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Path–Extensions and Long Cycles in Graphs

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Abstract

A notion of a path-extension in graphs is introduced and a new estimate for the length of a longest cycle of a graph is established. It is proved that in every graph G, $c \geq (q+2)(\delta - q)$ where C is a longest cycle in G of length c, q is the length of longest path in $G - V(C)$ and δ is the minimum degree of G.

Terminology

We consider only finite undirected graphs without loops or multiple edges. For unexplained terminology see [1]. The set of vertices of a graph G is denoted by $V(G)$ or just V; the set of edges by $E(G)$ or just E. If P is a path in a graph G then $\overrightarrow{P}$ denotes the path P with a given orientation. For $u, v \in V(P)$ let $u\overrightarrow{P}v$ denotes the consecutive vertices on P from u to v in the direction specified by $\overrightarrow{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. For $\overrightarrow{P} = x\overrightarrow{P}y$ and $u \in V(P)$ let $u^+(\overrightarrow{P})$ or just u^+ denotes the successor of $u (u \neq y)$ on $\overrightarrow{P}$ and u^- denotes its predecessor $(u \neq x)$. For cycles we will use analogical definitions. If $v \in V(G)$ then $N(v)$ is the set of all vertices in V adjacent to v. Let $d(v) = |N(v)|$ and $\delta = \min\{d(v)|v \in V\}$. For P is a path of G let $|P| = |V(P)| - 1$. If $|V(P)| = 0$ then $|P| = -1$. For C is a cycle of G let $|C| = |V(C)|$. Extending the notion of a cycle we will assume that every edge $e \in E$ is a cycle of length 2 and every vertex $v \in V$ is a cycle of length 0. And finally we let $|C| = 0$ if $V(C) = \emptyset$. The length of a longest cycle C in G is denoted by $c(G)$ or just c and the length of a longest path P by $p(G)$ or just p.

For H a subgraph of a graph G let $Q = u_0 \ldots u_q$ be a longest path in $G - V(H)$. A tree T is said to be an QH-extension with T-rays $\overrightarrow{T}(u_i)$ if T is a union of Q and vertex-disjoint paths $\overrightarrow{T}(u_i) = u_i\overrightarrow{T}(u_i)u_i^*, i = o, ..., q$ of a graph G-V(H).

Preliminary Results

The following statement will be useful and is easely verified:

Lemma 1 *Let $\overrightarrow{P} = u\overrightarrow{P}v$ be a longest path of a graph G, $\overrightarrow{\Omega} = x\overrightarrow{\Omega}y$ be a path of G with $|V(\Omega) \cap V(P)| = \{x\}$. Then*

(i) $|u\overrightarrow{P}x| \geq |\Omega|$, $|x\overrightarrow{P}v| \geq |\Omega|$.

Moreover, if $u_1, u_2, v_1, v_2 \in V(P)$ *and* $|x\overrightarrow{P}v_1| = |u_1\overrightarrow{P}x| = |u\overrightarrow{P}v_2| + 1 = |u_2\overrightarrow{P}v| + 1 = |\Omega|$, *then*

(ii) $N(y) \cap V(u_1\overrightarrow{P}x^-) = N(y) \cap V(x^+\overrightarrow{P}v_1) = \emptyset$,

(iii) $N(y) \cap V(u\overrightarrow{P}v_2) = N(y) \cap V(u_2\overrightarrow{P}v) = \emptyset$.

Lemma 2 *Let* C *be a longest cycle of a graph* G, Q *be a path in* $G - V(C)$ *and* $u_0, ..., u_q \in V(Q)$. *Then*

$$c \geq \sum_{i=0}^{q} |Z_i| + |\bigcup_{i=0}^{q} Z_i|,$$

where $Z_i = N(u_i) \cap V(C), i = 0, ..., q$.

