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Cycle-Extensions and Long Cycles in Graphs

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Abstract

Let G be a graph with minimum degree δ, C be a longest cycle in G and h be the length of a longest cycle in $G-C$. Then $|C| \geq (h+1)(\delta-h+1)$.

1 Introduction

Our main purpose is to give lower bounds for the length c (the circumference) of a longest cycle C in graph G in terms of the minimum degree $\delta(G)$ and some structures of $G-C$.

In 1998 a notion of path-extensions was introduced [4] and a lower bound for the circumference is obtained in terms of $\delta(G)$ and the length q of a longest path in $G-C$.

$(A) \qquad c \geq (q+2)(\delta-q)$.

In this paper we present a similar result in terms of $\delta(G)$ and the length h of a longest cycle in $G-C$.

Theorem. $\quad c \geq (h+1)(\delta-h+1)$.

These two results show how the path and cycle structures of $G-C$ impact on circumference and cycle structures of G. In view of the main purpose the following results can be considered as starting points for $h \leq 1$,

$(B) \qquad k \geq 2 \quad \Longrightarrow c \geq 2\delta$ or $h=0$ (1952,Dirac[2]),
$(C) \qquad k \geq 3 \quad \Longrightarrow c \geq 3(\delta-1)$ or $h \leq 1$ (1977,Voss[5]),

where k denote the vertex-connectivity of G. In the next result [3] some bounds of the type $c \geq k(\delta-k+2)$ are established for k-connected graphs with a rather strong condition with respect to $G-C$ (for any two vertices x, y in some component of $G-C$, there is a path of length at least $k-2$ with endvertices x and y).

The bound $(h+1)(\delta-h+1)$ in the theorem is shurp, as can be seen from the following family of graphs. Take $k+1$ disjoint copies of the complete graph $K_{\delta-k+1}$ and join each vertex in their union to every vertex of a disjoint complete graph K_k. This graph $G = (k+1)K_{\delta-k+1} + K_k$ is clearly not hamiltonian. Moreover, $h = \delta-k+1$ and hence $c = k(\delta-k+2) = (h+1)(\delta-h+1)$.

2 Terminology

We consider only finite undirected graphs without loops and multiple edges. For unexplained terminology see [1]. The vertex set of a graph G is denoted by $V(G)$ or just V; the set of edges by $E(G)$ or just E. We use $|G|$ as a symbol of the cardinality $|V(G)|$. For a subset S of V, $G-S$ denotes the subgraph $\langle V-S\rangle$ induced by $V-S$. If H is a subgraph of G, we also use the symbol $G-H$ for $G-V(H)$.

Paths and cycles in graph G are considered as subgraphs of G, they are connected and have maximum degree 0, 1 or 2. The length of path P is $|P|-1$. For convenience, every edge (resp., vertex) will be interpreted as a cycle of length 2 (resp.,1). By the definition, G is hamiltonian iff $h=0$. If $h=1$ then $V-V(C)$ is an independent set of vertices or, in other words, C is a dominating cycle of G. Let c (the circumference) denote the length of a longest cycle in G. An (x,y)-path is a path with endvertices x and y. Given an (x,y)-path L of G we denote by $\overrightarrow{L}$ the path L with an orientation from x to y. If $u,v\in V(L)$ then $u\overrightarrow{L}v$ denote the consequtive vertices on L from u to v in the direction specified by $\overrightarrow{L}$. The same vertices, in reverse order, are given by $v\overleftarrow{L}u$. For $\overrightarrow{L}=x\overrightarrow{L}y$ and $u\in V(L)$, let $u^+(\overrightarrow{L})$ (or just u^+) denote the successor of u $(u\neq y)$on $\overrightarrow{L}$ and u^- denote its predecessor $(u\neq x)$. If $A\subseteq V(L)$ then $A^+=\{v^+\mid v\in A-y\}$ and $A^-=\{v^-\mid v\in A-x\}$. If Q is a cycle in G and $A\subseteq V(Q)$ then $\overrightarrow{Q}, A^+$ and A^- are analogously defined. For $v\in V(Q)$, $v\overrightarrow{Q}v$ will be interpreted as a vertex v. For $v\in V$, put $N(v)=\{u\in V\mid uv\in E\}$, $d(v)=|N(v)|$ and $\delta=\min\{d(u)\mid u\in V\}$.

