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Generalized B-Spline Using for Signal Restoration

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Abstract

In this paper the new type of spline-functions is considered. The principal properties of these functions are given. The results of approximation of some elementary functions on the base of the generalized B-spline are also shown.

1 Introduction

The one of the important problems of image processing is the lack of good models for the representation of signals. From the computational point of view it is convenient to use B-splines which have many remarkable properties.[1, 2]. The main advantage of B-spline using is the information minimum keeping. But to solve many problems of image processing it is needed the approximation of high precision, which the B-spline using doesn't provide. Therefore, the new models for representation of signals are considered, e.g. β-splines [4], alternate splines [3], etc.

Now, we propose to consider generalized or parametric B-spline.

Let $\tau = \{\tau_i\}$ be a knot sequence, Δ: $a = \tau_0 < \ldots < \tau_N = b$.

Definition 1.1. *Generalized $B(p_1, \ldots, p_k)$-spline ($B(p_1, \ldots, p_k)$) of degree k (order $k+1, k \geq 0$) for the knot sequence τ and parameters $p_1, \ldots, p_k$ ($p_k \in N$) denoted by $G^{(p_1,\ldots,p_k)}_{i,k+1,\tau}$, is defined recursively by the following procedure:*

$$G_{i,1,\tau}(x) = B_{i,1,\tau}(x) = \begin{cases} 1, & x \in [\tau_i, \tau_{i+1}) \\ 0, & x \notin [\tau_i, \tau_{i+1}) \end{cases} \tag{1}$$

and

$$G^{(p_1,\ldots,p_k)}_{i,k+1,\tau}(x) = A_{i,k+1,\tau}(x) - A_{i+p_k,k+1,\tau}(x) \tag{2}$$

for $k \geq 1$ and $p_k \geq 1$, where

$$A_{i,k+1,\tau}(x) = \begin{cases} \int_{\tau_i}^{x} G^{(p_1,\ldots,p_{k-1})}_{i,k,\tau}(s)\, ds / \delta^{(q_{k-1})}_{i,k,\tau}, & \text{if } \delta^{(q_{k-1})}_{i,k,\tau} \neq 0 \\ \pi_{i,\tau}(x), & \text{if } \delta^{(q_{k-1})}_{i,k,\tau} = 0 \end{cases}$$

where

$$q_{k-1} = \sum_{i=1}^{k-1} p_i + 1$$

$$\delta_{i,k,\tau}^{(q_k-1)} = \int_{\tau_i}^{\tau_{i+q_k-1}} G_{i,k,\tau}^{(p_1,\dots,p_k-1)}(s)\,ds$$

and

$$\pi_{i,\tau}(x) = \begin{cases} 1, & x \geq \tau_i \\ 0, & x < \tau_i \end{cases}$$

The generalized $B(p_1 \dots, p_k)$–spline is an alternate spline, when $p_1 = \dots = p_k = 2$ [3],and it is the B-spline, when $p_1 = \dots = p_k = 1$ But, also this generalized B-spline has many remarkable properties: the positivity, finiteness of support, it is invariant to shift and consists of the $(k-1)$ and k degree polynomials

Now, let's consider the examples of $B(p_1, \dots, p_k)$–spline, when the knots τ are uniformly spaced.

Example 1.1.

$$G_{i,2}^{3}(x) = \begin{cases} (x-\tau_i)/\Delta\tau, & x \in [\tau_i, \tau_{i+1}) \\ 1, & x \in [\tau_{i+1}, \tau_{i+2}) \\ 1, & x \in [\tau_{i+2}, \tau_{i+3}) \\ (\tau_{i+4}-x)/\Delta\tau, & x \in [\tau_{i+3}, \tau_{i+4}) \\ 0, & otherwise \end{cases}$$

Example 1.2.

