Mathematical Problems of Computer Science 21, 2000, 17-22.

Generalized B-splines Using for Hybrid Filter Design

Hegine R. Tshughuryan

Institute for Informatics and Automation Problems of NAS RA and YSU

Abstract

This paper is dedicated to the design of the hybrid (linear - nonlinear) filters. It is introduced new type of spline-functions and the interconnection with the usual B-splines is obtained. In this paper it is also defined the discrete analogue of the proposed function. The algorithm for hybrid spline- FIR - median filtering is elaborated.

1 Intoduction

The fast development of theory of spline-functions is caused, in mainly, by two factors, namely, splines have fine convergence to the approximated objects and they are simple for realization. Therefore the apparatus of spline -functions is successfully used to solve some problems, such as approximate representation and restoration of signals, smoothing, information preliminary processing (filtering, compression)[1, 2, 3]

Due to some helpful properties (finiteness of support, smoothing of experimental distortion) the spline-functions have the wide application in digital filtering [2, 3, 6, 7].The B-splines are special interesting, particularly cubic B-spline, as it is the impulse response of known Parsen window[3]. However, in some cases the use of the B-splines does not provide required approximation precision. Therefore, new types and generalizations of the B-splines are studied. So, there were introduced β-splines[4], which are used to construct visually smooth (i.e. satisfying to the condition of second degree geometrical continuity) curves, alternate splines, being B-splines at the special choice of knots[5].

In section 2 new type of B-spline, namely the generalized or parametric $B(p_1, ..., p_k)$-spline is defined and the interconnection with the usual B-spline is obtained.

In section 3 it is introduced the discrete generalized $B(p_1, ..., p_k)$-spline and the algorithm of hybrid spline-FIR-median (Finite Impulse Response) filtering is given.

2 Parametric or generalized $B(p_1, ..., p_k)$-spline

Let's $\tau = \{\tau_i\}$ be the sequence of knots.

Definition 1.1

Generalized $B(p_1, \dots, p_k)$-spline $(B(p_1, \dots, p_k))$ degree k (order k+1, $k \geq 0$) for the knot sequence τ and parameters $p_1, \dots, p_k$ ($p_k \in N$) denoted by $G^{(p_1,\dots,p_k)}_{i,k+1,\tau}$, is defined recursively by the following procedure:

$$G_{i,1,\tau}(x) = B_{i,1,\tau}(x) = \begin{cases} 1, & x \in [\tau_i, \tau_{i+1}) \\ 0, & x \notin [\tau_i, \tau_{i+1}) \end{cases} \quad (1)$$

and

$$G_{i,k+1,\tau}^{(p_1,\ldots,p_k)}(x) = A_{i,k+1,\tau}(x) - A_{i+p_k,k+1,\tau}(x) \tag{2}$$

for $k \geq 1$ $p_k \geq 1$, where

$$A_{i,k+1,\tau}(x) = \begin{cases} \int_{\tau_i}^{x} G_{i,k,\tau}^{(p_1,\ldots,p_{k-1})}(s)\, ds / \delta_{i,k,\tau}^{(q_{k-1})}, & \text{if } \delta_{i,k,\tau}^{(q_{k-1})} \neq 0 \\ \pi_{i,\tau}(x), & \text{if } \delta_{i,k,\tau}^{(q_{k-1})} = 0 \end{cases}$$

where

$$q_{k-1} = \sum_{i=1}^{k-i} p_i + 1$$

$$\delta_{i,k,\tau}^{(q_{k-1})} = \int_{\tau_i}^{\tau_{i+q_{k-1}}} G_{i,k,\tau}^{(p_1,\ldots,p_{k-1})}(s)\, ds$$

and

$$\pi_{i,\tau}(x) = \begin{cases} 1, & x \geq \tau_i \\ 0, & x < \tau_i \end{cases}$$

The generalized $B(p_1 \ldots, p_k)$–spline is an alternate spline, when $p_1 = \ldots = p_k = 2$ [5],and it is the B-spline, when $p_1 = \ldots = p_k = 1$. The generalized $B(p_1, ..., p_k)$-spline has many remarkable properties: the positivity, finiteness of support, it is invariant to shift and consists of the $(k-1)$ and k degree polynomials[8] .