3 Special definitions

We begin introducing some special definitions and convenient notations. For the remainder of this section let a longest cycle C in graph G and a longest cycle $H=u_1...u_hu_1$ in $G-C$ be fixed.

Definition 3.1 T is an HC-extension; $T(u_i)$; $\mathring{u}$; $\hat{u}$.

Let $T(u_1),...,T(u_h)$ are vertex-disjoint $(u_i,\hat{u}_i)$-paths in $G-C$ for $i=1,...,h$ respectively. The union $T=\bigcup_{i=1}^{h}T(u_i)$ is called HC-extension if $N(\hat{u}_i)\subseteq V(T)\cup V(C)$ for each $i=\overline{1,h}$. An HC-extension T is called maximal if it is chosen so as to maximize $|\{u\in V(H)\mid u\neq\hat{u}\}|$. If $u\neq\hat{u}$ for some $u\in V(H)$ then we use $\mathring{u}$ to denote $u^+(\overrightarrow{T}(u))$.

Definition 3.2 $\Phi_u;\varphi_u;\Psi_u;\psi_u$.

Let T be a maximal HC-extension. For each $u\in V(H)$, put

$$\Phi_u=N(\hat{u})\cap V(T),\qquad \varphi_u=|\Phi_u|,$$
$$\Psi_u=N(\hat{u})\cap V(C),\qquad \psi_u=|\Psi_u|.$$

Definition 3.3 $U_0;U_1;U^*$.

For T a maximal HC-extension, put $U_0=\{u\in V(H)\mid u=\hat{u}\}$ and

$$U_1=\{u\in V(H)-U_0\mid \Phi_u\not\subseteq V(T(u))\},U^*=V(H)-(U_0\cup U_1).$$

Definition 3.4 $B_u;B_u^*;b_u;b_u^*$.

Let T be a maximal HC-extension. For each $u\in V(H)$, put $B_u=\left\{v\in U_0\mid v\,\mathring{u}\in E\right\}$. Clearly $B_u=\emptyset$ if $u\in U_0$. Furthermore, for each $u\in U_0$ put $B_u^*=\left\{v\in V(H)\mid u\,\mathring{v}\in E\right\}$. Clearly $B_u^*\subseteq V(H)-U_0$. Let $b_u=|B_u|$ and $b_u^*=|B_u^*|$.

4 Results

Our main purpose is to prove the following result.

Theorem. Let G be a graph with minimum degree δ, C be a longest cycle in G and h be the length of a longest cycle in $G-C$. Then $|C| \geq (h+1)(\delta-h+1)$.

We need the following two preliminary results.

Lemma 1. Let C be a longest cycle of a graph G, Q be a path in $G-C$ and $P_i = v_i\overrightarrow{P_i}w_i \quad (i=0,...,q)$ are vertex-disjoint paths in $G-C$ having only $v_0,...,v_q$ in common with Q. Then

$$c \geq \sum_{i=0}^{q} |Z_i| + \left|\bigcup_{i=0}^{q} Z_i\right|,$$

where $Z_i = N(w_i) \cap V(C) \qquad (i=0,...,q)$.

Lemma 2. Let C be a longest cycle of a graph G and $H = u_1...u_hu_1$ a longest cycle of $G-C$ with a maximal HC-extension T. Then for each $u \in U_1$,

(b1) $\quad |T(u)|-1 \geq 2 \implies h \geq 2(\varphi_u + b_u) \geq \varphi_u + b_u + 1,$

