + |A_u(v)| = \beta + |A_u(u)| + |A_u(\xi_{g+1})| \geq 2\sum_{i=1}^{t} |A_u(\xi_i)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| \geq |\Phi_u \cup B_u| + f - 2 \geq \varphi_u + b_u - 1 \geq \gamma_u - 1.$$

Case 1.3. $\sum_{i\notin\{g,t\}} \omega_i - | A_u(u) | + 1 \leq \beta < \sum_{i\notin\{g,t\}} \omega_i$.

Case 1.3.1. $\xi_1 \notin \{x_1, y_1\}$.

It follows that $A_u(u), A_u(\xi_g), A_u(\xi_t)$ and $A_u(\xi_{t+1})$ are pairwise different. Since $f \geq b_u + 1$,

$$\beta + |A_u(v)| = \beta + |A_u(\xi_{g+1})| \geq 2\sum_{i=1}^{f} |A_u(\xi_i)| - |A_u(u)| - |A_u(\xi_g)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| + 1 \geq \sum_{i=1}^{f} |A_u(\xi_i)| + f - 3 \geq \varphi_u + b_u - 2 \geq \gamma_u - 1 - |A_u(u)|.$$

Case 1.3.2. $\xi_1 \in \{x_1, y_1\}$.

Assume w.l.o.g. that $\xi_1 = x_1$, i.e. $M_t = M_f$. If $M_1 \notin \Omega^-$ then $\beta \geq \sum_{i\notin\{g,t\}} \omega_i$, a contradiction. Let $M_1 \in \Omega^-$. This implies $\xi_2 \in B_u$ and $M_s \in \Omega^+$. If $M_s \neq M_g$ then $\beta \geq \sum_{i\notin\{g,t\}} \omega_i$, a contradiction. So, assume $M_s = M_g$. Analogously, $M_{r-1} = M_f$. If $M_j \in \Omega^+$ for some $j \in \{1, ..., f\} - \{g\}$ then again $\beta \geq \sum_{i\notin\{g,t\}} \omega_i$, a contradiction. Let

$$i \in \{1, ..., f\} - \{g\} \implies M_i \in \Omega^0 \cup \Omega^-.$$

It follows that $B_u \cup \{u\} \subseteq V(x_1 \overrightarrow{H} y_1)$ and $\Lambda_u - (B_u \cup \{u\}) \subseteq V(x_2 \overrightarrow{H} y_2)$. Furthermore, noting that $A_u(\xi_g), A_u(\xi_t), A_u(\xi_{t+1})$ are pairwise different and $f \geq b_u + 1$,

$$\beta + |A_u(u)| + |A_u(v)| = \beta + |A_u(u)| + |A_u(\xi_{g+1})| \geq 2\sum_{i=1}^{f} |A_u(\xi_i)| -$$
$$|A_u(\xi_g)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| + 1 \geq \sum_{i=1}^{f} |A_u(\xi_i)| + f - 2 \geq$$
$$\varphi_u + b_u - 1 \geq \gamma_u - 1.$$

Case 1.4. $\sum_{i\notin\{g,t\}} \omega_i - 2|A_u(u)| + 2 \leq \beta < \sum_{i\notin\{g,t\}} \omega_i - |A_u(u)| + 1$.

If $|\Omega^-| \leq 1$ then clearly $\beta \geq \sum_{i\notin\{g,t\}} \omega_i - |A_u(u)| + 1$, a contradiction. Let $|\Omega^-| = 2$. This implies M_1, $M_f \in \Omega^-$ and $M_s, M_{r-1} \in \Omega^+$. If $|\Omega^+| \geq 3$ then again $\beta \geq \sum_{i\notin\{g,t\}} \omega_i - |A_u(u)| + 1$, a contradiction. Let $|\Omega^+| = 2$, i.e. $\Omega^+ = \{M_s, M_{r-1}\}$. By claim 4.3,

$$B_u \cup \{u\} \subseteq V(x_1 \overrightarrow{H} y_1), \quad \Lambda_u - (B_u \cup \{u\}) \subseteq V(x_2 \overrightarrow{H} y_2).$$

Recalling that $f \geq b_u + 1$,

$$\beta + |A_u(u)| + |A_u(v)| = \beta + |A_u(u)| + |A_u(\xi_{g+1})|$$
$$\geq 2\sum_{i=1}^{f} |A_u(\xi_i)| - |A_u(u)| - |A_u(\xi_g)| - |A_u(\xi_t)| - |A_u(\xi_{t+1})| + 2$$
$$\geq \sum_{i=1}^{f} |A_u(\xi_i)| + \sum_{i\notin\{1,g,t,t+1\}} |A_u(\xi_i)| + 2 \geq |\Phi_u \cup B_u| + (f-4) + 2 \geq \gamma_u - 1.$$

Case 2. $u \in U_2$.

Apply the arguments used in the proof of $(d1)$ (see case 2 and case 3).

(f2) Case 1. $u \in U_1$.

As shown in the proof of $(f1)$,

$$\beta + |A_u(u)| + |A_u(v)| \geq |\Phi_u \cup B_u| + f - 2 \geq |\Phi_u| + f - 2 \geq \varphi_u + b_u - 1.$$

Since $\beta + |A_u(u)| + |A_u(v)| = \varphi_u + b_u - 1 = \gamma_u - 1$, we have equations

$$\beta + |A_u(u)| + |A_u(v)| = |\Phi_u \cup B_u| + f - 2 = |\Phi_u| + f - 2 = \varphi_u + b_u - 1$$

implying that $f = b_u + 1$. If $|T(u)| - 1 \geq 2$ then $\Lambda_u - (B_u \cup \{u\}) \neq \emptyset$ and hence $f \geq |B_u| + |\{u\}| + 1 = b_u + 2$, a contradiction. Otherwise ($|T(u)| - 1 = 1$),

$$\Lambda_u = B_u \cup \{u\}, \quad |A_u(u)| = |A_u(v)| = 1,$$
$$\varphi_u = b_u + 1, \quad \beta = \varphi_u + b_u - 3 = 2\varphi_u - 4$$

and we deduce that $(\hat{u}, x_i \overrightarrow{H} y_i) \in \Delta$ $(i = 1, 2)$ and $\gamma_u - 1 = 2(\varphi_u - 1)$.

Case 2. $u \in U_2$.

Apply the arguments used in the proof of $(d1)$.□

Proof of lemma 7. (g1) Clearly $h \geq 2(b_u+1) = 2(\varphi_u' + b_u + 1) = 2(\gamma_u + 1)$ and therefore, $|O(x,y)| - 1 \geq h/2 \geq \gamma_u + 1$.□

(g2) By $(d1)$, $h \geq 2\gamma_u$ which implies $|O(x,y)| - 1 \geq h/2 \geq \gamma_u$.□

(g3), (g5) If $|\Lambda_u| = 1$ then $\gamma_u = 1$ and there is nothing to prove. Let $|\Lambda_u| \geq 2$, i.e. $u \in U_1$.

Case 1. $u \notin \{x,y\}$.

Assume w.l.o.g. that $u \in V(x^+\overrightarrow{H}y^-)$. We can assume also that $\Lambda_u \not\subseteq V(x\overrightarrow{H}y)$, since otherwise the result holds by $(e2)$. Let $x_1\overrightarrow{H}y_1$ be the longest segment in $x\overrightarrow{H}y^-$ with $x_1, y_1 \in \Lambda_u$ and $x_2\overrightarrow{H}y_2$ be the longest segmnet in $y\overrightarrow{H}x^-$ with $x_2, y_2 \in \Lambda_u$. Putting $\beta = | x_1\overrightarrow{H}y_1 | -1 + | x_2\overrightarrow{H}y_2 | -1$ we see (by lemma 6) that

$$\beta \geq \gamma_u - 1 - |A_u(u)| - |A_u(y_2)| = \gamma_u - 2 - |A_u(y_2)|$$

and therefore

$$\begin{aligned}|O_u(x,y)| - 1 &\geq | x\overrightarrow{H}y_1\Lambda_u(y_1,y_2)y_2\overleftarrow{H}y | -1 \geq \beta + |A_u(y_1)| + |A_u(y_2)| \\ &\geq \gamma_u - 2 + |A_u(y_1)| \geq \gamma_u - 1.\end{aligned}$$

Case 2. $u \in \{x,y\}$.