$$G_{i,3}^{(3,4)}(x) = \begin{cases} \frac{(x-\tau_i)^2}{6\Delta\tau^2}, & x \in [\tau_i, \tau_{i+1}) \\ \frac{1}{6} + \frac{x-\tau_{i+1}}{3\Delta\tau}, & x \in [\tau_{i+1}, \tau_{i+2}) \\ \frac{1}{2} + \frac{x-\tau_{i+2}}{3\Delta\tau}, & x \in [\tau_{i+2}, \tau_{i+3}) \\ 1 - \frac{(\tau_{i+4}-x)^2}{6\Delta\tau^2}, & x \in [\tau_{i+3}, \tau_{i+4}) \\ 1 - \frac{(x-\tau_{i+4})^2}{6\Delta\tau^2}, & x \in [\tau_{i+4}, \tau_{i+5}) \\ \frac{1}{2} + \frac{\tau_{i+6}-x}{3\Delta\tau}, & x \in [\tau_{i+5}, \tau_{i+6}) \\ \frac{1}{6} + \frac{\tau_{i+7}-x}{3\Delta\tau}, & x \in [\tau_{i+6}, \tau_{i+7}) \\ \frac{(\tau_{i+8}-x)^2}{6\Delta\tau^2}, & x \in [\tau_{i+7}, \tau_{i+8}) \\ 0, & otherwise \end{cases}$$

$\Delta\tau$ is the distance between two consecutive knots.For the case of the uniformly spaced knots $G_{i,k+1,\tau}^{(p_1,\dots,p_k)}$ we will denote as $G_{i,k+1}^{(p_1,\dots,p_k)}$.

2 Cubic generalized B-spline application for signal restoration

In [5] it is obtained the decomposition of generalized B-spline by the B-splines. It is held the theorem.

Theorem 2.1. *If $G_{i,k+1,\tau}^{(p_1,\dots,p_k)}$ is the generalized $B(p_1, \dots, p_k)$–spline of degree k $(k \geq 0)$, for the knot sequence $\tau = \{\tau_l\}$, then there exists $q_k - k$ real numbers $\alpha_{l,i}^{(k+1)}, l = 0, \dots, q_k - k - 1$ such, that*

$$G_{i,k+1,\tau}^{(p_1,\dots,p_k)}(x) = \sum_{l=0}^{q_k-k-1} \alpha_{l,i}^{(k+1)} B_{i+l,k+1,\tau}(x)$$

for all x, *where* $\alpha_{l,i}^{(k+1)}, l = \overline{0, q_k - k - 1}$ *can be recursively defined on* k *by the following way:*

$$\alpha_{0,i}^{(1)} = 1$$

and

$$\alpha_{l,i}^{(k+1)} = a_{l,i}^{(k+1)} - a_{l-p_k,i+p_k}^{(k+1)}, \quad l = \overline{0, q_k - k - 1}$$

for $k > 0$

$$a_{l,i}^{(k+1)} = \begin{cases} \sum_{j=i}^{i+l} \alpha_{j-i,i}^{(k)}(\tau_{j+k} - \tau_j)/\Delta_i^{(k)}, & \Delta_i^{(k)} \neq 0 \\ 1, & \Delta_i^{(k)} = 0 \end{cases}$$

$$\Delta_i^{(k)} = \sum_{j=i}^{i+q_{k-1}-k} \alpha_{j-i,i}^{(k)}(\tau_{j+k} - \tau_j)$$

and

$$a_{-1,i+p_k}^{(k+1)} = \ldots = a_{-p_k,i+p_k}^{(k+1)} = \alpha_{q_{k-1}-k+1,i}^{(k)} = \ldots = \alpha_{q_k-k-1,i}^{(k)} = 0$$

Using above theorem we try to restore the signal by the given values.

For this purpose we will use the cubic generalized B-spline.

Let $f = [f(\tau_1), ..., f(\tau_k)]^T$ be the input signal. Now, we consider the signal interpolation task, using cubic generalized B-spline:

$$\hat{f}(\varsigma) = \sum_{k=1}^{K} c_k G_{k,4,\tau}^{(p_1,p_2,,p_3)}(\varsigma) \tag{3}$$

$\hat{f}(\varsigma)$ is the interpolated signal, c_k are the coefficients, defined by input signal.

