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Random coding bound for E-capacity region of the channel with two inputs and two outputs

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Abstract

The channel with two inputs and two outputs is studied. A random coding bound for E-capacity region in the case of average error probability is constructed. When $E \to 0$ this bound coincides with the channel capacity region found by Ahlswede.

1 Introduction

The channel with two inputs and two outputs is defined by a matrix of transition probabilities

$$\mathcal{W} = \{W(y_1, y_2|x_1, x_2), x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2, y_1 \in \mathcal{Y}_1, y_2 \in \mathcal{Y}_2\},$$

where $\mathcal{X}_1, \mathcal{X}_2$ are the finite input and $\mathcal{Y}_1, \mathcal{Y}_2$ are the finite output alphabets of the channel.

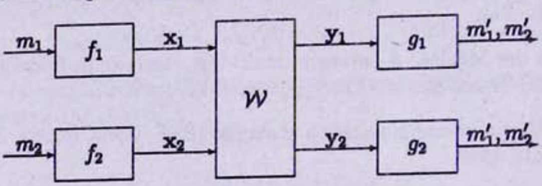

Fig. 1. Channel with two inputs and two outputs.

The channel is supposed to be memoryless, it means that for n-length sequences

$$\mathbf{x}_1 = (x_{11}, x_{12}, ..., x_{1N}) \in \mathcal{X}_1^N, \ \mathbf{x}_2 = (x_{21}, x_{22}, ..., x_{2N}) \in \mathcal{X}_2^N,$$

$$\mathbf{y}_1 = (y_{11}, y_{12}, ..., y_{1N}) \in \mathcal{Y}_1^N, \ \mathbf{y}_2 = (y_{21}, y_{22}, ..., y_{2N}) \in \mathcal{Y}_2^N,$$

the transition probabilities are given in the following way

$$W^N(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2) = \prod_{n=1}^{N} W(y_{1n}, y_{2n}|x_{1n}, x_{2n}).$$

Denote

$$W_i^N(y_i|x_1, x_2) = \sum_{y_j} W(y_1, y_2|x_1, x_2), i = 1, 2; j = 3 - i.$$

Let $\mathcal{M}_1 = \{1, 2, ..., |\mathcal{M}_1|\}$ and $\mathcal{M}_2 = \{1, 2, ..., |\mathcal{M}_2|\}$ be the message sets of corresponding sources. The code for $\mathcal{W}$ is collection of mappings (f_1, f_2, g_1, g_2), where $f_1 : \mathcal{M}_1 \to \mathcal{X}_1^N, f_2 : \mathcal{M}_2 \to \mathcal{X}_2^N$ are encodings and $g_1 : \mathcal{Y}_1^N \to \mathcal{M}_1 \times \mathcal{M}_2, g_2 : \mathcal{Y}_2^N \to \mathcal{M}_1 \times \mathcal{M}_2$ are decodings. The numbers

$$\frac{1}{N} \log |\mathcal{M}_i|, \ i = 1, 2,$$

are called code rates. Here and later we use the logarithmical and exponential functions to the base 2. Denote

$$f(m_1, m_2) = (f_1(m_1), f_2(m_2)),$$

$$g_i^{-1}(m_1, m_2) = \{\mathbf{y}_i : g_i(\mathbf{y}_i) = m_1, m_2\}, \ i = 1, 2,$$

then

$$e_i(m_1, m_2) = W_i^N\{\mathcal{Y}_i^N - g_i^{-1}(m_1, m_2) | f(m_1, m_2)\}, \ i = 1, 2$$

are the error probabilities of messages m_1 and m_2. We consider the average error probabilities of the code

$$e_i(f, g_i) = \frac{1}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \sum_{m_1, m_2} e_i(m_1, m_2), \ i = 1, 2.$$

Let $E = (E_1, E_2)$, $E_i > 0$, $i = 1, 2$. Nonnegative numbers R_1, R_2 are called E-achievable rates pair for $\mathcal{W}$, if for any $\delta > 0, i = 1, 2$, and for sufficiently large N there exists a code such that

$$\frac{1}{N} \log |\mathcal{M}_i| \geq R_i - \delta, \ i = 1, 2$$

and

$$e_i(f, g_i) \leq 2^{-NE_i}, \ i = 1, 2.$$

The region of all E-achievable rates pairs is called E-capacity region for average error probability and denoted $C(E)$. When $E_1 \to 0, E_2 \to 0$ we obtain the capacity region C of the channel $\mathcal{W}$ for average probability of error.