Assume w.l.o.g. that $u = x$. Let $x_1\overrightarrow{H}y_1$ be the longest segment in $x^+\overrightarrow{H}y$ with $x_1, y_1 \in \Lambda_x$ and $x_2\overrightarrow{H}y_2$ be the longest segmnet in $y^+\overrightarrow{H}x$ with $y_2 \in \Lambda_x$. Putting $\beta = | x_1\overrightarrow{H}y | + | x_2\overrightarrow{H}y | -2$ we see (by lemma 6) that

$$\beta \geq \gamma_x - 1 - |A_x(x)| - |A_x(x_1)| = \gamma_x - 2 - |A_x(x_1)|,$$

and therefore

$$\begin{aligned}|O_x(x,y)| - 1 &\geq | x\overleftarrow{H}x_2\Lambda_x(x_2,x_1)x_1\overrightarrow{H}y | -1 \geq \beta + |A_x(x_1)| + |A_x(x_2)| \\ &\geq \gamma_x + |A_x(x_2)| - 2 \geq \gamma_x - 1.\end{aligned}$$

Also, by $(e1)$ and $(e2)$, $| x_1\overrightarrow{H}x | -1 \geq \gamma_x - 1$, $| x_2\overrightarrow{H}y | -1 \geq \gamma_x - 1$ and hence

$$\begin{aligned}| O(\mathring{x}, x) | -1 &\geq | \mathring{x}\, \rho_x(x_1)\, T(x_1)\, x_1\overrightarrow{H}x | -1 \geq | x_1\overrightarrow{H}x | \geq \gamma_x, \\ | O(\mathring{x}, y) | -1 &\geq | \mathring{x}\, \rho_x(x_2)\, T(x_2)\, x_2\overrightarrow{H}y | -1 \geq | x_2\overrightarrow{H}y | \geq \gamma_x. \square\square\end{aligned}$$

(g4) We can suppose $u \notin \{x,y\}$, since otherwise the arguments are the same. Assume w.l.o.g. that $u \in V(x^+\overrightarrow{H}y^-)$. Clearly $|O_u(x,y)| = |O(x,y)|$. In order to prove $(g4.1)-(g4.4)$, we recall (by the hypothesis) that $\Lambda_u \subseteq V(x\overrightarrow{H}y)$ and $(\overset{0}{u}, H) \notin \Delta$.

(g4.1), (g4.2) By $(e2)$, $\gamma_u - 1 = |O(x,y)| - 1 \geq | x\overrightarrow{H}y | -1 \geq \gamma_u - 1$ which implies $|O(x,y)| - 1 = | x\overrightarrow{H}y | -1 = \gamma_u - 1$. Using $(e3)$, it is easy to see that $(\mathring{u}, x\overrightarrow{H}y) \in \Delta$, $\gamma_u - 1 = 2(\varphi_u - 1)$ and $B_u = \Lambda_u - u \subseteq U_0$.

(g4.3) **Case 1.** $z \in U_1$.

Case 1.1. $\Lambda_z \not\subseteq V(x\overrightarrow{H}y)$.

Choose $w \in \Lambda_z \cap V(y^+\overrightarrow{H}x^-)$. If $z = x^+$ then

$$|O(x,y)| - 1 \geq | y\overleftarrow{H}z\Lambda_z(z,w)w\overrightarrow{H}x | -1 \geq | x\overrightarrow{H}y |,$$

a contradiction. Otherwise we reach a contradiction by the following way

$$|O(x,y)|-1 \geq | y\overleftarrow{H}z^{+} \overset{\circ}{u}\, x^{++}\overrightarrow{H}z\Lambda_z(z,w)\, w\overrightarrow{H}x\,| -1 \geq | x\overrightarrow{H}y\,|.$$

Case 1.2. $\Lambda_z \subseteq V(x\overrightarrow{H}y)$.

Choose $w \in \Lambda_z - z$. Assume w.l.o.g. that $w \in V(x\overrightarrow{H}z^{-})$. Since $z \in U_1$, we have $|\Lambda_z(z,w)|-1 \geq 2$ and hence

$$|O(x,y)|-1 \geq | y\overleftarrow{H}z\Lambda_z(z,w)\, w\overrightarrow{H}z^{-} \overset{\circ}{u}\, w'\overleftarrow{H}x\,| -1 \geq | x\overrightarrow{H}y\,|$$

for some $w' \in \{w^{-}, w^{--}\}$, a contradiction.

Case 2. $z \in U_0 \cup U_2$.

If $z \in U_2$ then apply the arguments used in the proof of $(d1)$ (see case 2 and 3). Let $z \in U_0$. If there exists a vertex $w \in (\Lambda_z - z) - \Lambda_u$ then we can reach a contradiction as in case 1. Otherwise, $\Lambda_z \subseteq \Lambda_u \cup \{z\}$ and $\gamma_z \leq \varphi_u = (\gamma_u+1)/2$.

(g4.4) Suppose, to the contrary, that $\Lambda_z \not\subseteq V(x\overrightarrow{H}y)$. If $| y\overrightarrow{H}x\,| -1 = 2$ then clearly $(\overset{\circ}{u}, H) \in \Delta$, a contradiction. Let $| y\overrightarrow{H}x\,| -1 \geq 3$. Choose $w \in \Lambda_z \cap V(y^{+}\overrightarrow{H}x^{-})$. Assume w.l.o.g. that $| w\overrightarrow{H}x\,| -1 \geq 2$. If $z \notin \Lambda_u$ then by $(g4.3)$, we are done. Otherwise,

$$|O(x,y)|-1 \geq | y\overleftarrow{H}z^{++} \overset{\circ}{u}\, x^{++}\overrightarrow{H}z\Lambda_z(z,w)\, w\overrightarrow{H}x\,| -1 \geq | x\overrightarrow{H}y\,|,$$

a contradiction. So, $(g4.1)$–$(g4.4)$ are proved. A similar proof holds for $(g4.5)$–$(g4.7)$ when $\Lambda_u \not\subseteq V(x\overrightarrow{H}y)$ and $\Lambda_u \not\subseteq V(y\overrightarrow{H}x)$. So, $(g4)$ is proved.□

(g6), (g7) Let $u = x^{+}$. Choose $v \in \Lambda_u$ so as to maximize $| v\overrightarrow{H}y\,|$. Clearly, $v \in V(y\overleftarrow{H}u)$.

Case 1. $v = u$.

Case 1.1 $|T(u)|-1 \geq 1$.

By $(e1)$ and $(e2)$, $|O(x,y)|-1 \geq | u\overrightarrow{H}y\,| \geq \gamma_u$. If $| O(x,y)\,| -1 = \gamma_u$ then by $(e1)$, $|T(u)|-1 \leq 1$ and $| u\overrightarrow{H}y\,| = \gamma_u - 1$ which by $(e3)$ holds $(\hat{u}, u\overrightarrow{H}y) \in \Delta$.

Case 1.2 $|T(u)|-1 = 0$.

Clearly, $|O(x,y)|-1 \geq | x\overrightarrow{H}y\,| -1 \geq \gamma_u$. If $| O(x,y)\,| -1 = \gamma_u$ then $| x\overrightarrow{H}y\,| -1 = \gamma_u$ implying that $uw \in E$ for each $w \in V(x\overrightarrow{H}y) - u$, i.e. $(u, x\overrightarrow{H}y) \in \Delta$.

Case 2. $v \neq u$.

Case 2.1 $|T(u)|-1 \geq 1$.

By $(e1)$ and $(e2)$, $| v\overrightarrow{H}y\,| -1 \geq \gamma_u - 1$ and hence

$$|O(x,y)|-1 \geq | y\overleftarrow{H}u\Lambda_u(u,v)\, v\overrightarrow{H}x\,| -1 \geq | v\overrightarrow{H}y\,| -1 + |T(u)|-1 \geq \gamma_u + |T(u)| - 2.$$

If $|O(x,y)|-1 = \gamma_u$ then $|T(u)|-1 = 1$, $| v\overrightarrow{H}y\,| -1 = \gamma_u - 1$ and by $(e3)$, $(\hat{u}, v\overrightarrow{H}y) \in \Delta$.

Case 2.2 $|T(u)|-1 = 0$.

Clearly, $|O(x,y)|-1 \geq | y\overleftarrow{H}u\Lambda_u(u,v)\, v\overrightarrow{H}x\,| -1 \geq | v\overrightarrow{H}y\,| -1 \geq \gamma_u$. If $|O(x,y)|-1 = \gamma_u$ then $| v\overrightarrow{H}y\,| -1 = \gamma_u$ implying that $uw \in E$ for each $w \in V(v\overrightarrow{H}y) - u$, i.e. $(u, v\overrightarrow{H}y) \in \Delta$.□

(g8) By $(g4)$, $(\overset{\circ}{x}, H) \in \Delta$. Since $\{w, y\} \subseteq \Lambda_x$, we have $h \geq 6$. If $| O_x(x,y)\,| -1 \leq \gamma_x$ then by $(g4.6)$ and $(g4.7)$, $h - 2 = |O_x(x,y)| - 1 \leq \gamma_x \leq h/2$ implying that $h \leq 4$, a contradiction. So, $|O_x(x,y)| - 1 \geq \gamma_x + 1$. By symmetry, $|O_x(x,w)| - 1 \geq \gamma_x + 1$ and the result follows.□

Proof of lemma 8. Immediate from definition 3.11.□

Proof of lemma 9. By $(d3), h \geq \gamma_i + 1$ and $h \geq \gamma_{i+1} + 1$ for each $i \in \overline{1,h}$. In other words,

$$h - 1 \geq (\gamma_i + \gamma_{i+1})/2 \quad (i = 1, ..., h). \tag{8}$$

(i1) By lemma 8, it sufficies to prove $|O(x,y)| - 1 \geq \beta_i$. Assume w.l.o.g. that $i = 1$ and $u_1, u_2 \in V(x^+\overrightarrow{H}y^-)$.