Proof. If $U_{i=0}^{q} Z_i = \emptyset$ there is nothing to prove. Let $U_{i=0}^{q} Z_i = \{\xi_1, ..., \xi_m\}, m \geq 1$, where $\xi_1, ..., \xi_m$ occurs on $\overrightarrow{C}$ in consecutive order. Set

$$A_i = N(\xi_i) \cap \{u_0, ..., u_q\}, i = 1, ..., m.$$

Suppose $m = 1$. Choose $u, v \in A_1$ such that $|u\overrightarrow{Q}v|$ is maximum. If $u = v$ then $\sum_{i=0}^{q} |Z_i| = 1$ and hence there is nothing to prove. Otherwise,

$$c \geq |u\overrightarrow{Q}v\xi_1 u| \geq \sum_{i=0}^{q} |Z_i| + |\bigcup_{i=0}^{q} Z_i|.$$

So, we may assume $m \geq 2$. Set

$$f(\xi_i) = |\xi_i \overrightarrow{C} \xi_{i+1}|, \quad i = 1, ...m(\xi_{m+1} = \xi_1).$$

It is easy to see that

$$c = \sum_{i=1}^{m} f(\xi_i), \quad \sum_{i=1}^{m} |A_i| = \sum_{i=0}^{q} |Z_i|, \qquad m = |\bigcup_{i=0}^{q} Z_i|. \tag{1}$$

For every $i \in \{1, ..., m\}$ choose $x_i, y_i \in A_i \cup A_{i+1}$ such that $|x_i\overrightarrow{Q}y_i|$ is maximum ($A_{m+1} = A_1$). To show that

$$f(\xi_i) \geq (|A_i| + |A_{i+1}| + 2)/2, \quad i = 1, ..., m, \tag{2}$$

we distinguish two cases:

Case 1. Either $x_i \in A_i, y_i \in A_{i+1}$ or $x_i \in A_{i+1}, y_i \in A_i$.

Let $x_i \in A_i, y_i \in A_{i+1}$. Since C is extreme,

$$f(\xi_i) \geq |\xi_i x_i \overrightarrow{Q} y_i \xi_{i+1}| \geq |A_i \cup A_{i+1}| + 1 \geq$$
$$\max\{|A_i|, |A_{i+1}|\} + 1 \geq (|A_i| + |A_{i+1}|)/2 + 1 = (|A_i| + |A_{i+1}| + 2)/2.$$

If $x_i \in A_{i+1}, y_i \in A_i$ then

$$f(\xi_i) \geq |\xi_i y_i \overleftarrow{Q} x_i \xi_{i+1}| \geq |A_i \cup A_{i+1}| + 1,$$

which implies (2).

Case 2. Either $x_i, y_i \in A_i$ or $x_i, y_i \in A_{i+1}$.

First, suppose $x_i, y_i \in A_i$. Then $x_i, y_i \notin A_{i+1}$ since otherwise (2) holds as in case 1. Choose $x'_i, y'_i \in A_{i+1}$ such that $|x'_i \overrightarrow{Q} y'_i|$ is maximum. If $|x_i \overrightarrow{Q} x'_i| \geq (|A_i| - |A_{i+1}|)/2$ then

$$f(\xi_i) \geq |\xi_i x_i \overrightarrow{Q} y'_i \xi_{i+1}| \geq (|A_i| - |A_{i+1}|)/2 + |A_{i+1}| + 1,$$

which implies (2). Otherwise

$$f(\xi_i) \geq |\xi_i y_i \overleftarrow{Q} x'_i \xi_{i+1}| = |x'_i \overrightarrow{Q} y_i| + 2$$
$$= |x_i \overrightarrow{Q} y_i| - |x_i \overrightarrow{Q} x'_i| + 2 \geq |A_i| - (|A_i| - |A_{i+1}| - 1)/2 + 2,$$

which again implies (2). By symmetry, the case $x_i, y_i \in A_{i+1}$ requires the same argument. Thus (2) is proved. By (2),

$$\sum_{i=1}^{m} f(\xi_i) \geq \sum_{i=1}^{m} (|A_i| + |A_{i+1}| + 2)/2 = \sum_{i=1}^{m} |A_i| + m,$$

which by (1) completes the proof of lemma 2.

By the same argument to the above we obtain:

Lemma 3 *Let C be a longest cycle of a graph G, $Q = u_0 ... u_q$ be a longest path of $G - V(C)$ and T be a maximal QC-extension with T-rays $\overrightarrow{T}(u_i) = u_i \overrightarrow{T}(u_i) u_i^*, i = 0, ..., q$. Then*

$$c \geq \sum_{i=0}^{q} |Z_i| + |\bigcup_{i=0}^{q} Z_i|,$$

where $Z_i = N(u_i^*) \cap V(C), i = 0, ..., q$.