Now, let's consider the examples of $B(p_1, \ldots, p_k)$–spline, when the knots τ are uniformly spaced.

Example 1.1

$$G_{i,2}^{3}(x) = \begin{cases} (x-\tau_i)/\Delta\tau, & x \in [\tau_i, \tau_{i+1}) \\ 1, & x \in [\tau_{i+1}, \tau_{i+2}) \\ 1, & x \in [\tau_{i+2}, \tau_{i+3}) \\ (\tau_{i+4}-x)/\Delta\tau, & x \in [\tau_{i+3}, \tau_{i+4}) \\ 0, & \text{otherwise} \end{cases}$$

Example 1.2

$$G_{i,3}^{(3,4)}(x) = \begin{cases} \frac{(x-\tau_i)^2}{6\Delta\tau^2}, & x \in [\tau_i, \tau_{i+1}) \\ \frac{1}{6} + \frac{x-\tau_{i+1}}{3\Delta\tau}, & x \in [\tau_{i+1}, \tau_{i+2}) \\ \frac{1}{2} + \frac{x-\tau_{i+2}}{3\Delta\tau}, & x \in [\tau_{i+2}, \tau_{i+3}) \\ 1 - \frac{(\tau_{i+4}-x)^2}{6\Delta\tau^2}, & x \in [\tau_{i+3}, \tau_{i+4}) \\ 1 - \frac{(x-\tau_{i+4})^2}{6\Delta\tau^2}, & x \in [\tau_{i+4}, \tau_{i+5}) \\ \frac{1}{2} + \frac{\tau_{i+6}-x}{3\Delta\tau}, & x \in [\tau_{i+5}, \tau_{i+6}) \\ \frac{1}{6} + \frac{\tau_{i+7}-x}{3\Delta\tau}, & x \in [\tau_{i+6}, \tau_{i+7}) \\ \frac{(\tau_{i+8}-x)^2}{6\Delta\tau^2}, & x \in [\tau_{i+7}, \tau_{i+8}) \\ 0, & \text{otherwise} \end{cases}$$

$\Delta\tau$ is the distance between two consecutive knots. For the case of the uniformly spaced knots $G_{i,k+1,\tau}^{(p_1,\ldots,p_k)}$ we also denote as $G_{i,k+1}^{(p_1,\ldots,p_k)}$.

It is obtained the decomposition of generalized B-spline by B-splines[8]. It is held the theorem.

Theorem 2.1. *If $G_{i,k+1,\tau}^{(p_1,\ldots,p_k)}$ is the generalized $B(p_1, \ldots, p_k)$–spline of degree k ($k \geq 0$), for the knot sequence $\tau = \{\tau_l\}$, then there exists $q_k - k$ real numbers $\alpha_{l,i}^{(k+1)}, l = 0, \ldots, q_k - k - 1$ such, that*

$$G^{(p_1,\dots,p_k)}_{i,k+1,\tau}(x)=\sum_{l=0}^{q_k-k-1}\alpha^{(k+1)}_{l,i}B_{i+l,k+1,\tau}(x)$$

for all x, *where* $\alpha^{(k+1)}_{l,i}, l=\overline{0,q_k-k-1}$ *can be recursively defined on* k *by the following way:*

$$\alpha^{(1)}_{0,i}=1$$

and

$$\alpha^{(k+1)}_{l,i}=a^{(k+1)}_{l,i}-a^{(k+1)}_{l-p_k,i+p_k},\quad l=\overline{0,q_k-k-1}$$

for $k>0$

$$a^{(k+1)}_{l,i}=\begin{cases}\sum_{j=i}^{i+l}\alpha^{(k)}_{j-i,i}(\tau_{j+k}-\tau_j)/\Delta^{(k)}_i, & \Delta^{(k)}_i\neq 0\\ 1, & \Delta^{(k)}_i=0\end{cases}$$

$$\Delta^{(k)}_i=\sum_{j=i}^{i+q_{k-1}-k}\alpha^{(k)}_{j-i,i}(\tau_{j+k}-\tau_j)$$

and

$$a^{(k+1)}_{-1,i+p_k}=\ldots=a^{(k+1)}_{-p_k,i+p_k}=\alpha^{(k)}_{q_{k-1}-k+1,i}=\ldots=\alpha^{(k)}_{q_k-k-1,i}=0$$