Some scientific works are devoted to different types of two-way channels [1-8]. The capacity region of the channel with two inputs and two outputs is found in [4]. There are detailed surveys in [9-11]. Our bound when $E \to 0$ coincides with the capacity region of the channel constructed in [4]. The proof is similar to construction of random coding bound for restricted two-way channel [12].

2 Formulation of results

Let U, X_1, X_2, Y_1, Y_2 be random variables with values in alphabets $\mathcal{U}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ respectively where $\mathcal{U}$ is a set with $|\mathcal{U}| \leq 6$ and $X_1 \ominus U \ominus X_2$, $U \ominus X_1X_2 \ominus Y_1Y_2$. The restriction $|\mathcal{U}| \leq 6$ can be proved with the help of restriction technique for the number of auxiliary RV's values, suggested independently by Ahlswede R., Korner J. [13] and Wyner A. D. [14]. Consider following probability distributions :

$$P_0 = \{P_0(u), \ u \in \mathcal{U}\},$$

$$P = \{P(u, x_1, x_2) = P_0(u) P(x_1, x_2 | u), \ x_1 \in \mathcal{X}_1, \ x_2 \in \mathcal{X}_2\},$$

$$P_i = \{P_i(x_i | u) = \sum_{x_{3-i}} P(x_1, x_2 | u), \ x_i \in \mathcal{X}_i\}, \ i = 1, 2,$$

$$P^* = \{P_0(u)P^*(x_1,x_2|u) = P_0(u)P_1(x_1|u)P_2(x_2|u),\ x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\},$$

$$P \circ V_i = \{P_0(u)P(x_1,x_2|u)V_i(y_i|x_1,x_2),\ x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2,\ y_i \in \mathcal{Y}_i\},\ i=1,2,$$

where V_1, V_2 are probability matrices. We use also matrix

$$Q = \{Q(x_2|x_1,u) = P(x_1,x_2|u)/P^*(x_1,x_2|u),\ x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\}.$$

The following notations are used for entropies:

$$H_{P,V_i}(Y_i|X_1,X_2) = H_{P,V_i}(Y_i|X_1,X_2,U) =$$

$$= -\sum_{u,x_1,x_2,y_i} P(u,x_1,x_2)V_i(y_i|x_1,x_2)\log V_i(y_i|x_1,x_2),\ i=1,2,$$

$$H_{P,V_i}(Y_i|U) = -\sum_{u,x_1,x_2,y_i} P(u,x_1,x_2)V_i(y_i|x_1,x_2)\log \sum_{x_1,x_2} P(u,x_1,x_2)V_i(y_i|x_1,x_2),\ i=1,2,$$

for mutual informations:

$$I_{P,V_i}(X_1 \wedge Y_i|X_2,U) = H_{P,V_i}(Y_i|X_2,U) - H_{P,V_i}(Y_i|X_1,X_2,U),\ i=1,2,$$

$$I_{P,V_i}(Y_i \wedge X_1X_2|U) = I_{P,V_i}(X_1 \wedge Y_i|X_2U) + I_{P,V_i}(Y_1 \wedge X_2|U),\ i=1,2,$$

and for divergences:

$$D(P||P^*) = \sum_{u,x_1,x_2} P(u,x_1,x_2)\log\frac{P(x_1,x_2|u)}{P^*(x_1,x_2|u)},$$

$$D(V_i||W_i|P) = \sum_{u,x_1,x_2,y_i} P(u,x_1,x_2)V_i(y_i|x_1,x_2)\log\frac{V_i(y_i|x_1,x_2)}{W_i(y_i|x_1,x_2)},\ i=1,2,$$

$$D(P \circ V_i||P^* \circ W_i) = D(P||P^*) + D(V_i||W_i|P).$$

Let us consider *random coding* region $\mathcal{R}_r(E)$ ($|a|^+ = \max(a,0)$) :