(b2) $\quad |T(u)|-1 = 1 \implies h \geq 2\varphi_u \geq \varphi_u + b_u + 1.$

5 Proofs

Proof of lemma 1. We shall prove the result for the case $v_i = w_i \quad (i=0,...,q)$, since otherwise the arguments are the same. The result is immediate if $\bigcup_{i=0}^q Z_i = \emptyset$.Let $\bigcup_{i=0}^q Z_i \neq \emptyset$ and let $\xi_1,...,\xi_m \quad (m \geq 1)$ be the elements of $\bigcup_{i=0}^q Z_i$ occurring on $\overrightarrow{C}$ in consequtive order. Set

$$F_i = N(\xi_i) \bigcap \{w_0,...,w_q\} \qquad (i=1,...,m).$$

Suppose that $m=1$. If $|F_1|=1$ then $q=0$ and $Z_0 = Z_q = \{\xi_1\}$ implying that

$$c \geq 2 = \sum_{i=0}^{q} |Z_i| + \left|\bigcup_{i=0}^{q} Z_i\right|.$$

Let $|F_1| \geq 2$. Choosing $u,v \in F_1 \quad (u \neq v)$ such that $\left|u\overrightarrow{Q}v\right|$ is maximum,

$$c \geq \left|\xi_1 u\overrightarrow{Q}v\xi_1\right| \geq \sum_{i=0}^{q} |Z_i| + 1 = \sum_{i=0}^{q} |Z_i| + \left|\bigcup_{i=0}^{q} Z_i\right|.$$

Thus we may assume $m \geq 2$. It means, in particular, that $c \geq 3$. For $i=1,...,m$, put $f(\xi_i) = \left|\xi_i\overrightarrow{C}\xi_{i+1}\right| - 1 \quad$ (indices mod m). It is easy to see that

$$c = \sum_{i=1}^{m} f(\xi_i), \quad \sum_{i=1}^{m} |F_i| = \sum_{i=0}^{q} |Z_i|, \quad m = \left|\bigcup_{i=0}^{q} Z_i\right|. \tag{1}$$

For every $i \in \overline{1,m}$ choose $x_i, y_i \in F_i \cup F_{i+1}$ such that $\left|x_i\overrightarrow{Q}y_i\right|$ is maximum (indices mod m).

Claim 1.1 $\quad f(\xi_i) \geq (|F_i| + |F_{i+1}| + 2)/2 \qquad (i=1,...,m)$.

Proof of claim 1.1 **Case 1** Either $x_i \in F_i,\ y_i \in F_{i+1}$ or $x_i \in F_{i+1},\ y_i \in F_i$.

If $x_i \in F_i,\ y_i \in F_{i+1}$ then
$$f(\xi_i) \geq \left|\xi_i x_i \overrightarrow{Q} y_i \xi_{i+1}\right| - 1 \geq \max(|F_i|, |F_{i+1}|) + 1 \geq (|F_i| + |F_{i+1}| + 2)/2.$$
Otherwise, the result holds from $f(\xi_i) \geq \left|\xi_i y_i \overleftarrow{Q} x_i \xi_{i+1}\right| - 1$.

Case 2 Either $x_i,\ y_i \in F_i$ or $x_i,\ y_i \in F_{i+1}$.

First suppose $x_i,\ y_i \in F_i$. We can assume $x_i,\ y_i \notin F_{i+1}$, since otherwise we are in case 1. Choose $x_i',\ y_i' \in F_{i+1}$ such that $\left|x_i' \overrightarrow{Q} y_i'\right|$ is maximum. If $\left|x_i \overrightarrow{Q} x_i'\right| - 1 \geq (|F_i| - |F_{i+1}|)/2$ then
$$f(\xi_i) \geq \left|\xi_i x_i \overrightarrow{Q} y_i' \xi_{i+1}\right| - 1 \geq (|F_i| - |F_{i+1}|)/2 + |F_{i+1}| + 1 = (|F_i| + |F_{i+1}| + 2)/2.$$
Otherwise,
$$\begin{aligned} f(\xi_i) &\geq \left|\xi_i y_i \overleftarrow{Q} x_i' \xi_{i+1}\right| - 1 = \left|x_i' \overrightarrow{Q} y_i\right| + 1 = \left|x_i \overrightarrow{Q} y_i\right| - \left|x_i \overrightarrow{Q} x_i'\right| + 2 \\ &\geq |F_i| - (|F_i| - |F_{i+1}| + 1)/2 + 2 > (|F_i| + |F_{i+1}| + 2)/2. \end{aligned}$$
By symmetry, the case $x_i,\ y_i \in F_{i+1}$ requires the same arguments. Claim 1.1 is proved.◁