Case 1. $u_1, u_2 \in U_0$.

Putting $\Gamma_i = \Phi_i \cap V(H)$ $(i = 1, 2)$ we see that $|\Gamma_i| = \varphi_i - b_i^* = \gamma_i$ $(i = 1, 2)$.

Case 1.1. $\Gamma_1 \cup \Gamma_2 \subseteq V(x\overrightarrow{H}y)$.

Clearly $|O(x,y)| - 1 \geq | x\overrightarrow{H}y | -1 \geq \max(|\Gamma_1|, |\Gamma_2|) \geq (\gamma_1 + \gamma_2)/2$.

Case 1.2. $\Gamma_1 \cup \Gamma_2 \not\subseteq V(x\overrightarrow{H}y)$.

Assume w.l.o.g. that $\Gamma_1 \cap (V(y^+\overrightarrow{H}x^-)) \neq \emptyset$. Let $z_1\overrightarrow{H}z_2$ be the longest segment in $y^+\overrightarrow{H}x^-$ with $z_1, z_2 \in \Gamma_1$.

Case 1.2.1. $\Gamma_2 \cap V(y^+\overrightarrow{H}x^-) = \emptyset$.

Choose $w \in V(x^+\overrightarrow{H}u_1)$ such that $u_2w \in E$ and $| x\overrightarrow{H}w |$ is minimum. Then

$$|O(x,y)| - 1 \geq | x\overrightarrow{H}y | -1 \geq | w\overrightarrow{H}y | + | x\overrightarrow{H}w | -2 \geq \gamma_2 + | x\overrightarrow{H}w | -2,$$
$$|O(x,y)| - 1 \geq | x\overrightarrow{H}z_1u_1\overleftarrow{H}wu_2\overrightarrow{H}y | -1 \geq \gamma_1 - | x\overrightarrow{H}w | +2.$$

Combining these two inequalities yields $|O(x,y)| - 1 \geq (\gamma_1 + \gamma_2)/2$.

Case 1.2.2. $\Gamma_2 \cap V(y^+\overrightarrow{H}x^-) \neq \emptyset$.

Let $w_1\overrightarrow{H}w_2$ be the longest segment in $y^+\overrightarrow{H}x^-$ with $w_1, w_2 \in \Gamma_2$.

Case 1.2.2.1. $z_1, w_2 \in V(w_1\overrightarrow{H}z_2)$.

It follows that $|O(x,y)| - 1 \geq | x\overrightarrow{H}u_1z_2\overleftarrow{H}w_1u_2\overrightarrow{H}y | -1 \geq \max(\gamma_1, \gamma_2) \geq (\gamma_1 + \gamma_2)/2$.

Case 1.2.2.2. $z_2, w_1 \in V(z_1\overrightarrow{H}w_2)$.

Clearly $|O(x,y)| - 1 \geq | x\overrightarrow{H}u_1z_1\overrightarrow{H}w_2u_2\overrightarrow{H}y | -1 \geq \max(\gamma_1, \gamma_2) \geq (\gamma_1 + \gamma_2)/2$.

Case 1.2.2.3. Either $w_1, w_2 \in V(z_1\overrightarrow{H}z_2)$ or $z_1, z_2 \in V(w_1\overrightarrow{H}w_2)$.

Assume w.l.o.g. that $w_1, w_2 \in V(z_1\overrightarrow{H}z_2)$. If $w_1 = z_1$ (resp. $w_2 = z_2$) then we could argue as in case 1.2.2.1. (resp. 1.2.2.2.). Otherwise ($w_1 \neq z_1$ and $w_2 \neq z_2$),

$$|O(x,y)| - 1 \geq | x\overrightarrow{H}u_1z_1\overrightarrow{H}w_2u_2\overrightarrow{H}y | -1 \geq \gamma_2 + | z_1\overrightarrow{H}w_1 | -1,$$
$$|O(x,y)| - 1 \geq | x\overrightarrow{H}u_1z_2\overleftarrow{H}w_1u_2\overrightarrow{H}y | -1 \geq \gamma_1 - | z_1\overrightarrow{H}w_1 | +1.$$

Combining these two inequalities yields $|O(x,y)| - 1 \geq (\gamma_1 + \gamma_2)/2$.

Case 2. $u_1, u_2 \in \overline{U}_0$.

By $(g2)$ and $(g3)$, $|O(x,y)| - 1 \geq \gamma_i - 1$ $(i = 1, 2)$. If either $u_1 \in U_*$ or $u_2 \in U_*$ then by $(g1)$ we are done. Let $u_1, u_2 \in U_1 \cup U_2$.

Case 2.1. Either $|O(x,y)| - 1 = \gamma_1 - 1$ or $|O(x,y)| - 1 = \gamma_2 - 1$.

Assume w.l.o.g. that $|O(x,y)| - 1 = \gamma_1 - 1$. Using $(g2)$ we see that $|T(u)| - 1 = 1$ and by $(g4.1)$ and $(g4.3)$, $u_2 \in U_0 \cup U_*$, a contradiction.

Case 2.2. $|O(x,y)| - 1 \geq \gamma_1$ and $|O(x,y)| - 1 \geq \gamma_2$.

Clearly, $|O(x,y)| - 1 \geq \max(\gamma_1, \gamma_2) \geq (\gamma_1 + \gamma_2)/2$.

Case 3. $u_1 \in U_0$, $u_2 \in \overline{U}_0$.

By $(g2)$ and $(g3)$,

$$|O(x,y)| - 1 \geq \gamma_2 - 1. \tag{9}$$

Case 3.1. $\Phi_1 - B_1^* \subseteq V(x\overrightarrow{H}y)$.

Clearly $|O(x,y)|-1 \geq |x\overrightarrow{H}y|-1 \geq \gamma_1$. The result is immediate if either $|O(x,y)|-1 > \gamma_1$ or $|O(x,y)|-1 > \gamma_2-1$. Thus we can assume $|O(x,y)|-1 = |x\overrightarrow{H}y|-1 = \gamma_1 = \gamma_2-1$. Since $|x\overrightarrow{H}y|-1 = \gamma_1$, we have $\Lambda_1 = V(x\overrightarrow{H}y)-u_1$. On the other hand, by (g4.3), $\Lambda_1 \subseteq \Lambda_2 \cup \{u_1\}$, a contradiction.

Case 3.2. $\Phi_1 - B_1^* \nsubseteq V(x\overrightarrow{H}y)$.

Let $z_1\overrightarrow{H}z_2$ be the maximal segment in $y^+\overrightarrow{H}x^-$ with $z_1, z_2 \in N(u_1)$.

Case 3.2.1. $\Lambda_2 \subseteq V(x\overrightarrow{H}y)$.

For each $v \in \Lambda_2 \cap V(x\overrightarrow{H}u_2)$, put $P_v = x\overleftarrow{H}z_1u_1\overrightarrow{H}y$ if $v = u_2$ and

$$P_v = x\overleftarrow{H}z_1u_1\overleftarrow{H}v\Lambda_2(v,u_2)u_2\overrightarrow{H}y$$

if $v \neq u_2$. Choose $w_1 \in \Lambda_2 \cap V(x\overrightarrow{H}u_2)$ so as to maximize $|w_1\overrightarrow{H}u_2|$. By (g2) and (g3), $|w_1\overrightarrow{H}y|-1 \geq \gamma_2-1$. If $w_1 \neq x$ then clearly $|z_1\overrightarrow{H}y|-1 \geq \gamma_1$ and hence

$$|O(x,y)|-1 \geq |P_{w_1}|-1 \geq |z_1\overrightarrow{H}y| - |x\overrightarrow{H}w_1| + 2 \geq \gamma_1 - |x\overrightarrow{H}w_1| + 3,$$
$$|O(x,y)|-1 \geq |x\overrightarrow{H}y|-1 \geq |w_1\overrightarrow{H}y| + |x\overrightarrow{H}w_1| - 2 \geq \gamma_2 + |x\overrightarrow{H}w_1| - 2.$$

Combining these two inequalities yields the results. Now let $w_1 = x$. Choose $w_2 \in \Lambda_2 \cap V(x^+\overrightarrow{H}u_2)$ so as to maximize $|w_2\overrightarrow{H}u_2|$. Since H is extreme,

$$h \geq |u_2\Lambda_2(u_2,x)x\overrightarrow{H}u_1z_2\overleftarrow{H}u_2|-1$$

which implies that

$$\overline{p}_2(x) = \hat{u}_2 \implies |z_2\overrightarrow{H}x|-1 \geq |A_2(x)| + |A_2(u_2)|, \tag{10}$$

$$\overline{p}_2(x) = \overset{0}{u}_2 \implies |z_2\overrightarrow{H}x|-1 \geq |A_2(x)| + 1 \geq 2. \tag{11}$$

Also, we have

$$|O(x,y)|-1 \geq |x\overrightarrow{H}y|-1 \geq \gamma_1 - |z_1\overrightarrow{H}z_2|. \tag{12}$$

Claim 9.1. $|O(x,y)|-1 \geq \gamma_2 + |z_1\overrightarrow{H}z_2|$.