$\mathcal{R}_r(P^*,E) = \{(R_1,R_2):$

$$0 \le R_1 \le \min_i \min_{Q,V_i:D(P\circ V_i||P^*\circ W_i)\le E_i} |I_{P,V_i}(X_1 \wedge Y_i|X_2,U) + D(P\circ V_i||P^*\circ W_i) - E_i|^+, \quad (1)$$

$$0 \le R_2 \le \min_i \min_{Q,V_i:D(P\circ V_i||P^*\circ W_i)\le E_i} |I_{P,V_i}(X_2 \wedge Y_i|X_1,U) + D(P\circ V_i||P^*\circ W_i) - E_i|^+, \quad (2)$$

$$R_1 + R_2 \le$$

$$\min_i \min_{Q,V_i:D(P\circ V_i||P^*\circ W_i)\le E_i} |I_{P,V_i}(Y_i \wedge X_1X_2|U) + D(P\circ V_i||P^*\circ W_i) - E_i|^+, i=1,2, \quad (3)$$

and

$$\mathcal{R}_r(E) = \bigcup_{P^*} \mathcal{R}_r(P^*,E).$$

The main result of the paper is formulated in the

Theorem. For all $E=(E_1,E_2), E_1>0, E_2>0,$

$$\mathcal{R}_r(E) \subset C(E).$$

We recall here some necessary combinatorial notions and relations. The type of vector $\mathbf{u} \in \mathcal{U}^N$ is the PD P_0 on $\mathcal{U}$ defined by the equalities

$$P_0(u) = \frac{1}{N} N(u|\mathbf{u}), u \in \mathcal{U},$$

where $N(u|\mathbf{u})$ is the number of occurences of u in the sequence $\mathbf{u}$. The set of all sequences of type P_0 in $\mathcal{U}$ we denote by $T_{P_0}(U)$. Similarly can be defined conditional types $T_{P_1}(X_1|\mathbf{u})$ for given $\mathbf{u} \in T_{P_0}(U)$, $T_{P,V_1}(Y_1|\mathbf{x}_1, \mathbf{x}_2)$ for given $(\mathbf{x}_1, \mathbf{x}_2) \in T_{P,V_1}(X_1, X_2|\mathbf{u})$. We shall use the fact that the quantity of all different types $P \circ V_1$ of the vectors in $\mathcal{X}_1^N \times \mathcal{X}_2^N \times \mathcal{Y}_1^N$ do not exceed $(N+1)^{|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}_1|}$, the inequality

$$(N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}_1|} \exp\{N H_{P,V_1}(Y_1|X_1, X_2)\} \leq |T_{P,V_1}(Y_1|\mathbf{x}_1, \mathbf{x}_2)| \leq$$

$$\leq \exp\{N H_{P,V_1}(Y_1|X_1, X_2)\}, \tag{4}$$

and the following formula: when $(\mathbf{x}_1, \mathbf{x}_2) \in T_P(X_1, X_2|\mathbf{u})$ and $\mathbf{y}_1 \in T_{P,V_1}(Y_1|\mathbf{x}_1, \mathbf{x}_2)$, then

$$W_1^N(\mathbf{y}_1|\mathbf{x}_1, \mathbf{x}_2) = \exp\{-N(D(V_1||W_1|P) + H_{P,V_1}(Y_1|X_1 X_2))\}.$$

3 Proof of theorem

The proof is based on the following lemma. (For the second output result can be stated similarly)