By claim 1.1,
$$\sum_{i=1}^{m} f(\xi_i) \geq \sum_{i=1}^{m} (|F_i| + |F_{i+1}| + 2)/2 = \sum_{i=1}^{m} |F_i| + m,$$
and the result holds from (1).◁

Proof of lemma 2. For each $u, v \in V(H)$, put $A_u(v) = (\Phi_u \cup B_u) \cap V(T(v))$. Let $\rho_u(v)$ denote the vertex in $A_u(v)$ maximizing $\left|v \overrightarrow{T}(v) \rho_u(v)\right|$. In particular, $\rho_u(u) = \hat{u}^-$. Put $\overline{\rho}_u(v) = \hat{u}$ if $\rho_u(v) \in \Phi_u$ and $\overline{\rho}_u(v) = \mathring{u}$ if $\rho_u(v) \in B_u$. Clearly $\overline{\rho}_u(u) = \hat{u}$. Putting $\Lambda_u = \{v \in V(H) \mid A_u(v) \neq \emptyset\}$, we use also the following abbreviation
$$\Lambda_u(v, w) = vT(v)\rho_u(v)\overline{\rho}_u(v)T(u)\overline{\rho}_u(w)\rho_u(w)T(w)w$$
for each $v, w \in \Lambda_u \quad (v \neq w)$. Let $\Lambda_u = \{\xi_1, ..., \xi_f\}$. Assume w.l.o.g that $u = \xi_1$ and $\xi_1, ..., \xi_f$ occurs on H in consequtive order. For each integer $i \quad (1 \leq i \leq f)$ let
$$M_i = \xi_i \overrightarrow{H} \xi_{i+1}, \quad \omega_i = |A_u(\xi_i)| + |A_u(\xi_{i+1})| \qquad \text{(indices mod } f).$$
Since H is extreme,
$$|M_i| \geq |\Lambda_u(\xi_i, \xi_{i+1})| \qquad (i = 1, ..., f). \tag{2}$$
Let $\xi_r \overrightarrow{H} \xi_s$ be the longest segment on H with
$$\xi_1 \in V\left(\xi_r \overrightarrow{H} \xi_s\right), \qquad \{\xi_r, \xi_{r+1}, ..., \xi_s\} \subseteq B_u \cup \{u\}.$$
Put
$$\begin{aligned} \Omega^+ &= \{M_i \in \{M_2, ..., M_{f-1}\} \mid \overline{\rho}_u(\xi_i) \neq \overline{\rho}_u(\xi_{i+1})\}, \\ \Omega^- &= \{M_i \in \{M_1, M_f\} \mid \overline{\rho}_u(\xi_i) \neq \overline{\rho}_u(\xi_{i+1})\}, \\ \Omega^0 &= \{M_1, ..., M_f\} - (\Omega^+ \cup \Omega^-). \end{aligned}$$
Observe that $|\Omega^-| \leq 2$ and $|M_i| - 1 \geq |\Lambda_u(\xi_i, \xi_{i+1})| - 1$ for each $i \in \overline{1, f}$. Then clearly
$$M_i \in \Omega^+ \implies |M_i| - 1 \geq \omega_i + |A_u(u)| - 1, \tag{3}$$
$$M_i \in \Omega^- \implies |M_i| - 1 \geq \omega_i - |A_u(u)| + 1, \tag{4}$$
$$M_i \in \Omega^0 \implies |M_i| - 1 \geq \omega_i. \tag{5}$$

Claim 2.1 If $|\Omega^-| = 0$ then $|M_i| - 1 \geq \omega_i \quad (i = 1, ..., f)$.

Proof of claim 2.1 Immediate from (3), (4) and (5)$\triangleleft$

Claim 2.2 If $|\Omega^-| = 1$, say $\Omega^- = \{M_1\}$, then $M_s \in \Omega^+$.

Proof of claim 2.2 By the definition, $\{\xi_2, ..., \xi_s\} \subseteq B_u$ and $\xi_{s+1} \in \Lambda_u - (B_u \cup \{u\})$ and the result follows.$\triangleleft$

Claim 2.3 If $|\Omega^-| = 2$, i.e. $\Omega^- = \{M_1, M_f\}$, then $M_s, M_{r-1} \in \Omega^+$.