From the corollary of theorem 2.1 [8], we have the relation for generalized $B(p_1,\dots,p_k)$-spline, when the knots τ_i are uniformly spaced

$$G^{(p_1,\dots,p_k)}_{i,k+1,\tau}(x)=\sum_{\iota=0}^{q_k-k-1}\alpha^{(k+1)}_{\iota}B_{i+\iota,k+1,\tau}(x)$$

where

$$\alpha^{(k+1)}_{l}=\frac{e_{p_1}*\dots*e_{pk}}{P}$$

$$p=\prod_{j=1}^{k-1}p_j$$

$*$—means the discrete convolution operation,
$e_{p_j}-(1,\dots,1),(p_j$ units)
p_j are the parameters, $(p_j\in N)$.

3 Discrete parametric or generalized $B(p_1,\dots,p_k)$-spline using for spline-FIR-median filter design

Using the corollary of theorem 2.1, let's introduce the discrete $B(p_1,\dots,p_k)$-spline.

Definition 3.1. Let $g_{1,t}(j)$- the discrete $B(p_1,\dots,p_k)$-spline of the first degree

$$g_{1,t}(j)=\begin{cases}1, & j=0,\dots,t-1\\ 0, & \text{otherwise}\end{cases}$$

$g^{(p_1,\dots,p_k)}_{k+1,t}(j)$ is expressed on the reccurence relation throug $g^{(p_1,\dots,p_{k-1})}_{k,t}(j)$ by the following way:

$$g_{k+1,t}^{(p_1,\ldots,p_k)}(j) = g_{k,t}^{(p_1,\ldots,p_{k-1})}(j) * e_{p_k} * g_{1,t}$$

for $k \geq 2$.

The length of vector $g_{k+1,t}^{(p_1,\ldots,p_k)}(j)$ depends on the parameters $p_1, ..., p_k$ spline degree and step t as

$$d = d(p_1, \ldots, p_k, t, k) = t(k+1) + q_k - 2k - 1$$

So, for example, under $p_1 = 3, t = 4, k = 1$

$$g_{2,4}^{(3)} = [1, 3, 6, 9, 10, 9, 6, 3, 1]^T$$

As it was noted in introduction, the spline functions are widely used for the problems of digital filtering. Now, let's consider the problem of nonlinear generalized filtering, using the discrete generalized $B(p_1, \ldots, p_k)$-spline, but at first we give the conception of median filter.

The class of nonlinear filters, which is used to smooth impulsive noise, keeping at that edge of image, contains median filters. At the first time the median filters were proposed by J.W.Tukey for the smoothing of statistical time series [7].

Let's assume that M is the number of samples, which are arranged by increasing (decreasing). Then median X_{med} is defined as

$$Y_M = X_{\text{med}}[X_1, \ldots, X_M] = \begin{cases} X_{k+1}, & M = 2k+1 \\ \frac{1}{2}(X_k + X_{k+1}), & M = 2k \end{cases}$$

So, for example

$med\,[0, 1, 5, 3, 8] = 3$

$med\,[0, 2, 4, 6, 7, 0] = \frac{2+4}{2} = 3$

The median filter is defined by the following way:

$Y_i = med\,[X_{i-k,\ldots,} X_{i,\ldots,} X_{i+k}]$

where *med*-median operation and the sequence to which median operation is applied- filter window.