Proof of claim 9.1. Clearly,

$$|O(x,y)|-1 \geq |P_{w_2}|-1 \geq |w_2\overrightarrow{H}y| + |z_2\overrightarrow{H}x| + |z_1\overrightarrow{H}z_2|-1. \tag{13}$$

By (f1), $|w_2\overrightarrow{H}y|-1 \geq \gamma_2 - 2 - |A_2(x)|$ if $\overline{p}_2(x) = \overset{0}{u}_2$ and $|w_2\overrightarrow{H}y|-1 \geq \gamma_2 - 1 - |A_2(u_2)| - |A_2(x)|$ if $\overline{p}_2(x) = \hat{u}_2$. Using also (10) and (11), we obtain $|w_2\overrightarrow{H}y| + |z_2\overrightarrow{H}x| - 2 \geq \gamma_2 - 1$, which by (13) implies $|O(x,y)|-1 \geq \gamma_2 + |z_1\overrightarrow{H}z_2|$.□

Claim 9.1 with together (12) implies the result.

Case 3.2.2. $\Lambda_2 \nsubseteq V(x\overrightarrow{H}y)$.

Let $y_1\overrightarrow{H}y_2$ be the maximal segment in $y^+\overrightarrow{H}z^-$ with $y_1, y_2 \in \Lambda_2$.

Case 3.2.2.1. Either $z_1, y_2 \in V(y_1\overrightarrow{H}z_2)$ or $z_2, y_1 \in V(z_1\overrightarrow{H}y_2)$.

Assume w.l.o.g that $z_1, y_2 \in V(y_1\overrightarrow{H}z_2)$. Then

$$|O(x,y)|-1 \geq |y\overleftarrow{H}u_2\Lambda_2(u_2,y_1)y_1\overrightarrow{H}z_2u_1\overleftarrow{H}x|-1 \geq \gamma_1 + 1,$$

and the result follows by (9).

Case 3.2.2.2. $z_1, z_2 \in V(y_1 \overrightarrow{H} y_2)$.

Apply the arguments in case 3.2.2.1.

Case 3.2.2.3. $y_1, y_2 \in V(z_1 \overrightarrow{H} z_2)$.

Putting $\beta = | x\overrightarrow{H} y | + | y_1 \overrightarrow{H} y_2 | - 2$ and

$$P_1 = y\overleftarrow{H} u_2 \Lambda_2 (u_2, y_2) y_2 \overleftarrow{H} z_1 u_1 \overleftarrow{H} x,$$
$$P_2 = y\overleftarrow{H} u_2 \Lambda_2 (u_2, y_1) y_1 \overrightarrow{H} z_2 u_1 \overleftarrow{H} x,$$

we obtain

$$|O(x,y)| - 1 \geq |P_2| - 1 \geq | x\overrightarrow{H} y | + | y_1 \overrightarrow{H} z_2 | \geq \gamma_1 - | z_1 \overrightarrow{H} y_1 | + 2. \tag{14}$$

Claim 9.2. $|O(x,y)| - 1 \geq \gamma_2 + | z_1 \overrightarrow{H} y_1 | - 1$.

Proof of claim 9.2. If $\overline{p}_2 (y_2) = \hat{u}_2$ then by $(f1)$, $\beta \geq \gamma_2 - 1 - |A_2 (u_2)| - |A_2 (y_2)|$ and

$$\begin{aligned}|O(x,y)| - 1 \geq |P_1| - 1 &\geq \beta + |A_2 (u_2)| + | A_2 (y_2) | + | z_1 \overrightarrow{H} y_1 | - 1\\ &\geq \gamma_2 + | z_1 \overrightarrow{H} y_1 | - 2.\end{aligned}$$

Otherwise $(\overline{p}_2(y_2) = \mathring{u}_2)$, $\beta \geq \gamma_2 - 2 - |A_2 (y_2)| = \gamma_2 - 3$ and

$$|O(x,y)| - 1 \geq |P_1| - 1 \geq \beta + | z_1 \overrightarrow{H} y_1 | + 1 \geq \gamma_2 + | z_1 \overrightarrow{H} y_1 | - 2. \square$$

Claim 9.2 together with (14) implies the result.□

(i2) By $(d1)$, $h \geq 2\gamma_x$ imlying that $|O(x,y)| - 1 \geq h/2 \geq \gamma_x$. Also, $|O(x,y)| - 1 \geq \gamma_z$ by $(g6)$. Recalling lemma 8, $|\Omega (x,y)| - 1 \geq |O(x,y)| - 1 \geq (\gamma_x + \gamma_z)/2$.□

(i3) By $(g1)$, $|O(x,y)| - 1 \geq \gamma_x + 1$ and by $(g6)$, $|O(x,y)| - 1 \geq \gamma_z$. Using lemma 8, we obtain the result immediately.□

(i4) **Claim 9.3.** $\max(|O_x(x,y)| - 1, |O_x(x,w)| - 1) \geq (\gamma_x + \gamma_z)/2$.

Proof of claim 9.3. By $(g6)$, $\min(|O(x,y)| - 1, |O(x,w)| - 1) \geq \gamma_z$. If either $|O_x(x,y)| - 1 \geq \gamma_x$ or $|O_x(x,w)| - 1 \geq \gamma_x$ then clearly we are done. Otherwise, by $(g3)$, $|O_x(x,y)| - 1 = |O_x(x,w)| - 1 = \gamma_x - 1$ and the result holds by $(g8)$ and lemma 8.□

Claim 9.4. $\min(| O(\mathring{x},y) | - 1, | O(\mathring{x},w) | - 1) \geq (\gamma_x + \gamma_z + 1)/2$.

Proof of claim 9.4. By $(g5)$ and $(g6)$, $| O(\mathring{x},y) | - 1 \geq \gamma_x$ and $|O(x,y)| - 1 \geq \gamma_z$ respectively. Since $V(O(x,y)) \cap \{\mathring{x}\} = \emptyset$ (by definition 2.10), $\left|O(\mathring{x},y)\right| - 1 \geq |O(x,y)| \geq \gamma_z + 1$, implying that $| O(\mathring{x},y) | - 1 \geq (\gamma_x + \gamma_z + 1)/2$. Analogously, $| O(\mathring{x},w) | - 1 \geq (\gamma_x + \gamma_z + 1)/2$.□

Claim 9.5. $| O(\mathring{x},x) | - 1 \geq (\gamma_x + \gamma_z + 1)/2$.

Proof of claim 9.5. Let $v \in \Lambda_x - x$. By $(g5)$ and $(g6)$, $| O(\mathring{x},x) | - 1 \geq \gamma_x$ and $| O(x,v) | - 1 \geq \gamma_z$ respectively. Hence

$$| O(\mathring{x},x) | - 1 \geq |O(x,v)| + | vT(v)\, \hat{v}\mathring{x}| - 2 \geq \gamma_z + 1$$

which implies $| O(\mathring{x},x) | - 1 \geq (\gamma_x + \gamma_z + 1)/2$.□

The result holds from claims 9.3-9.5 and lemma 8.□

(i5) By $(g6)$, $|O(x,y)| - 1 \geq \gamma_z$ and $|O(z,w)| - 1 \geq \gamma_x$ and the result follows from lemma 8.□

(i6) By $(g6)$, $|O(x,y)|-1 \geq \max(\gamma_{x^+},\gamma_{x^-})$. If $|T(x)|-1 \geq 2$ then by $(g2)$, $|O(x,y)|-1 \geq \gamma_x$ and the result holds immediately. Thus we can assume $|T(x)|-1=1$. Put $z=x^+$ and $w=x^-$. By $(g3)$ and $(g6)$, $|O_z(x,y)|-1 \geq \max(\gamma_z-1,\gamma_z,\gamma_w)$. If $|O_z(x,y)|-1 \geq \min(\gamma_z,\gamma_z+1,\gamma_w+1)$ then clearly we are done. Now let $|O_z(x,y)|-1=\gamma_z-1=\gamma_z=\gamma_w$. Since $u \notin U_*$ (by $(g1)$), we have by $(g4.3)$ and $(g4.7)$, $\gamma_z \leq (\gamma_z+1)/2=(\gamma_z+2)/2$ implying that $\gamma_z \leq 2$ and $|O_z(x,y)|-1 \leq 2$. It means that $h \leq 4$. Recalling also $(g4.1)$ and $(g4.5)$, we conclude that $h=4$, a contradiction.□