Lemma 4 *For H a subgraph of a graph G, let $Q = u_0 ... u_q$ be a longest path of $G - V(H)$ and T be a maximal QH-extension with T-rays $\overrightarrow{T}(u_i) = u_i \overrightarrow{T}(u_i) u_i^*, i = 0, ..., q$. Then*

$$\sum_{i=0}^{q} |\hat{Z}_i| \leq q(q+1),$$

where $\hat{Z}_i = N(u_i^*) \cap V(T), i = 0, ..., q$.

Proof. Set

$$U_0 = \{u \in V(Q) | u = u^*\}, U_1 = V(Q) - U_0.$$

For every $u \in U_1$ put $u' = u^+(\overrightarrow{T}(u))$. Assume, w.l.o.g., that T is choosen such that $|U_1|$ is maximum. If $U_1 = \emptyset$ then clearly $|\hat{Z}_i| \leq q, i = 0, ..., q$, which immediately implies $\sum_{i=0}^{q} |\hat{Z}_i| \leq q(q+1)$. Let $U_1 \neq \emptyset$. If $vw \in E$ for some $w \in V(T(u)) - \{u, u'\}$ and $u \in U_1, v \in U_0$ then the system

$$\{Q, T(u_0), ..., T(u_q), u\overrightarrow{T}(u)w^-, vw\overrightarrow{T}(u)u^*\} - \{T(u), T(v)\}$$

creates another QH-extension contradicting the maximality of $|U_1|$. Hence

$$u \in U_1, v \in U_0 \Rightarrow N(v) \cap V(T(u) - u) \subseteq \{u'\}.$$

Putting

$$B(u) = \{v \in U_0 | vu' \in E\} \quad \text{if} \quad u \in U_1,$$
$$B^*(u) = \{v \in U_1 | uv' \in E\} \quad \text{if} \quad u \in U_0,$$

we see that

$$\sum_{u \in U_1} |B(u)| = \sum_{u \in U_0} |B^*(u)|. \tag{3}$$

For every $u \in U_1$ set

$$U(u) = \{u\} \bigcup \{v \in U_1 | N(u^*) \cap V(T(v) - v) \neq \emptyset\}.$$

Now let z be an arbitrary vertex of U_1. For convenience, renumber the vertices of $U(z)$ as $U(z) = \{\xi_1, ..., \xi_f\}$. Let $z = \xi_t$ for some $t \quad (1 \le t \le f)$. Assume, w.l.o.g., that $\xi_1, ..., \xi_f$ occurs on $\overrightarrow{Q}$ in consecutive order. For each $i \in \{1, ..., f\}$ choose $\Psi_i \in V(T(\xi_i))$ such that $\xi_i^* \Psi_i \in E$ and $|\xi_i \overrightarrow{T}(\xi_i) \Psi_i|$ is maximum. In particular, we have $\Psi_t = (\xi_t^*)^-$. Set

$$F_i = V(\xi_i' \overrightarrow{T}(\xi_i) \Psi_i), \quad i = 1, ..., f.$$

For every $f \ge 1$ we let

$$\overrightarrow{M}_0 = u_0 \overrightarrow{Q} \xi_1^- \quad \text{and} \quad \overrightarrow{M}_f = \xi_f^+ \overrightarrow{Q} u_q.$$

For $f \ge 2$, in addition, we let

$$\overrightarrow{M}_i = \xi_i^+ \overrightarrow{Q} \xi_{i+1}^-, \quad i = 1, ..., f - 1.$$

For convenience, for each $u \in V(Q)$ we let

$$T(u) = uT(u)u^*, \hat{Z}(u^*) = N(u^*) \bigcap V(T).$$

To prove that $| \hat{Z}(\xi_i^*)| \le q - |B(\xi_i)|$ we distinguish the following cases:
Case 1. $f \ge 2$.
Since Q is extreme,

$$|V(M_i)| \ge |\xi_i \overrightarrow{T}(\xi_i) \Psi_i \xi_i^* \Psi_{i+1} \overleftarrow{T}(\xi_{i+1}) \xi_{i+1}| - 2 \ge |F_i| + |F_{i+1}| + 1, \quad i = 1, ..., f - 1.$$