Lemma. For each $E_1 > 0, \delta \in (0, E_1)$ and any type P^* if

$$0 \leq N^{-1} \log |\mathcal{M}_1| \leq \min_{Q,V_1: D(P \circ V_1||P^* \circ W_1) \leq E_1} |I_{P,V_1}(X_1 \wedge Y_1|X_2 U) +$$

$$+ D(P \circ V_1||P^* \circ W_1) - E_1|^+ - \delta,$$

$$0 \leq N^{-1} \log |\mathcal{M}_2| \leq \min_{Q,V_1: D(P \circ V_1||P^* \circ W_1) \leq E_1} |I_{P,V_1}(X_2 \wedge Y_1|U) + D(P \circ V_1||P^* \circ W_1) - E_1|^+ - \delta,$$

then for any $\mathbf{u} \in T_{P_0}(U)$ there exist $|\mathcal{M}_1|$ vectors $f_1(m_1) \in T_{P_1}(X_1|\mathbf{u})$ and $|\mathcal{M}_2|$ vectors $f_2(m_2) \in T_{P_2}(X_2|\mathbf{u})$ such that for matrices $V_1 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1, V_1' : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$ and for sufficiently large N the following inequalities are valid

$$\sum_{f(\mathcal{M}_1, \mathcal{M}_2) \in T_P(X_1, X_2|\mathbf{u})} \left| T_{P,V_1}(Y_1|f(m_1, m_2)) \bigcap \bigcup_{(m_1', m_2') \neq (m_1, m_2)} T_{P',V_1'}(Y_1|f(m_1', m_2')) \right| \leq$$

$$\leq |\mathcal{M}_1| \times |\mathcal{M}_2| \times |T_{P,V_1}(Y_1|f(m_1, m_2))| \times \exp\{-N|E_1 - D(P' \circ V_1'||P^* \circ W_1)|^+\} \times$$

$$\times \exp\{-N(D(P||P^*) - \delta)\}. \tag{5}$$

The proof of the lemma is given in the appendix.

To prove the theorem we must show the existence of the code (f_1, f_2, g_1, g_2) for which $e_i(f, g_i) \leq 2^{-NE_i}$, $i = 1, 2$ and (1-3) saticfies for any fixed P^*. The existence of codewords $f_1(m_1) \in T_{P_1}(X_1|\mathbf{u})$ and $f_2(m_2) \in T_{P_2}(X_2|\mathbf{u})$ satisfying (5) is stated by the lemma. Let us use the following decoding method: to each $\mathbf{y}_1$we put into accordance such (m_1, m_2) for which $\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_1, m_2))$ with such P, V_1 that $D(P \circ V_1||P^* \circ W_1)$ is mininum. A decoding error during the transmission of the message (m_1, m_2) appears if there exist some $(m_1', m_2') \neq (m_1, m_2)$ and Q', V_1' such that

$$\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_1,m_2)) \bigcap T_{P',V_1'}(Y_1|f(m_1',m_2'))$$

and $D(P' \circ V_1' || P^* \circ W_1) \leq D(P \circ V_1 || P^* \circ W_1)$. Denote

$$\mathcal{D} = \{Q', V_1, V_1' : D(P' \circ V_1' || P^* \circ W_1) \leq D(P \circ V_1 || P^* \circ W_1)\}.$$

The average error probability can be upper bounded in the following way:

$$\frac{1}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \sum_{f(\mathcal{M}_1,\mathcal{M}_2)} e_1(m_1,m_2) \leq$$

$$\leq \frac{1}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \sum_{f(\mathcal{M}_1,\mathcal{M}_2)} W_1^N [\bigcup_{\mathcal{D}} T_{P,V_1}(Y_1|f(m_1,m_2)) \bigcap$$

$$\bigcup_{(m_1',m_2') \neq (m_1,m_2)} T_{P',V_1'}(Y_1|f(m_1',m_2')|f(m_1,m_2))] \leq$$

$$\leq \frac{1}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \sum_{Q} \sum_{f(\mathcal{M}_1,\mathcal{M}_2) \in T_P(X_1,X_2|\mathbf{u})} \sum_{\mathcal{D}} W_1^N(\mathbf{y}_1 | f(m_1,m_2)) \times$$

$$\left| T_{P,V_1}(Y_1|f(m_1,m_2)) \bigcap \bigcup_{(m_1',m_2') \neq (m_1,m_2)} T_{P',V_1'}(Y_1|f(m_1',m_2')) \right|.$$