(i7) If either $|O_x(x,y)|-1=\gamma_x-1$ or $|O_y(x,y)|-1=\gamma_y-1$, say $|O_x(x,y)|-1=\gamma_x-1$, then by $(g1)$–$(g3)$, $|T(x)|-1=1$. By $(g4)$, either $(\overset{\circ}{x}, x\overrightarrow{H}y) \in \Delta$ or $(\overset{\circ}{x}, y\overrightarrow{H}x) \in \Delta$ or $(\overset{\circ}{x},H) \in \Delta$. This implies by $(g4.2)$ and $(g4.6)$ that $y \in U_0$, a contradiction. Thus $|O_z(x,y)|-1 \geq \gamma_z$ and $|O_y(x,y)|-1 \geq \gamma_y$. Using $(g6)$ with lemma 8, we obtain $|\Omega(x,y)|-1 \geq (\gamma_x+\gamma_z)/2$ for each $z \in \{x^+,x^-\}$ and $|\Omega(x,y)|-1 \geq (\gamma_y+\gamma_w)/2$ for each $w \in \{y^+,y^-\}$. Then the result follows by $(i1)$.□

(i8) Observing that $|O(x,y)|-1 \geq |y\overrightarrow{H}x|-1=h-1$, we obtain the result from (8) immediately.□

(i9) Put $z=x^+$. We can assume $h \geq 4$, since otherwise the result holds from $(i8)$. By $(d3)$, $|O(x,y)|-1 \geq h-2 \geq \gamma_z-1$. If $|O(x,y)|-1 \geq \gamma_z+1$ then clearly we are done. Let $|O(x,y)|-1=\gamma_z$. By $(g7)$, $|T(z)|-1 \leq 1$. If $\Lambda_z \cap V(y^+\overrightarrow{H}x^-) \neq \emptyset$ then by $(\mathbf{g}7)$, $\hat{z}\,x^- \in E$. Hence $|O(x,y)|-1 \geq h-1$ and by (8) we are done. Now let $\Lambda_z \subseteq V(x\overrightarrow{H}y)$. It means that $\gamma_z=2$ and $|O(x,y)|-1=2$. But then $h=4$, a contradiction.□

Proof of the theorem. Let C be a longest cycle of a graph G and $H=u_1...u_hu_1$ a longest cycle of $G-C$ with a maximal HC–extension T. Putting $U_*=\{v_1^*,...,v_r^*\}$ and using definition 3.3, we let for each $i \in \overline{1,r}$,

$$\Theta(\overleftarrow{T}(v_i^*),V_{neut},V_{fin}^{(i)})=(P_0^{(i)},...,P_{\pi(i)}^{(i)}),$$
$$R_i=<(V(\hat{v}_i^*\,\overleftarrow{T}(v_i^*)z_{\pi(i)}^{(i)}) \cup \bigcup_{j=0}^{\pi(i)} V(P_j^{(i)}))-z_{\pi(i)}^{(i)}>,$$

where $V_{neut}=V-(V(C)\cup V(T))$ and $V_{fin}^{(i)}=V(T)-V\overline{T(v_i^*)}$. Since $c \geq \delta+1 \geq k+1$, a variation of Menger's theorem [7] asserts that for each $i \in \overline{1,r}$ there are $k-1$ internally disjoint paths $E_i^{(1)},...,E_i^{(k-1)}$ in $(k-1)$–connected graph $G-z_{\pi(i)}^{(i)}$, starting at R_i, passing through V_{neut} and terminating on C at $k-1$ different vertices. Let $E_j^{(a)}$ have a common vertex v with $E_e^{(b)}$ for some $a,b \in \overline{1,k-1}$ and $j,e \in \overline{1,r}$ $(j \neq e)$. If $v \notin V(C)$ then there is a path starting in R_j, passing through V_{neut} and terminating in R_e, contradicting the fact that $v_j^*,v_e^* \in U_*$. So, $v \in V(C)$. Choose vertex-disjoint paths $E_1^{(i_1)},...,E_t^{(i_t)}$ $\left(i_j \in \overline{1,k-1} \text{ for each } j \in \overline{1,t}\right)$ so as to maximize t and put $E_j^{(i_j)}=x_j\overrightarrow{E}_j^{(i_j)}w_j^*$ $(j=1,...,t)$, where $x_j \in V(R_j)$ and $w_j^* \in V(C)$. It is easy to see that $t \geq \min(r,k-1)$. By $(a2)$, for each $j \in \overline{1,t}$ there is an $\left(x_j,v_j^*\right)$–path $F_j^{(i_j)}$ passing through $V(R_j)\cup V(T(v_j^*))$ and having length at least $\varphi_{v_j^*}$. Denoting $E_j^*=v_j^*F_j^{(i_j)}x_jE_j^{(i_j)}w_j^*$ $(j=1,...,t)$, we see that $E_1^*,...,E_t^*$ are vertex disjoint (H,C)–paths with $|E_i^*|-1 \geq \varphi_{v_i^*}+1$ $(i=1,...,t)$.

Case 1. $k \geq 4$, $h \geq 5$.

Case 1.1. $r \geq k$.

It follows that $t \geq k-1$. Let $\xi_1,...,\xi_t$ be the elements of $\{w_1^*,...,w_t^*\}$ occuring on C in consequtive order.

Case 1.1.1. $t \geq k$.

Assume w.l.o.g. that $\varphi_{v_1^*} \geq ... \geq \varphi_{v_r^*}$. Since $r \geq k$, we have

$$\frac{1}{k}\sum_{i=1}^{k}\varphi_{v_i^*} \geq \frac{1}{r}\sum_{i=1}^{r}\varphi_{v_i^*}$$

implying that

$$\sum_{i=1}^{k}\varphi_{v_i^*} \geq \frac{k}{r}\sum_{i=1}^{r}\varphi_{v_i^*} \geq \frac{k}{h}\sum_{i=1}^{r}\varphi_{v_i^*}. \quad (15)$$

By $(i1)$ and $(i3)$, $|\Omega(v_a^*, v_b^*)| - 1 \geq \beta_i$ for each $a, b \in \overline{1,t}$ and $i \in \overline{1,h}$. Hence

$$|\Omega(v_a^*, v_b^*)| - 1 \geq \frac{1}{h}\sum_{i=1}^{h}\beta_i = \mu(T).$$

Then for each $i, j \in \overline{1,t}$,

$$|w_i^*\overrightarrow{C}w_j^*| - 1 \geq |E_i^*| - 1 + \left|E_j^*\right| - 1 + \left|\Omega\left(v_i^*, v_j^*\right)\right| - 1 \geq \varphi_{v_i^*} + \varphi_{v_j^*} + 2 + \mu(T). \quad (16)$$

Using (15),(16) and recalling that $t \geq k$, we obtain

$$\begin{aligned} c = &\sum_{i=1}^{t}(|\xi_i\overrightarrow{C}\xi_{i+1}| - 1) \geq 2\sum_{i=1}^{t}\varphi_{v_i^*} + 2t + t\mu(T) \\ \geq &\sum_{i=1}^{t}\varphi_{v_i^*} + 2t + t\mu(T) \geq \tfrac{k}{h}\sum_{i=1}^{r}\varphi_{v_i^*} + k\mu(T) + 2k \\ &\geq \tfrac{k}{h}\left(\sum_{i=1}^{r}\varphi_{v_i^*} + \sum_{i=1}^{h}\varphi_i^{'} + 2h\right) = \tfrac{k}{h}\left(\sum_{i=1}^{h}\varphi_i + 2h\right), \end{aligned}$$

where $\xi_{t+1} = \xi_1$. It follows that $\sum_{i=1}^{h}\varphi_i \leq h(c/k - 2)$. Since $\varphi_i + \psi_i = d(u_i) \geq \delta$ $(i = 1, ..., h)$,

$$\sum_{i=1}^{h}\psi_i \geq h\delta - \sum_{i=1}^{h}\varphi_i \geq h\delta - ch/k + 2h.$$

In particular, $\max_i \psi_i \geq \delta - c/k + 2$. Using lemma 3, we obtain

$$c \geq \sum_{i=1}^{h}\psi_i + \max_i \psi_i \geq h\delta - ch/k + 2h + \delta - c/k + 2,$$

and the result follows immediately.

Case 1.1.2. $t = k - 1$.