From the corollary of the theorem 2.1 [5], we have a relation for cubic generalized B-spline, when the knots τ_k are uniformly spaced

$$G_{k,4,\tau}^{(p_1,p_2,p_3)}(\varsigma) = \sum_{\iota=0}^{q_3-4} \alpha_\iota^{(4)} B_{k+\iota,4,\tau}(\varsigma),$$

where

$$\alpha^{(4)} = (\alpha_0^{(4)}, ..., \alpha_{q_3-4}^{(4)}),\ \alpha^{(4)} = \frac{e_{p_1} * e_{p_2} * e_{p_3}}{P},\ p = \prod_{j=1}^{2} p_j$$

$*$—means the discrete convolution operation, $e_{p_j} - (1, ..., 1)$, (p_j units) p_j are the parameters, $(p_j \in N)$.

Without generality limitation we will consider the case, when $p_1 = p_2 = 1, p_3 = 3$. Taking into account the theorem 2.1 the generalized B-spline can be represented by means of the truncated degree functions as:

$$G_{k,4,\tau}^{(1,1,3)}(\varsigma) = 4 \sum_{r=0}^{2} \alpha_r^4 \sum_{j=0}^{4} \frac{(\tau_{k+j+r} - \varsigma)_+^3}{\prod_{l=0_{(l\neq j)}} (\tau_{k+j+r} - \tau_{k+l+r})} \tag{4}$$

Let $\zeta = \tau_k + x\Delta$, $0 \leq x \leq 1$ Δ is the interpolation step. Then (3) can be written as

$$\hat{f}(\tau_k + x\Delta) = \frac{1}{6\Delta}\sum_{i=0}^{3} b_i x^{3-i} \tag{5}$$

where

$$b = \Omega c_k \tag{6}$$

$b = [b_0, b_1, b_2, b_3]^T$, $c_k = [c_{k-3}, c_{k-2}, c_{k-1}, c_k, c_{k+1}, c_{k+2}]^T$.
and

$$\Omega = \begin{pmatrix} -1 & 2 & -1 & 1 & -2 & 1 \\ 3 & -3 & 0 & -3 & 3 & 0 \\ -3 & -3 & 0 & 3 & 3 & 0 \\ 1 & 5 & 6 & 5 & 1 & 0 \end{pmatrix}$$

(5) allows to find the interpolated value in an arbitrary point between the knots. Particulary, when $\zeta = \tau_k$, then

$$\hat{f}(\tau_k) = \frac{1}{6\Delta}(c_{k-3} + 5c_{k-2} + 6c_{k-1} + 5c_k + c_{k+1})$$

or

$$\hat{f} = Ec \tag{7}$$

where

$$E = \frac{1}{6\Delta}\begin{pmatrix} 6 & 5 & 1 & 0 & 0 & 0 \\ 5 & 6 & 5 & 1 & 0 & 0 \\ 1 & 5 & 6 & 5 & 1 & 0 \\ 0 & 1 & 5 & 6 & 5 & 1 \\ 0 & 0 & 1 & 5 & 6 & 5 \\ 0 & 0 & 0 & 1 & 5 & 6 \end{pmatrix}$$

The values of $\hat{f}$ in (7) they are considered as the values in the knots τ_k.From (7) coefficients c_k can be found, but b_k by (6). Using the proposed algorithm we have obtained the following results for the approximation of some elementary functions given in Table 1.

Function	Approx.by cubic gener.B-spline ($p_1 = p_2 = 1, p_3 = 3$) ϵ	Approx. by cubic B-spline ϵ
$y = \sqrt{x}$	0,08	0,08
$y = \log x$	0,282	0,294
$y = A + x + \sin x$	0,069	0,085
$y = 3\sin 2x + 4\cos x$	0,043	0,140
$y = 3\sin x + 4\cos x$	0,033	0,075
$y = \sin x$	0,03	0,03
$y = \cos x$	0,03	0,03
$y = \sin x + \cos 2x$	0,028	0,077

Table 1: Approximation of elementary functions by cubic generalized$B$-spline.

ϵ is the mean-absolute error of approximation. As it is easy to see from the table the approximation is more successful for the oscillatory functions.

References

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