By lemma 1 (see(i)),

$$|V(M_0)| \ge |V(\xi_1 \overrightarrow{T}(\xi_1) \Psi_1 \xi_t^* \Psi_f \overleftarrow{T}(\xi_f) \xi_f'| - 1 \ge |F_1| + |F_f| + 1,$$
$$|V(M_f)| \ge |V(\xi_f \overrightarrow{T}(\xi_f) \Psi_f \xi_t^* \Psi_1 \overleftarrow{T}(\xi_1) \xi_1'| - 1 \ge |F_1| + |F_f| + 1.$$

However,

$$|V(M_i)| \ge |F_i| + |F_{i+1}| + 1, \quad i = 0, ..., f, \tag{4}$$

where $F_0 = F_f, F_{f+1} = F_1$. For every $i \in \{0, ..., f\}$ let X_i' be the first $|F_i| + 1$vertices set of $\overrightarrow{M}_i$, X_i'' be the last $|F_{i+1}| + 1$ vertices set and $Y_i = V(\overrightarrow{M}_i) - (X_i' \cup X_i'')$. By definition,

$$|X_i'| = |F_i| + 1, \quad |X_i''| = |F_{i+1}| + 1, \quad i = 0, ..., f. \tag{5}$$

Set

$$Y_i' = \{u \in Y_i | \xi_i^* u \notin E\}, \quad Y_i'' = \{u \in Y_i | \xi_i' u \in E\}, i = 0, ..., f,$$
$$X_t = \{u \in X_t' \bigcup X_{t-1}'' | \xi_t' u \in E\}, \quad W_i = X_i' \bigcup X_i'' \bigcup Y_i', i = 0, ..., f.$$

For every $i \in \{0, ..., f\}$ choose $\tau_i \in Y_i$ such that $|\xi_i \overrightarrow{Q} \tau_i|$ is minimum, where $\xi_0 = u_0$. If $v \in Y_i'' - \{\tau_i\}$ then by lemma 1 (see(ii) with $\Omega = v\xi_t' \overrightarrow{T}(\xi_t)\xi_t^*$), $v^- \in Y_i'$, which implies

$$|Y_i'| \geq |Y_i''| - 1, \quad i = 0, ..., f. \tag{6}$$

For every $v \in X_t$, by lemma 1 (see(ii)), $v^-, v^+ \notin X_t$, implying that $|X_t \bigcap X_t'| \leq [|X_t'|/2]$ and $|X_t \bigcap X_t''| \leq [|X_{t-1}''|/2]$. Since $X_t \subseteq X_t' \cup X_t''$, by (5) we obtain

$$|X_t| \leq |F_t| + 1. \tag{7}$$

Prove that

$$|W_i| \geq |F_i| + |F_{i+1}| + |Y_i''| + 1, \quad i = 0, ..., f. \tag{8}$$

If $Y_i = \emptyset$ for some $i \in \{0, ..., f\}$ then $|W_i| = |V(M_i)|$ and therefore (8) holds by (4). Otherwise X_i', X_i'' and Y_i' are vertex-disjoint sets and therefore (8) holds by the definition of W_i using (5) and (6). So , (8) is proved.

By lemma 1 (see(ii) with $\Omega = \xi_t' \overrightarrow{T}(\xi_t)\xi_t^* \Psi_i \overleftarrow{T}(\xi_i)\xi_i, i \neq t$) we have

$$N(\xi_t') \bigcap (\bigcup_{i=0}^{f} X_i'' - X_{t-1}'') = \emptyset.$$

By symmetry,

$$N(\xi_t') \bigcap (\bigcup_{i=0}^{f} X_i' - X_t') = \emptyset.$$

So , recalling the definition of $B(\xi_t)$ and X_t we deduce that

$$B(\xi_t) \subseteq \bigcup_{i=0}^{f} Y_i'' \bigcup X_t. \tag{9}$$

By lemma 1 (see(ii) with $\Omega = \xi_t^* \Psi_i \overleftarrow{T}(\xi_i)\xi_i, i = 0, ..., f$),

$$N(\xi_t^*) \bigcap X_i' = \emptyset, i = 1, ..., f; \quad N(\xi_t^*) \bigcap X_i'' = \emptyset, i = 0, ..., f-1.$$