The hybrid FIR-median (linear-nonlinear) filters, which were elaborated by Heinonene, Y.Neuvo[6],are classified to the class of generalized median filters. The principle of hybrid FIR-median filter is: the filter window is splitted on N subwindows (N is odd) in each impulse response is finite (it may be done, as FIR filters are linear and superposition principe is held), then median from N outputs of subwindows is taken. Thus, outputs of hybrid FIR-median filter is expressed as:

$Y(x) = med\,[Y_1(x), Y_2(x), \ldots, Y_N(x)]$

where $Y_i(x)$ - output of subwindow.

Usually, the number of subwindows is taken equaled to three, because of the number of operations required to obtain median is significantly decreased. This filter is defined by means of the following three structures:

$Y_1(x) = \frac{1}{M}(X_1 + \ldots + X_M)$;

$Y_2(x) = 1$;

$Y_3(x) = \frac{1}{M}(X_{M+2} + \ldots + X_{2M+1})$.

Now, we consider the wiegted hybrid FIR-median filters, where the normalized discrete generalized $B(p_1, \ldots, p_k)$-splines are used as weights. Then, weigthed by splines hybrid

FIR-median filters are defined by follows:

$$\begin{aligned} Y_1(x) &= \frac{1}{L_1}(\alpha_1 X_1 + \ldots + \alpha_M X_M); \\ Y_2(x) &= 1; \\ Y_3(x) &= \frac{1}{L_1}(\alpha_M X_{M+2} + \ldots + \alpha_1 X_{2M+1}). \end{aligned} \quad (3)$$

where

$$L_1 = \frac{2}{t^{k+1}\prod_{j=1}^{k} p_j - g_{k+1,t}^{(p_1,\ldots,p_k)}(M+1)}$$

$\alpha_i = g_{k+1,t}^{(p_1,\ldots,p_k)}(i)$
and $t, p_1, \ldots, p_k$ are taken odd. Filter, defined by (3) we call as spline -FIR-median filter. For α_i normalizatiom condition is held:

$$\frac{1}{L_1}\sum_{i=1}^{M} \alpha_i = 1$$

Under $k = 0$, $t = M$ spline-FIR-median filter coincides with hybrid FIR-median one. The results of the experiments of hybrid spline - FIR - median filtering of the signal, corrupted with Gaussian noise, are given in table 3.1. As we can see from the table, the absolute fiitering error is minimal, when

$$p_1 = p_2 = 7$$

Table 3.1

P_1	P_2	P_3	absolute error
5	3		1.189474
5	5	1	1.129032
7	7		1.089888
7	5		1.098901
5	1		1.268041
3	1		1.454545
7	7	7	1.096386
9	9	9	1.116883

References

[1] V. M. Shumilov, The use of approximation by local splines for the problem of information preliminary processing, Computing systems, 93, Novosibirsk, 1982.

[2] V. A. Vasilenko, M. V. Zeuzin, A. V. Kovalkov, Spline-functions and digital filters, Novosibirsk 1984.

[3] H. Hou, H. Andrews, Cubic splines for image interpolation and filtering, IEEE Trans. on acoustics, speech and signal processing, vol. ASSP-26, No. 6, December 1987, pp. 204-207.

[4] T. N. T. Goodman, K. Unsworth. Generation of β-spline curves using a reccurence relation, NATO ASI Series, vol. F 17, Springer-Verlag, Berlin Heidelberg, 1985.

[5] A. Bien, F. Cheng, Alternate spline: a generalized B -spline, Journal of Approximation Theory 51, 1987, pp. 138-159.

[6] P. Heinonen, Y. Neuvo, FIR-median hybrid filters, IEEE Trans. on acoustics, speech and signal processing, vol. ASSP-35, No. 6, 1987, pp. 832-838.

[7] J. W. Tukey, Nonlinear (nonsuperposable) methods for smoothing data, Cong. Rec. EASCON 74, 1974.

[8] H. R. Tshughuryan, Generalized B-splines, Journal of Contemporary Mathematical Analysis (Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika), Allerton Press Inc., vol. XXXI, No.2, New-York, 1996, pp. 72-79.