At least from (4) and (5) we obtain

$$\frac{1}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \sum_{f(\mathcal{M}_1,\mathcal{M}_2)} e_1(m_1,m_2) \leq$$

$$\leq \sum_{Q,\mathcal{D}} \exp\{-N(D(V_1||W_1|P) - D(P' \circ V_1' || P^* \circ W_1) + E_1)\} \times \exp\{-N(D(P||P^*) - \delta) \leq$$

$$\leq (N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}_1||\mathcal{U}|} \exp\{-N(E_1 - \delta)\} \leq \exp\{-N(E_1 - 2\delta)\}.$$

Appendix

Proof of Lemma. For some fixed sequence $\mathbf{u} \in T_{P_0}(U)$ let us randomly take $|\mathcal{M}_1|$ vectors from $T_{P_1}(X_1|\mathbf{u})$ and $|\mathcal{M}_2|$ vectors from $T_{P_2}(X_2|\mathbf{u})$. If $D(P' \circ V_1' || P^* \circ W_1) > E_1$, then (5) is equivalent to

$$\frac{|T_P(X_1,X_2|\mathbf{u}) \cap f(m_1,m_2)|}{|\mathcal{M}_1| \times |\mathcal{M}_2|} \leq \exp\{-n(D(P||P^*) - \delta\}.$$

Then it is enough to prove that

$$\sum_{Q,V_1} \sum_{Q',V_1' : D(P' \circ V_1' || P^* \circ W_1) \leq E_1} \mathbf{E} \left| T_{P,V_1}(Y_1|f(m_1,m_2)) \bigcap \bigcup_{(m_1',m_2') \neq (m_1,m_2)} T_{P,V_1'}(Y_1|f(m_1',m_2')) \right| \times$$

$$\times \exp\{N(D(P||P^*) - H_{P,V_1}(Y_1|X_1X_2U) + E_1 - D(P' \circ V_1' || P^* \circ W_1))\} +$$

$$+\sum_{Q,V_1}\sum_{Q',V_1':D(P'\circ V_1'\|P^*\circ W_1)>E_1}\mathbf{E}\frac{|T_P(X_1,X_2|\mathbf{u})\cap f(m_1,m_2)|}{|\mathcal{M}_1|\times|\mathcal{M}_2|}\times\exp\{N(D(P\|P^*)-\delta)\}\leq 1. \tag{6}$$

For this we estimate expectation

$$\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2))\bigcap\bigcup_{(m_1',m_2')\neq(m_1,m_2)}T_{P',V_1'}(Y_1|f(m_1',m_2'))\right|=$$

$$=\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2))\bigcap\bigcup_{m_2'\neq m_2}\bigcup_{m_1'}T_{P',V_1'}(Y_1|f(m_1',m_2'))\right|+$$

$$+\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2))\bigcap\bigcup_{m_1'\neq m_1}T_{P',V_1'}(Y_1|f(m_1',m_2))\right|.$$