Furthermore, by lemma 1 (see(iii) with $\Omega_1 = \xi_t^* \Psi_1 \overleftarrow{T}(\xi_1)\xi_1$ and $\Omega_2 = \xi_t^* \Psi_f \overleftarrow{T}(\xi_f)\xi_f$) we have

$$N(\xi_t^*) \bigcap (X_0' \bigcup X_f'') = \emptyset.$$

By the definition of Y_i',

$$N(\xi_t^*) \bigcap Y_i' = \emptyset, \quad i = 0, ..., f.$$

Thus, recalling the definition of W_i we deduce that

$$N(\xi_t^*) \bigcap W_i = \emptyset, \quad i = 0, ..., f. \tag{10}$$

Using (7), (8), (9), (10) and recalling the definition of $\hat{Z}(\xi_i^*)$,

$$|\hat{Z}(\xi_i^*)| = |N(\xi_i^*) \cap V(T)| \leq |V(Q)| - \sum_{i=0}^{f} |W_i| + \sum_{i=1}^{f} |F_i|$$
$$\leq q + 1 - \sum_{i=0}^{f} (|F_i| + |F_{i+1}| + |Y_i''| + 1) + \sum_{i=0}^{f} |F_i|$$
$$\leq q - (\sum_{i=0}^{f} |Y_i''| + |X_t|) + |X_t| - \sum_{i=1}^{f} |F_i| - |F_1| - |F_f| - f \leq q - |B(\xi_t)|.$$

Case 2. $f = 1$.
By lemma 1(see(i)), $|T(\xi_1)| \leq |M_0| + 1$ and $|T(\xi_1)| \leq |M_1| + 1$, which implies

$$|T(\xi_1)| \leq (|M_0| + |M_1| + 2)/2 = q/2. \tag{11}$$

On the other hand,

$$u \in U_1 \Rightarrow B(u) \leq q/2, \tag{12}$$

since $v \in B(u)$ implies $v^+, v^- \notin B(u)$.

Case 2.1. $\hat{Z}_1(\xi_1^*) \cap (V(M_0) \cup V(M_1)) = \emptyset$.
By (11) and (12),

$$|\hat{Z}(\xi_1^*)| \leq |T(\xi_1)| \leq q/2 \leq q - B(\xi_1).$$

Case 2.2. $\hat{Z}_1(\xi_1^*) \cap (V(M_0) \cup V(M_1)) \neq \emptyset$
Case 2.2.1. $\hat{Z}_1(\xi_1^*) \cap V(M_0) \neq \emptyset$, $\hat{Z}_1(\xi_1^*) \cap V(M_1) \neq \emptyset$.
Let $\xi_1^*\xi \in E$ for some $\xi \in V(M_0)$. By lemma 1 (see(i)),

$$|u_0 \overrightarrow{Q} \xi| \geq |\xi \xi_1^* \overleftarrow{T}(\xi_1) \xi_1'| = |T(\xi_1)|.$$

On the other hand, since C is extreme,

$$|\xi \overrightarrow{Q} \xi_1| \geq |\xi \xi_1^* \overleftarrow{T}(\xi_1) \xi_1| = |T(\xi_1)| + 1.$$

Then

$$|V(M_0)| \geq |u_0 \overrightarrow{Q} \xi| + |\xi \overrightarrow{Q} \xi_1| \geq 2|T(\xi_1)| + 1 = 2|F_1| + 3.$$

By symmetry, $|V(M_1)| \geq 2|F_1| + 3$. Then we could argue exactly as in case 1 by the same definitions $X_i', X_i'', Y_i', Y_i'', X_1$ and W_i for $i = 0, 1$.