Proof of claim 2.3 By the definition, $\{\xi_2, \xi_f, \xi_s, \xi_r\} \subseteq B_u$ and $\xi_{s+1}, \xi_{r-1} \in \Lambda_u - (B_u \cup \{u\})$ and the result follows.$\triangleleft$

Claim 2.4 $\sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i$.

Proof of claim 2.4 Immediate from (3), (4),(5) and claims 2.1, 2.2, 2.3$\triangleleft$

Claim 2.5 $|T(u)| - 1 \geq 2 \implies \Phi_u \cap B_u = \emptyset$.
Proof of claim 2.5 **Case 1.** $u \in U_1$.
Suppose, to the contrary, that $\Phi_u \cap B_u \neq \emptyset$. If $z \in \Phi_u \cap B_u$ then by definitions 3.1 and 3.2, the collection of paths

$$\left\{T(u_1), ..., T(u_h), u\overset{\circ}{u},\ z\hat{u}\right\} - \{T(u), T(z)\}$$

generates some $HC-$extension, contradicting the maximality of T.
Case 2. $u \in U^*$.
By definition 3.3, $\Phi_u \subseteq V(T(u))$ and the result follows immediately.$\triangleleft$
(b1) By claim 2.5, $|\Phi_u \cup B_u| = \varphi_u + b_u$. Observing that $\sum_{i=1}^{f} |A_u(\xi_i)| = |\Phi_u \cup B_u|$, we obtain by claim 2.4,

$$\begin{aligned} h &= \sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i = \sum_{i=1}^{f} (|A_u(\xi_i)| + |A_u(\xi_{i+1})|) = \\ &= 2\sum_{i=1}^{f} |A_u(\xi_i)| = 2|\Phi_u \cup B_u| = 2(\varphi_u + b_u) \geq \varphi_u + b_u + 1. \end{aligned}$$

(b2) Clearly $\varphi_u \geq b_u + |\{u\}| = b_u + 1$. By claim 2.4,

$$h = \sum_{i=1}^{f} (|M_i| - 1) \geq \sum_{i=1}^{f} \omega_i = 2\left|\Phi_u \bigcup B_u\right| = 2\varphi_u \geq \varphi_u + b_u + 1$$

which completes the proof of lemma 2.$\triangleleft$

Proof of the theorem. Let $H = u_1...u_h u_1$ be a longest cycle in $G - C$ with a maximal $HC-$extension T. If $h = 0$ then clearly $c = |V| \geq \delta + 1 \geq (h+1)(\delta - h + 1)$. So we can assume that $h \geq 1$. Let $u \in U_0$ and $v \in V(H) - U_0$. If $uz \in E$ for some $z \in V(T(v))$ then $z \in \left\{v, \overset{\circ}{v}\right\}$, since otherwise the collection of paths

$$\left\{T(u_1), ..., T(u_h), v\overset{\circ}{v},\ uz\right\}$$

generates another HC–extension, contracting the maximality of T. In other words,

$$u \in U_0, v \in V(H) - U_0 \implies \Phi_u \bigcap V(T(v)) \subseteq \left\{v, \overset{\circ}{v}\right\}.$$

Recalling definitions 3.2 and 3.4, we obtain

$$u \in U_0 \implies \varphi_u \leq h - 1 + b_u^*. \tag{6}$$

Also, for each $u \in U_1$, $h \geq \varphi_u + b_u + 1$ (by lemma 2), i.e.

$$u \in U_1 \implies \varphi_u \leq h - 1 - b_u. \tag{7}$$

Case 1 $U^* = \emptyset$.
By (6) and (7),

$$\sum_{u \in U_0} \varphi_u \leq |U_0|(h-1) + \sum_{u \in U_0} b_u^*, \quad \sum_{u \notin U_0} \varphi_u \leq (h - |U_0|)(h-1) - \sum_{u \notin U_0} b_u.$$

Observing that $\sum_{u \in U_0} b_u^* = \sum_{u \notin U_0} b_u$, we obtain

$$\sum_{u \in V(H)} \varphi_u = \sum_{i=1}^{h} \varphi_{u_i} \leq h(h-1). \tag{8}$$