Observe that $E_k^{(i)}$ terminates in $\left\{w_1^*, ..., w_{k-1}^*\right\}$ for each $i \in \overline{1, k-1}$, since otherwise $E_1^*, ..., E_{k-1}^*, E_k^{(j)}$ contradict the maximality of t for some $j \in \overline{1, k-1}$. By the same arguments, $E_j^{(i)}$ terminates in $\left\{w_1^*, ..., w_{k-1}^*\right\}$ for each $i \in \overline{1, k-1}$ and $j \in \overline{1, k}$. By Menger's theorem [7], there is a path $E = vE\xi_{t+1}$ starting in

$$< (V(T) \cup \bigcup_{i=1}^{k} R_i \cup \bigcup_{j=1}^{k}\bigcup_{i=1}^{k-1} V(E_j^{(i)})) - \left\{w_1^*, ..., w_{k-1}^*\right\} >$$

and terminating in $C - \{w_1^*, ..., w_{k-1}^*\}$. Assume w.l.o.g. that $\xi_1, ..., \xi_{t+1}$ occurs on $\overrightarrow{C}$ in consequtive order. Then it is easy to see that

$$c = \sum_{i=1}^{t+1}(|\xi_i\overrightarrow{C}\xi_{i+1}| - 1) \geq \sum_{i=1}^{k}\varphi_{v_i^*} + 2k + k\mu(T)$$

where $\xi_{t+2} = \xi_1$. Further we can argue exactly as in case 1.1.1.

Case 1.2. $r \le k-1$.

It follows that $t = r$. By Menger's theorem [7], there are k vertex-disjoint (H, C)–paths $E_i = v_i E_i w_i \quad (i = 1, ..., k)$. Assume w.l.o.g. that $w_1, ..., w_k$ occurs on $\overrightarrow{C}$ in consequtive order. Put

$$W = \{w_1, ..., w_k\}, \quad W^* = \{w_1^*, ..., w_r^*\}.$$

Let $a, b \in \overline{1,k}$. Denoting

$$W^*(a,b) = W^* \cap V(w_a \overrightarrow{C} w_b)$$

we will say that $w_a \overrightarrow{C} w_b$ is a suitable segment if

$$| w_a \overrightarrow{C} w_b | - 1 \ge \sum_{v_i^* \in W^*(a,b)} \varphi_{v_i^*} + 2(b-a) + \sum_{i=1}^{b-a} (|\Omega(\overline{v}_{a+i-1}, \overline{\overline{v}}_{a+i})| - 1),$$

where $\overline{v}_j, \overline{\overline{v}}_j \in \{v_j\} \cup \overline{U}_0 \ (j = 1, ..., k)$.

Claim 1. Let $i \in \overline{1,k}$. If either $|W^*(i,i+1)| \ne 1$ or $|W^*(i,i+1)| = 1$ and $W^*(i,i+1) \cap \{w_i, w_{i+1}\} = \emptyset$ then $w_i \overrightarrow{C} w_{i+1}$ is suitable.

Proof of claim 1. **Case a1** $|W^*(i,i+1)| = 0$.

Let $T_{tr}(E_i, E_{i+1}) = (E_i', E_{i+1}')$ and $T_{tr}(v_i, v_{i+1}) = (\overline{v}_i, \overline{\overline{v}}_{i+1})$. Then $w_i \overrightarrow{C} w_{i+1}$ is suitable, since by $(a1)$,

$$| w_i \overrightarrow{C} w_{i+1} | - 1 \ge 2 + (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1).$$

Case a2. $|W^*(i,i+1)| \ge 2$.

Let E, F be any two elements of $\{E_1^*, ..., E_r^*\}$ with $E = xEv, F = yFw$ for some $v, w \in W_i^*$. Since $T_{tr}(E,F) = (E,F)$ and $\{x,y\} \subseteq \overline{U}_0$, we have by $(a1)$, $| v \overrightarrow{C} w | - 1 \ge \varphi_x + \varphi_y + 2 + (|\Omega(x,y)| - 1)$ implying that $w_i \overrightarrow{C} w_{i+1}$ is suitable.

Case a3. $|W^*(i,i+1)| = 1$.

Assume w.l.o.g that $W^*(i,i+1) = \{w_1^*\}$. If either E_i or E_{i+1} (say E_i) has no common vertex with E_1^* then using transformation $T_{tr}(E_i, E_1^*) = (E_i', E_1^*)$, we obtain by $(a1)$,

$$| w_i \overrightarrow{C} w_{i+1} | - 1 \ge | w_i \overrightarrow{C} w_1^* | - 1 \ge \varphi_{v_1^*} + 2 + (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1)$$

for some appropriate $\overline{v}_i \in \{v_i\} \cup \overline{U}_0$ and $\overline{\overline{v}}_{i+1} = v_1^*$. It means that $w_i \overrightarrow{C} w_{i+1}$ is suitable. Now let both E_i and E_{i+1} have common vertices with E_1^*. Walking along E_1^* from w_1^* to v_1^* we stop at the first vertex $v \in V(E_i) \cup V(E_{i+1})$. Assume.w.l.o.g. that $v \in V(E_{i+1})$. Putting $E_{i+1}' = w_1^* E_1^* v E_{i+1} v_{i+1}$ and $T_{tr}(E_i, E_{i+1}') = (E_i', E_{i+1}'')$, we see by $(a1)$ that for some appropriate $\overline{v}_i \in \{v_i\} \cup \overline{U}_0$ and $\overline{\overline{v}}_{i+1} \in \{v_{i+1}\} \cup \overline{U}_0$,

$$| w_i \overrightarrow{C} w_{i+1} | - 1 \ge | w_i \overrightarrow{C} w_1^* | - 1 \ge 2 + \varphi_{v_1^*} + (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1).$$

So, again $w_i \overrightarrow{C} w_{i+1}$ is suitable.□

Claim 2. If $w_a \overrightarrow{C} w_b$ and $w_b \overrightarrow{C} w_c$ are suitable segments then $| w_a \overrightarrow{C} w_c |$ is suitable as well.

Proof of claim 2. Immediate from the definition.□

Claim 3. Let $w_a \overrightarrow{C} w_b$ is a suitable segment. If $w_b \overrightarrow{C} w_{b+1}$ is not suitable and $W^*(b,b+1) = \{w_b\}$ then $w_a \overrightarrow{C} w_{b+1}$ is suitable.

Proof of claim 3. Immediate from the definition.□

Claim 4. Let $i \in \overline{1,k}$. If $W^*(i,i+1) \subseteq \{w_i, w_{i+1}\}$ and $|W^*(i,i+1)| = 1$ (say $W^*(i,i+1) = \{w_i\}$) then $w_{i-1}\overrightarrow{C}w_{i+1}$ is suitable.

Proof of claim 4. Assume w.l.o.g. that $W^*(i,i+1) = \{w_1^*\}$, i.e. $w_1^* = w_i$. If $|W^*(i-1,i)| \geq 2$ then by claims 1 and 3, $w_{i-1}\overrightarrow{C}w_{i+1}$ is suitable. Let $|W^*(i-1,i)| = 1$, i.e $W^*(i-1,i) = \{w_1^*\}$. If either E_{i-1} or E_{i+1} (say E_{i-1}) has no common vertices with E_1^* then using transformations $T_{tr}(E_{i-1}, E_1^*)$ and $T_{tr}(E_i, E_{i+1})$, we see that $w_{i-1}\overrightarrow{C}w_i$ is suitable and by claim 3, $w_{i-1}\overrightarrow{C}w_{i+1}$ is suitable as well. Now let both E_{i-1} and E_{i+1} have common vertices with E_1^*. Walking along E_1^* from w_1^* to v_1^* we stop at the first vertex $v \in V(E_{i-1}) \cup V(E_{i+1})$. Assume w.l.o.g. that $v \in V(E_{i+1})$. If $v = \overset{o^*}{v}_1$i. e. $v_{i-1} = v_1^*$ then using $T_{tr}(E_{i-1}, w_iE_1^*vE_{i+1}v_{i+1})$ and $T_{tr}(E_i, w_{i+1}E_{i+1}vv_1^*)$ we see that $w_{i-1}\overrightarrow{C}w_i$ is suitable. By claim 3, $w_{i-1}\overrightarrow{C}w_{i+1}$ is suitable as well. Finally, if $v \neq \overset{o^*}{v}_i$ (i. e. $v_{i+1} \notin U_0$) then using $T_{tr}(E_{i-1}, w_iE_1^*vE_iv_{i+1})$ and $T_{tr}(E_i, E_{i+1})$, we see that $w_{i-1}\overrightarrow{C}w_i$ is suitable, implying by claim 3 that $w_{i-1}\overrightarrow{C}w_{i+1}$ is suitable as well.□

Claim 5. For appropriate $\overline{v}_i, \overline{\overline{v}}_i \in \{v_i\} \cup \overline{U}_0$,

$$c = \sum_{i=1}^{k} (|w_i\overrightarrow{C}w_{i+1}| - 1) \geq \sum_{i=1}^{r} \varphi_{v_i^*} + 2k + \sum_{i=1}^{k} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1).$$