Case 2.2.2. Either $\hat{Z}_1(\xi_1^*) \cap V(M_0) = \emptyset$ or $\hat{Z}_1(\xi_1^*) \cap V(M_1) = \emptyset$.
Assume, w.l.o.g., that $\hat{Z}_1(\xi_1^*) \cap V(M_1) = \emptyset$. Let X_0', $X_0'', Y_0', Y_0'' W_0$ are defined exactly as in case 1 and let $X_1 = \{u \in X_0'' | \xi_1' u \in E\}$. As in case 1,

$$|X_1| = \left|X_1 \bigcap X_0''\right| \leq |X_0'|/2. \tag{13}$$

Since Q is extreme, $|B(\xi_1) \cap V(M_1)| \leq |V(M_1)|/2$. Then

$$|B(\xi_1)| \leq |Y_0''| + |X_1| + |V(M_1)|/2. \tag{14}$$

Thus, using (13) and (14) by a similar argument to the above (case 1) we obtain $|\hat{Z}(\xi_1^*)| \leq q - |B(\xi_1)|$. We have thus shown that $|\hat{Z}(\xi_t^*)| \leq q - |B(\xi_t)|$ for all $f \geq 1$, i.e.

$$u \in U_1 \Rightarrow |\hat{Z}(u^*)| \leq q - |B(u)|. \tag{15}$$

By (15),

$$\sum_{u\in U_1} |\hat{Z}(u^*)| \le |U_1|q - \sum_{u\in U_1} B(u). \tag{16}$$

Noting that $\hat{Z}(u) \subseteq (V(Q) - \{u\}) \bigcup B^*(u)$ for every $u \in U_0$, we see that

$$u \in U_0 \Rightarrow |\hat{Z}(u)| \le q + |B^*(u)|,$$

which by (3) gives

$$\sum_{u\in U_0} |\hat{Z}(u)| \le |U_0|q + \sum_{u\in U_0} |B^*(u)| = |U_0|q + \sum_{u\in U_1} |B(u)|. \tag{17}$$

By (16) and (17),

$$\sum_{i=0}^{q} |\hat{Z}_i| = \sum_{u\in U_1} |\hat{Z}(u^*)| + \sum_{u\in U_0} |\hat{Z}(u)| \le q(q+1).$$

Lemma 4 is proved.

The main result

Theorem. Every graph G contains a cycle of length at least $(q+2)(\delta-q)$, where q denotes the length of a longest path in $G-V(C)$, C denotes a longest cycle in G and δ the minimum degree of G.

Proof. We can assume that $q \ge 0$ since otherwise $(q=-1)$ G is hamiltonian and $C = |V(G)| \ge \delta + 1 = (q+2)(\delta - q)$. Let $Q = u_0 ... u_q$ be the longest path in $G - V(C)$ and T be a maximal QC-extension with T-rays $\overrightarrow{T}(u_i) = u_i \overrightarrow{T}(u_i) u_i^*, \quad i = 0, ..., q$. Set

$$Z_i = N(u_i^*) \bigcap V(C), \quad \hat{Z}_i = N(u_i^*) \bigcap V(T), \quad i = 0, ..., q.$$

By lemma 4, $\sum_{i=0}^{q} |\hat{Z}_i| \le q(q+1)$. Since

$$|Z_i| + |\hat{Z}_i| = d(u_i^*) \ge \delta, i = 0, ..., q,$$

it follows that

$$\sum_{i=0}^{q} |Z_i| \ge \sum_{i=0}^{q} (\delta - |\hat{Z}_i|) = (q+1)\delta - \sum_{i=0}^{q} |\hat{Z}_i| \ge (q+1)(\delta - q).$$

In particular, $\max_i |Z_i| \ge \delta - q$. Since $|U_{i=0}^{q} Z_i| \ge \max_i |Z_i|$, it follows by lemma 2 that

$$c \ge \sum_{i=0}^{q} |Z_i| + |\bigcup_{i=0}^{q} Z_i| \ge (q+1)(\delta - q) + \max_i |Z_i| \ge (q+2)(\delta - q).$$

Theorem is proved.

The following example of a graph shows that the theorem 1 is best possible. Let G be a $(\delta - q)$-connected graph, S be a minimum vertex cut of G with $< S >= K_{\delta - q}$ and $H_i = K_{q+1}, i = 1, ..., \delta - q + 1$ be the components of $G - S$, where every vertex of H_i is adjacent to all vertices of S. It is easy to see that $|C| = (q+2)(\delta - q)$ and $|Q| = q$, where C denotes a longest cycle of G and Q a longest path of $G - V(C)$.

References

[1] J.A. Bondy and U.S.R.Murty, Graph Theory with Application, MacMillan & co., London and Amer. Elsevier, New York, 1976.