By definition 3.2, $\varphi_{u_i} + \psi_{u_i} = d\left(\hat{u}_i\right) \geq \delta \quad (i = 1, ..., h)$. Using (8),

$$\sum_{i=1}^{h} \psi_{u_i} = \sum_{i=1}^{h} \left(d\left(\hat{u}_i\right) - \varphi_{u_i}\right) \geq \sum_{i=1}^{h} d\left(\hat{u}_i\right) - h(h-1).$$

It follows, in particular, that $\max_i \{\psi_{u_i}\} \geq \frac{1}{h} \sum_{i=1}^{h} d\left(\hat{u}_i\right) - h + 1$. Thus

$$\sum_{i=1}^{h} \psi_{u_i} + \max_i \{\psi_{u_i}\} \geq (h+1)\left(\frac{1}{h}\sum_{i=1}^{h} d\left(\hat{u}_i\right) - h + 1\right) \geq (h+1)(\delta - h + 1)$$

which by lemma 1 gives the desired result immediately.

Case 2 $U^* \neq \emptyset$.

Assume that T is chosen so as to minimize $|U^*|$. Let $v \in U^*$. By the definition, $\Phi_v \subseteq V(T(v))$. Let $x_1, ..., x_t$ be the elements of Φ_v^- occurring on $\overrightarrow{T}(v)$ in consequtive order such that $x_1 = \hat{v}$. Clearly $t = |\Phi_v| = \varphi_v$. Since $T(v)$ can be replaced by $v\overrightarrow{T}(v)x_i^+ \hat{v} \overleftarrow{T}(v)x_i$ for each $i \in \overline{1,t}$, we can assume w.l.o.g. that $N(x_i) \subseteq V(T(v)) \quad (i = 1, ..., t)$ and

$$\varphi_v = \max_i \left\{N(x_i) \bigcap V(T)\right\}.$$

Finally, assume that v is chosen from U^* so as to maximize φ_v. Putting $Z_i = N(x_i) \bigcap V(C)$ $(i = 1, ..., t)$ we see that $\varphi_v \geq d(x_i) - |Z_i| \geq \delta - |Z_i| \quad (i = 1, ..., t)$, i.e.

$$|Z_i| \geq \delta - \varphi_v \quad (i = 1, ..., t). \tag{9}$$

Claim 1. $c \geq (h+1)(\delta - h + 1) + (\varphi_v - 1)(\delta - \varphi_v) - (h+1)|U^*|(\varphi_v - 1)/2h.$

Proof of claim 1 If $u \in U^*$ then clearly $h \geq \varphi_u + 1$ and $h \geq 2(b_u + 1)$, implying that $h \geq b_u + 1 + (\varphi_u + 1)/2$ or, equivalently,

$$u \in U^* \implies \varphi_u \leq h - b_u - 1 + (\varphi_u - 1)/2.$$

Combining this with (6) and (7), as in case 1,

$$\sum_{i=1}^{h} \psi_{u_i} + \max_i \{\psi_{u_i}\} \geq (h+1)(\delta - h + 1) - (h+1)|U^*|(\varphi_v - 1)/2h. \tag{10}$$

Using lemma 1 for $Q = \hat{v}\overleftarrow{T}(v)v\overrightarrow{H}v^-$,

$$c \geq \sum_{i=1}^{h} \psi_{u_i} + \max_i \{\psi_{u_i}\} + \sum_{i=2}^{t} |Z_i|,$$

which by (9) and (10) completes the proof of claim 1.◁

Clearly $h \geq \varphi_v+1$. On the other hand, $h \leq \delta$, since otherwise $c \geq 0 \geq (h+1)(\delta - h + 1)$. So,

$$u \in U^* \implies \varphi_u + 1 \leq h \leq \delta. \tag{11}$$

By (11), $\varphi_u(\varphi_u - \delta + 1) \leq 0$ for each $u \in U^*$, which is equivalent to

$$u \in U^* \implies \delta^2/4 - \varphi_u \geq (\varphi_u - \delta/2)^2. \tag{12}$$