Proof of claim 5. Suppose not. Let $i \in \overline{1,k}$. If $w_i\overrightarrow{C}w_{i+1}$ is not suitable then by claims 1 and 4, either $w_{i-1}\overrightarrow{C}w_{i+1}$ or $w_i\overrightarrow{C}w_{i+2}$ is suitable. Thus there exist some suitable segment on C and let $w_a\overrightarrow{C}w_b$ be the longest one for some $a, b \in \overline{1,k}$ $(a \neq b)$. If $w_b\overrightarrow{C}w_{b+1}$ is suitable then by claim 2, $w_a\overrightarrow{C}w_{b+1}$ is suitable as well, a contradiction. Otherwise, by claims 3 and 4, $w_b\overrightarrow{C}w_{b+2}$ is suitable and hence (by claim 2) $w_a\overrightarrow{C}w_{b+2}$ is suitable as well, a contradiction.□

Claim 6. If $k \geq 4$ and $h \geq 5$ then for appropriate $\overline{v}_i, \overline{\overline{v}}_i \in \{v_i\} \cup \overline{U}_0$ $(i = 1, ..., k)$,

$$\sum_{i=1}^{k} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1) \geq k\mu(T).$$

Proof of claim 6. Assume w.l.o.g that $\beta_1 = \max_i \{\beta_i\}$. Put

$$A_0 = \{i \,||\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_1\}, \quad A_1 = \overline{1,k} - A_0,$$
$$A_{1j} = \{i \in A_1 \,|u_j \in \{\overline{v}_i, \overline{\overline{v}}_{i+1}\}\} \quad (j = 1, 2).$$

We can assume that $A_1 \neq \emptyset$, since otherwise by $(i1)$, $\sum_{i=1}^{k} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1) \geq k\beta_1 \geq k\mu(T)$. If $\{\overline{v}_i, \overline{\overline{v}}_{i+1}\} = \{u_1, u_2\}$ then by $(i8)$, $i \in A_0$. It means that $A_{11} \cap A_{12} = \emptyset$. On the other hand, by $(i5)$, either $A_{11} = \emptyset$ or $A_{12} = \emptyset$. Assume w.l.o.g. that $A_{12} = \emptyset$, i.e. $A_1 = A_{11}$.

Case b1. $|A_{11}| \geq 4$.

Recalling definition 3.9, it is not hard to see that there are at least two paths among $E_1, ..., E_k$ having common vertices with $V(T(u_1)) - u_1$, i.e. $|T(u_1)| - 1 \geq 2$. By $(i2)$, $A_1 = \emptyset$, a contradiction.

Case b2. $|A_{11}| = 3$.

If follows that at least one of the paths $E_1, ..., E_k$ has a common vertex with $V(T(u_1)) - u_1$, i.e. $|T(u_1)| - 1 \geq 1$. Clearly $|T(u_1)| - 1 = 1$, since otherwise $|A_{11}| \geq 4$. Assume w.l.o.g. that $A_1 = \{1, 2, 3\}$ and $\overline{v}_1 = \overline{v}_2 = \overline{v}_3 = u_1$. If $\overline{\overline{v}}_2 = \overline{\overline{v}}_3 = \overline{\overline{v}}_4$ then clearly $\overline{v}_1, \overline{\overline{v}}_2 \in \overline{U}_0$ and by (i7), $|\Omega(\overline{v}_1, \overline{\overline{v}}_2)| - 1 \geq \beta_1$, a contradiction. Otherwise, by (i4), $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_1$ for some $i \in A_1$, again a contradiction. So, $1 \leq |A_{11}| \leq 2$.

Case b3. $|A_{11}| = 2$.

Case b3.1. $h \geq 8$.

Let $i \in A_1$. Assume w.l.o.g. that $\overline{v}_i = u_1, \overline{\overline{v}}_{i+1} = u_s$ for some $s \in \overline{1,h}$. By (i1), $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_j$ for each $j \in \overline{1,h} - \{1, h, s-1, s\}$. Since $h \geq 8$, there are at least four pairwise different integers f_1, f_2, f_3, f_4 in $\overline{1,h} - \{1, h, s-1, s\}$. By (i1), $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_{f_j}$ $(j = 1, 2, 3, 4)$. So,

$$i \in A_1 \implies |\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{h} \beta_i - \beta_1 - \beta_h - \beta_{s-1} - \beta_s + \sum_{i=1}^{4} \beta_{f_i}).$$

On the other hand,

$$i \in A_0 \implies |\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_1 = \frac{1}{h}(\sum_{i=1}^{h} \beta_i - \sum_{i=1}^{h} \beta_i + h\beta_1). \quad (17)$$

Since $h\beta_1 - \beta_1 - \beta_h - \beta_{s-1} - \beta_s \geq \sum_{i=1}^{h} \beta_i - \sum_{i=1}^{4} \beta_{f_i}$, we have

$$i \in A_0,\ j \in A_1 \implies |\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 + |\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq 2\mu(T).$$

Observing that $|A_0| \geq |A_1|$, we obtain

$$\sum_{i=1}^{k} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1) = \sum_{i\in A_0} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1) + \sum_{i\in A_1} (|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1)$$
$$\geq (|A_0| - |A_1|)\mu(T) + 2|A_1|\mu(T) = (|A_0| + |A_1|)\mu(T) = k\mu(T).$$

Case b3.2. $6 \leq h \leq 7$.

Let $i \in A_1$. Assume w.l.o.g that $\overline{v}_i = u_1, \overline{\overline{v}}_{i+1} = u_s$ for some $s \in \overline{1,h}$. We will write $i \in A_1^*$ if and only if $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_j$ for some $j \in \{1, h, s-1, s\}$.

Case b3.2.1. $A_1 = A_1^*$.

Let $i \in A_1^*$ and let $\overline{v}_i = u_1, \overline{\overline{v}}_{i+1} = u_s$ $\left(s \in \overline{1,h}\right)$. By the definition, $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_j$ for some $j \in \{h, s-1, s\}$, say $j = s$. Since $6 \leq h \leq 7$, there are at least three pairwise different integers f_1, f_2, f_3 in $\overline{1,h} - \{1, h, s-1\}$. By (i1), $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \max(\beta_{f_1}, \beta_{f_2}, \beta_{f_3})$. So,

$$i \in A_1 \implies |\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{h} \beta_i - \beta_1 - \beta_h - \beta_{s-1} + \beta_{f_1} + \beta_{f_2} + \beta_{f_3})$$

and hence we can argue exactly as in case $h \geq 8$.

Case b3.2.2. $A_1 \neq A_1^*$.

Let $A_1 = \{i, j\}$, where $i \notin A_1^*$ and let $\overline{v}_i = \overline{v}_j = u_1$, $\overline{\overline{v}}_{i+1} = u_s$, $\overline{\overline{v}}_{j+1} = u_r$ for some $s, r \in \overline{1,h}$ $(s \leq r)$. By (i8) and (i9), $4 \leq s \leq r \leq h-1$. If $s = r$ then it is easy to see (by definition 3.9) that either $u_1 \in \overline{U}_0$ or $u_s \in \overline{U}_0$ implying by (i6) that $i \in A_1^*$, a contradiction. So, assume $s \neq r$, i. e. $4 \leq s < r \leq h-1$.

Case b3.2.2.1. $h = 7$.

Case b3.2.2.1.1. $s = 4, r = 5$.

By (i5), either $|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \beta_4$ or $|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \beta_4$. Since $i \notin A_1^*$, we have $|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \beta_4$. Using (i1),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \tfrac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_7 - \beta_3 - \beta_4 + 2\beta_2 + \beta_5 + \beta_6),$$
$$|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \tfrac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_7 - \beta_5 + 2\beta_3 + \beta_6).$$

Using also all $k-2$ inequalities of type (17), we obtain the desired result as in case $h \geq 8$.

Case b3.2.2.1.2. $s=4, r=6$.

By ($i1$) and ($i9$),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_7 - \beta_3 - \beta_4 + 2\beta_2 + \beta_5 + \beta_6),$$
$$|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_5 + \beta_3 + \beta_4).$$

Apply the arguments in case b3.2.2.1.1.

Case b3.2.2.1.3. $s=5, r=6$.

By ($i1$),($i5$) and ($i9$),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_7 - \beta_4 - \beta_5 + 2\beta_2 + \beta_3 + \beta_6),$$
$$|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{7} \beta_i - \beta_1 - \beta_5 + \beta_3 + \beta_4).$$

Apply the arguments in case b3.2.2.1.1.

Case b3.2.2.2. $h=6$.

Clearly $s=4, r=5$. By ($i1$),($i5$) and ($i9$),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{6} \beta_i - \beta_1 - \beta_6 - \beta_3 - \beta_4 + 2\beta_2 + 2\beta_5),$$
$$|\Omega(\overline{v}_j, \overline{\overline{v}}_{j+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{6} \beta_i - \beta_1 + \beta_3).$$

Apply the arguments in case b3.2.2.1.1.