The first summand can be estimated in the following way

$$\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2))\bigcap\bigcup_{m_2'\neq m_2}\bigcup_{m_1'}T_{P,V_1'}(Y_1|f(m_1',m_2'))\right|\leq$$

$$\leq\sum_{\mathbf{y}_1\in\mathcal{Y}_1^N}\Pr\{\mathbf{y}_1\in T_{P,V_1}(Y_1|f(m_1,m_2))\}\times\sum_{m_2'\neq m_2}\Pr\{\mathbf{y}_1\in\bigcup_{m_1'}T_{P',V_1'}(Y_1|f(m_1',m_2'))\},$$

which follows from the independence of $f(m_1,m_2)$ and $f(m_1',m_2')$. The first probability will be positive only when $\mathbf{y}_1\in T_{P,V_1}(Y_1|\mathbf{u})$. It can be estimated by the following way

$$\Pr\{\mathbf{y}_1\in T_{P,V_1}(Y_1|f(m_1,m_2))\}=$$

$$=\frac{|T_{P,V_1}(X_1X_2|\mathbf{y}_1,\mathbf{u})|}{|T_{P_1}(X_1|\mathbf{u})|\,|T_{P_2}(X_2|\mathbf{u})|}=\frac{|T_{P,V_1}(X_1|\mathbf{y}_1,\mathbf{u})|\,|T_{P,V_1}(X_2|\mathbf{y}_1,\mathbf{x}_1,\mathbf{u})|}{|T_{P_1}(X_1|\mathbf{u})|\,|T_{P_2}(X_2|\mathbf{u})|}\leq$$
$$\leq(N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1\wedge Y_1|U)+I_{P,V_1}(X_2\wedge X_1Y_1|U))\}=$$
$$=(N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1X_2\wedge Y_1|U)+I_{P,V_1}(X_2\wedge X_1|U))\}.$$

The second probability will be

$$\Pr\{\mathbf{y}_1\in\bigcup_{m_1'}T_{P',V_1'}(Y_1|f(m_1',m_2'))\}=\Pr\{\mathbf{y}_1\in T_{P',V_1'}(Y_1|f(m_2'))\}=$$

$$=\frac{\left|T_{P',V_1'}(X_2|\mathbf{y}_1,\mathbf{u})\right|}{\left|T_{P',V_1'}(X_2|\mathbf{u})\right|}\leq(N+1)^{-|\mathcal{X}_2||\mathcal{U}|}\exp\{-NI_{P',V_1'}(X_2\wedge Y_1|U)\}.$$

Let us estimate the second expectation:

$$\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2))\bigcap\bigcup_{m_1'\neq m_1}T_{P',V_1'}(Y_1|f(m_1',m_2))\right|\leq$$

$$\leq\sum_{\mathbf{y}_1\in\mathcal{Y}_1^N}\Pr\{\mathbf{y}_1\in T_{P,V_1}(Y_1|f(m_1,m_2))\}\times\sum_{m_1'\neq m_1}\Pr\{\mathbf{y}_1\in T_{P,V_1'}(Y_1|f(m_1',m_2))\}.$$

The first probability will be positive when $\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_2),\mathbf{u})$. It is upper bounded by

$$\Pr\{\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_1,m_2))\} = \frac{|T_{P,V_1}(X_1|\mathbf{x}_2,\mathbf{y}_1,\mathbf{u})|}{|T_{P_1}(X_1|\mathbf{u})|} \leq$$
$$\leq (N+1)^{-|\mathcal{X}_1||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1 \wedge Y_1X_2|U))\} =$$
$$= (N+1)^{-|\mathcal{X}_1||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1 \wedge Y_1|X_2U) + I_{P,V_1}(X_2 \wedge X_1|U))\}.$$

The second probability can be upper bounded by

$$\Pr\{\mathbf{y}_1 \in T_{P',V_1'}(Y_1|f(m_1',m_2))\} = \frac{\left|T_{P,V_1'}(X_1|\mathbf{y}_1,f(m_2),\mathbf{u})\right|}{\left|T_{P,V_1'}(X_1|f(m_2),\mathbf{u})\right|} \leq$$
$$\leq (N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-NI_{P',V_1'}(X_1 \wedge Y_1|X_2U)\}.$$