Claim 2 If $h \geq 2$ and $\delta - h + 1 \geq 2$ then

$$h(\delta - h + 1)(\delta - h - 1 - \frac{1}{h}) + \frac{\delta + 1}{2} \geq 0. \tag{13}$$

Proof of claim 2 Clearly $\delta \geq h + 1 \geq 3$. If $\delta - h + 1 = 2$ then the desired result is equivalent to $\delta \geq 3$. Otherwise, $\delta - h - 1 - 1/h > 0$ and the result follows.◁

Claim 3 $c \geq \frac{(\delta+1)}{2}|U^*| \implies c \geq (h+1)(\delta - h + 1)$.

Proof of claim 3 If either $h \leq 1$ or $\delta - h + 1 \leq 1$ then it is not hard to see that $c \geq (h+1)(\delta - h + 1)$. Let $h \geq 2$ and $\delta - h + 1 \geq 2$. By claim 2, we have inequality (13) which is equivalent to

$$\frac{h(\delta+1)}{h+1}(\delta - h + 1 + \frac{1}{2h}) \geq (h+1)(\delta - h + 1). \tag{14}$$

If $(h+1)|U^*| \leq 2h(\delta - \varphi_v)$ then we are done by claim 1. Otherwise, using $h \geq \varphi_v + 1$ (by (11)), or, in other words, $\delta - \varphi_v \geq \delta - h + 1$, we have $(h+1)|U^*| \geq 2h(\delta - h + 1) + 1$. But this is equivalent to

$$\frac{\delta+1}{2}|U^*| \geq \frac{h(\delta+1)}{h+1}(\delta - h + 1 + \frac{1}{2h})$$

and the result follows from (14).◁

If $|U^*| = 1$ then by (11), $(\varphi_v - 1)(\delta - \varphi_v) \geq (\varphi_v - 1)/2$ and the result holds by claim 1 immediately. So, we assume $|U^*| \geq 2$. Recalling, how v is chosen from U^* and how x_i, Z_i are defined for $T(v)$, we now choose $w \in U^* - v$ and define y_i, Z_i' $(i = 1, ..., r)$ for $T(w)$ by the same way. Also, choose $\varphi \in \{\varphi_v, \varphi_w\}$ such that

$$\varphi(\delta - \varphi) = \min(\varphi_v(\delta - \varphi_v), \quad \varphi_w(\delta - \varphi_w)).$$

Claim 4 $c \geq (2\varphi + 1)(\delta - \varphi)$.

Proof of claim 4 Using (9) with lemma 1 for $Q = \hat{v}\overleftarrow{T}(v)v\overrightarrow{H}w\overrightarrow{T}(w)\hat{w}$,

$$c \geq \sum_{i=1}^{t} |Z_i| + \sum_{i=1}^{r} \left|Z_i'\right| + \max\left(|Z_1|, ..., |Z_t|, \left|Z_1'\right|, ..., \left|Z_r'\right|\right)$$
$$\geq 2\varphi(\delta - \varphi) + \max(\delta - \varphi_v, \delta - \varphi_w) \geq (2\varphi + 1)(\delta - \varphi). \lhd$$

If $|N(\hat{u}) \cap V(C)| \geq (\delta + 1)/2$ for all $u \in U^*$ except one, then by lemma 1,

$$c \geq (|U^*| - 1)(\delta + 1)/2 + (\delta + 1)/2 = |U^*|(\delta + 1)/2,$$

which by claim 3 gives the desired result. Otherwise, $\varphi_{u_1} \geq \delta/2$ and $\varphi_{u_2} \geq \delta/2$ for some $u_1, u_2 \in U^*$ implying that $\varphi \geq \delta/2$ as well. Recalling also that $h \geq \varphi + 1$ (from (11)) and using (12) we obtain

$$(\varphi - \delta/2)^2 < (h - \delta/2)^2 \leq (h - \delta/2)^2 + \left(\delta^2/4 - \varphi\right) - (\varphi - \delta/2)^2$$

which is equivalent to $(2\varphi + 1)(\delta - \varphi) \geq (h + 1)(\delta - h + 1)$. Then the desired result holds from claim 4 immediately. $\lhd$

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