Case b3.3. $h=5$.

Let $A_1 = \{i,j\}$ and $\overline{v}_i = \overline{v}_j = u_1$, $\overline{\overline{v}}_{i+1} = u_s$, $\overline{\overline{v}}_{j+1} = u_r$ for some $s, r \in \overline{1,h}$ ($s \leq r$). By ($i8$) and ($i9$),$s=r=4$ and we can reach a contradiction as in case b3.2.2.

Case b4. $|A_{11}| = 1$.

Let $A_{11} = \{i\}$ and $\overline{v}_i = u_1$, $\overline{\overline{v}}_{i+1} = u_s$ for some $s \in \overline{1,h}$.

Case b4.1. $h=5$.

By ($i8$) and ($i9$), $s=4$. Also, by ($i1$) and ($i9$),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{5}(\sum_{i=1}^{5} \beta_i - \beta_1 - \beta_3 + \beta_2 + \beta_4).$$

Apply the arguments in case b3.2.2.1.1.

Case b4.2. $h \geq 6$.

There are at least two distinct integers f_1, f_2 in $\overline{1,h} - \{1, h, s-1, s\}$. By ($i1$),

$$|\Omega(\overline{v}_i, \overline{\overline{v}}_{i+1})| - 1 \geq \frac{1}{h}(\sum_{i=1}^{h} \beta_i - \beta_1 - \beta_h - \beta_{s-1} - \beta_s + 2\beta_{f_1} + 2\beta_{f_2}).$$

Since $|A_0| \geq 4 - |A_1| = 3$, we have at least two inequalities of type (17). So, we can argue as in case b3.2.2.1.1.□

By claims 5 and 6, $c \geq \sum_{i=1}^{r} \varphi_{v_i^*} + 2k + k\mu(T)$ and the result follows as in case 1.1.1.1.

Case 2. $k \geq 4, h \leq 4$.

Since $h \geq k$, we have $h = k = 4$. By a variation of Menger's theorem [7], there are four vertex-disjoint $(H,C)-$path. It can be easily cheeked that $c \geq 18$. If $\delta \leq 6$ then $c \geq 18 \geq 20(\delta+2)/9 = (h+1)k(\delta+2)/(h+k+1)$. Let $\delta \geq 7$. Using $(d3)$ we can show that $\varphi_i \leq 3$ for some $i \in \overline{1,4}$, i.e. $\max_i \psi_i \geq \delta - 3$. Then by lemma 3, $c \geq \sum_{i=1}^{4}(\delta - \varphi_i) + \delta - 3 = 5\delta - 3 - \sum_{i=1}^{4}\varphi_i$. If $\sum_{i=1}^{4}\varphi_i \leq 12$ then $c \geq 5\delta - 15 \geq (h+1)k(\delta+2)/(h+k+1)$. So, it sufficies to prove $\sum_{i=1}^{4}\varphi_i \leq 12$.

Case 2.1. Either $|U_0| = 0$ or $|U_0| = 4$.

It follows that $\varphi_i \leq 3$ for each $i \in \overline{1,4}$.

Case 2.2. $|U_0| = 3$.

Assume w.l.o.g. that $\overline{U}_0 = \{u_1\}$. If $u_3 \overset{o}{u}_1 \notin E$ then it is easy to see that $\varphi_i \leq 3$ for each $i \in \overline{1,4}$. Otherwise ($u_3 \overset{o}{u}_1 \in E$), $u_2 u_4 \notin E$ and hence $\varphi_1 \leq 3, \varphi_3 \leq 4, \varphi_2 \leq 2$ and $\varphi_4 \leq 2$.

Case 2.3. $|U_0| = 2$.

By symmetry, we can distinguish the following two cases.

Case 2.3.1. $\overline{U}_0 = \{u_1, u_4\}$.

If $u_3 \overset{0}{u}_1 \notin E$ and $u_2 \overset{0}{u}_4 \notin E$ then clearly $\varphi_i \leq 3$ for each $i \in \overline{1,4}$. Assume w.l.o.g. that $u_3 \overset{0}{u}_1 \in E$. We can assume also $u_2 \overset{0}{u}_4 \notin E$, since otherwise the cycle $u_1 \overset{0}{u}_1 u_3 u_4 \overset{0}{u}_4 u_2 u_1$ is larger than H, which is impossible. Then clearly $\varphi_1 \leq 3, \varphi_4 \leq 3, \varphi_3 \leq 4$ and $\varphi_2 \leq 2$.

Case 2.3.2. $\overline{U}_0 = \{u_1, u_3\}$.

It is easy to see that $\varphi_i \leq 3$ for each $i \in \overline{1,4}$.

Case 2.4. $|U_0| = 1$. Returning to the proof of lemma 3, we can see that in this special case the lower bound in lemma 3 can be improved by a unit. So, it suffices to show $\sum_{i=1}^{4}\varphi_i \leq 13$. Denoting $U_0 = \{u_4\}$, we see that $\varphi_1 \leq 3, \varphi_2 \leq 3, \varphi_3 \leq 3, \varphi_4 \leq 4$ and the result holds immediately.

Case 3. $k \leq 3$.

Claim 7. Let G be a $k-$connected ($k \in \{2,3\}$) graph with $h \geq k$ and without $k+1$ vertex-disjoint $(H,C)-$paths. Then

$$c \geq \min(k(h+1), k(\delta - k + 4)).$$

Proof of claim 7. Case d1. $k = 3$.

Assume w.l.o.g. that E_1, E_2 and E_3 are $T-$transformed. We now prove that $|w_1 \overrightarrow{C} w_2| - 1 \geq \min(h+1, \delta+1)$. If $v_2 = v_1^+$ then clearly

$$|w_1 \overrightarrow{C} w_2| - 1 \geq |w_1 E_1 v_1 \overleftarrow{H} v_2 E_2 w_2| - 1 \geq h+1.$$

Let $v_2 \neq v_1^+$. Walking along $\overrightarrow{H}$ from v_1 to v_2^- we stop at the first vertex z with either $\hat{z} w_2 \in E$ or $\hat{z} w_1 \notin E$ or $z = v_2^-$. If $\hat{z} w_2 \in E$ or $z = v_2^-$ then clearly $|w_1 \overrightarrow{C} w_2| - 1 \geq h+1$. Let $\hat{z} w_2 \in E$ and $\hat{z} w_1 \notin E$. If $\hat{z} w \in E$ for some $w \in V(C) - \{w_1, w_2, w_3\}$ then there are 4 vertex-disjoint $(H,C)-$paths, contradicting our assumption. So, $N(\hat{z}) \cap V(C) \subseteq \{w_3\}$, i.e. $\varphi_z \geq \delta - 1$ and $h \geq \varphi_z + 1 \geq \delta$. By $(g6)$, $|O(z^-, v_2)| - 1 \geq \gamma_z \geq \varphi_z \geq \delta - 1$ implying that $|w_1 \overrightarrow{C} w_2| - 1 \geq \delta + 2$. Thus we have proved $|w_1 \overrightarrow{C} w_2| - 1 \geq \min(h+1, \delta+1)$. By symmetry, we have similar inequalities for segments $w_2 \overrightarrow{C} w_3$ and $w_3 \overrightarrow{C} w_1$ and the result holds from $h+1 \geq \delta+1$.

Case d2. $k = 2$.

Apply the arguments in case 1. Claim 7 is proved.□

Case 3.1. $k = 3$.

We can assume that there are no 4 vertex-disjoint (H, C)−paths, since otherwise

$$c \geq \frac{(h+1)\,4}{h+4+1}\,(\delta+2) > \frac{(h+1)\,k}{h+k+1}\,(\delta+2).$$

Then by claim 7 we can distinguish the following two cases.

Case 3.1.1. $c \geq 3\,(h+1)$.

If $h \geq \delta - 2$ then $c \geq 3\,(h+1) \geq 3\,(h+1)\,(\delta+2)\,/\,(h+4)$. Otherwise, the result holds from $c \geq 3\,(\delta-1)$ (see [12]).

Case 3.1.2. $c \geq 3\,(\delta+1)$.

If $h \leq 3\delta+2$ then $c \geq 3\,(\delta+1) \geq 3\,(h+1)\,(\delta+2)\,/\,(h+4)$. Let $h \geq 3\,(\delta+1)$. Observing that $c \geq 3\,(h/2+2)$ (by standard arguments) we obtain the result immediately.

Case 3.2. $k = 2$.

Apply the arguments in case 1.

Thus we have proved the theorem for h a longest cycle in $G - C$. Observing that

$$c \geq \frac{(h+1)\,k}{h+k+1}\,(\delta+2) > \frac{\left(h'+1\right)k}{h'+k+1}\,(\delta+2)$$

for any $h' < h$, we complete the proof of the theorem.□

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