Now we can write

$$\mathbf{E}\left|T_{P,V_1}(Y_1|f(m_1,m_2)) \bigcap \bigcup_{(m_1',m_2')\neq(m_1,m_2)} T_{P',V_1'}(Y_1|f(m_1',m_2'))\right| \leq$$
$$\leq |T_{P,V_1}(Y_1|\mathbf{u})| \times \Pr\{\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_1,m_2))\} \times |\mathcal{M}_2 - 1| \times$$
$$\times \Pr\{\mathbf{y}_1 \in \bigcup_{m_1'} T_{P',V_1'}(Y_1|f(m_1',m_2'))\} + |T_{P,V_1}(Y_1|\mathbf{x}_2,\mathbf{u})| \times \Pr\{\mathbf{y}_1 \in T_{P,V_1}(Y_1|f(m_1,m_2))\} \times$$
$$|\mathcal{M}_1 - 1| \times \Pr\{\mathbf{y}_1 \in T_{P',V_1'}(Y_1|f(m_1',m_2))\} \leq$$
$$\leq (N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_2 \wedge Y_1|X_1,U) + I_{P,V_1}(Y_1 \wedge X_1|U) + I_{P,V_1}(X_2 \wedge X_1|U) -$$
$$- H_{P,V_1}(Y_1|U)\} \times (N+1)^{-|\mathcal{X}_1||\mathcal{U}|}\exp\{-NI_{P',V_1'}(X_2 \wedge Y_1|U)\} \times$$
$$\times \exp\{N \min_{Q,V_1: D(P\circ V_1||P^*\circ W_1)\leq E_1} |I_{P,V_1}(X_2 \wedge Y_1|U) + D(P \circ V_1||P^* \circ W_1) - E_1|^+ - \delta\} +$$
$$+ (N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1 \wedge Y_1|X_2,U) + I_{P,V_1}(X_2 \wedge X_1|U) - H_{P,V_1}(Y_1|X_2,U)\} \times$$
$$\times (N+1)^{-|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-NI_{P',V_1'}(X_1 \wedge Y_1|X_2,U)\} \times$$
$$\times \exp\{N \min_{Q,V_1: D(P\circ V_1||P^*\circ W_1)\leq E_1} |I_{P,V_1}(X_1 \wedge Y_1|X_2,U) + D(P \circ V_1||P^* \circ W_1) - E_1|^+ - \delta\}.$$

By using the following inequality $\min_x f(x) \leq f(x')$ the last expression will be not greater than

$$(N+1)^{-|\mathcal{X}_1||\mathcal{U}|(|\mathcal{X}_2|+1)}\exp\{-N(I_{P,V_1}(X_1X_2 \wedge Y_1|U) + I_{P,V_1}(X_2 \wedge X_1|U)\} \times$$
$$\times \exp\{-N(-H_{P,V_1}(Y_1|U) - D(P' \circ V_1'||P^* \circ W_1) + E_1) + \delta\} +$$
$$+ (N+1)^{-2|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}|}\exp\{-N(I_{P,V_1}(X_1 \wedge Y_1|X_2,U) + I_{P,V_1}(X_2 \wedge X_1|U)\} \times$$
$$\times \exp\{-N(-H_{P,V_1}(Y_1|X_2,U) - D(P' \circ V_1'||P^* \circ W_1) + E_1 + \delta)\}.$$

Now note that

$$\mathbf{E}\frac{|T_P(X_1,X_2|\mathbf{u}) \cap f(m_1,m_2)|}{|\mathcal{M}_1| \times |\mathcal{M}_2|} = \frac{|T_P(X_1,X_2|\mathbf{u})|}{|T_{P_1}(X_1|\mathbf{u})|\,|T_{P_1}(X_2|\mathbf{u})|} \leq$$
$$\leq (N+1)^{-(|\mathcal{X}_1||\mathcal{U}|+|\mathcal{X}_2||\mathcal{U}|)}\exp\{-ND(P||P^*)\}.$$

Placing the obtained results in (6) we prove the lemma.

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